MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2008
Problem Set 4 due Friday 10/3/2008

Readings: Notes for lectures 6 and 7. Pay special attention to the section on indicator variables in the lecture 7 notes.

Optional readings:
(a) Sections 3.1-3.7 of [GS]
(b) Chapter 2 of [BT] for a more "elementary" view of the subject, with examples, especially if you have not had a solid undergraduate course before.

Exercise 1. let N be a Poisson random variable with parameter λ. Let the conditional PMF $p_{X|N}(\cdot \mid n)$ be binomial with parameters n and p, and let $Y = N - X$. Prove that the random variables X and Y are independent and that $X \stackrel{d}{=} \text{Pois}(\lambda p)$. ***Hint:*** Just check the definition of independence.

Exercise 2. A 4-sided die has its four faces labeled as a, b, c, d. Each time the die is rolled, the result is a, b, c, or d, with probabilities p_a, p_b, p_c, p_d, respectively. Different rolls are statistically independent. The die is rolled n times. Let N_a and N_b be the number of rolls that resulted in a or b, respectively. Find the covariance of N_a and N_b. ***Hint:*** Use indicator variables.

Exercise 3. We have m married couples ($2m$ individuals). After a number of years, each person has died, independently, with probability p. Let N be the number of surviving individuals. Let C be the number of couples in which both individuals are alive. Find $\mathbb{E}[C \mid N = n]$.

Exercise 4. Let N be a random variable that takes nonnegative integer values. Let $X_1, X_2, \ldots$, be a sequence of i.i.d. discrete random variables that have finite expectation and are independent from N. Show that the expected value of $\sum_{i=1}^{N} X_i$ is $\mathbb{E}[N]\mathbb{E}[X_1]$.

Exercise 5. There are n persons, numbered 1 to n. Each person i is assigned a seat number X_i. The seat numbers are distinct integers in the range $1, \ldots, n$. We assume that the seating is "completely random", that is, the sequence $(X_1, \ldots, X_n)$ is a permutation of the numbers $1, \ldots, n$, and all permutations are equally likely.

(a) Find the probability that the first three persons are seated in the first three seats. (Mathematically, this is the event that the set $\{X_1, X_2, X_3\}$ is the set $\{1, 2, 3\}$.)

(b) Are the events $\{X_1 < X_2\}$ and $\{X_3 < X_4\}$ independent? Provide a brief justification.

(c) Are X_1 and X_2 conditionally independent, given the random variables X_3 and X_4? Provide a brief justification.

(d) Consider the first 10 people, $i = 1, \ldots, 10$. Find the probability that exactly 5 of the first 10 people are seated in seats with numbers in the range $1, \ldots, 8$. (You do not need to simplify your answer.)

(e) For any i and j, with $1 \leq i < j \leq n$, we say that we have an inversion if $X_i > X_j$. Let N be the number of inversions. Find $\mathbb{E}[N]$ and $\mathbb{E}[N^2]$.

The following formula may prove useful:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Exercise 6. (Another application of the "probabilistic method" in combinatorics.) Let G be an undirected graph with neither loops nor multiple edges, and write d_v for the degree of the vertex v. An independent set is a set of vertices no pair of which is joined by an edge. Let $\alpha(G)$ be the size of the largest independent set of G. Use the probabilistic method to show that $\alpha(G) \geq \sum_v 1/(1+d_v)$. *Hint:* Order the nodes at random, and examine the nodes one at a time, putting them in the independent set as long as there are no conflicts with previously examined nodes. Find the expected value of the resulting set.

Exercise 7. A factory has produced n robots, each of which is faulty with probability ϕ. To each robot a test is applied which detects the fault (if present) with probability δ. Let X be the number of faulty robots, and Y the number detected as faulty. Show that

$$\mathbb{E}[X \mid Y] = \frac{n\phi(1-\delta) + (1-\phi)Y}{1-\phi\delta},$$

after stating explicitly any independence assumptions you are making.

In addition, you need to be sure that you can solve elementary problems. As a check, make sure you are able to solve the next problem (not to be handed in).

Drill problem: At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up one, two, or three pens with equal probability 1/3. If he picks up three pens, he does not return to the supply room again that day. If he picks up one or two pens, he will make one additional trip to the supply room, where he again will pick up one, two, or three pens with equal probability 1/3. (The number of pens taken in one trip will not affect the number of pens taken in any other trip.) Calculate the following:

(a) The probability that Oscar gets a total of three pens on any particular day.

(b) The conditional probability that he visited the supply room twice on a given day, given that it is a day in which he got a total of three pens.

(c) $\mathbb{E}[N]$ and $\mathbb{E}[N \mid C]$, where $\mathbb{E}[N]$ is the unconditional expectation of N, the total number of pens Oscar gets on any given day, and $\mathbb{E}[N \mid C]$ is the conditional expectation of N given the event $C = \{N > 3\}$.

(d) $\sigma_{N|C}$, the conditional standard deviation of the total number of pens Oscar gets on a particular day, where N and C are as in part (c).

(e) The probability that he gets more than three pens on each of the next 16 days.

(f) The conditional standard deviation of the total number of pens he gets in the next 16 days given that he gets more than three pens on each of those days.

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