6.013 - Electromagnetics and Applications Fall 2005

Lecture 9 - Oblique Incidence of Electromagnetic Waves

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I. Wave Propagation at an Arbitrary Angle

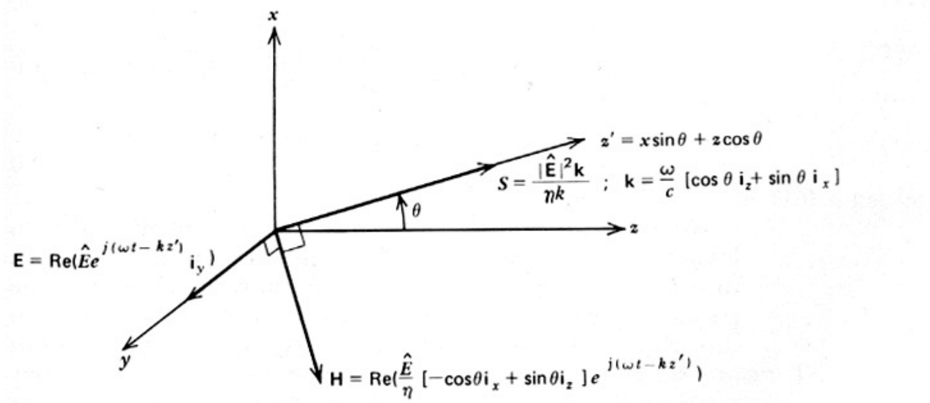

Figure 7-16 The spatial dependence of a uniform plane wave at an arbitrary angle θ can be expressed in terms of a vector wavenumber **k** as $\mathbf{e}^{-j\mathbf{k}\cdot\mathbf{r}}$, where **k** is in the direction of power flow and has magnitude ω/c.

$$z' = x\sin(\theta) + z\cos(\theta)$$
$$kz' = k_x x + k_z z, k_x = k\sin(\theta), k_z = k\cos(\theta), k = \omega\sqrt{\mu\epsilon}$$

$$\bar{E}(x,z,t) = \mathrm{Re}\left[\hat{E}e^{j(\omega t - kz')}\right]\bar{i}_y = \mathrm{Re}\left[\hat{E}e^{j(\omega t - k_x x - k_z z)}\right]\bar{i}_y$$
$$\nabla\times\bar{E} = -j\omega\mu\bar{H} \Rightarrow \bar{H} = -\frac{1}{j\omega\mu}\nabla\times\bar{E} = -\frac{1}{j\omega\mu}\left[-\bar{i}_x\frac{\partial E_y}{\partial z} + \bar{i}_z\frac{\partial E_y}{\partial x}\right]$$
$$\hat{\bar{H}} = -\frac{1}{\not{j}\omega\mu}\left[\not{j}k_z\hat{E}\bar{i}_x - \not{j}k_z\hat{E}\bar{i}_z\right]e^{-j(k_x x + k_z z)}$$
$$= -\frac{\hat{E}}{\eta}\left[\cos(\theta)\bar{i}_x - \sin(\theta)\bar{i}_z\right]e^{-j(k_x x + k_z z)}$$

$$\bar{H}(x,z,t) = \mathrm{Re}\left[-\frac{\hat{E}}{\eta}\left(\cos(\theta)\bar{i}_x - \sin(\theta)\bar{i}_z\right)e^{j(\omega t - k_x x - k_z z)}\right]$$

In general:

$$\bar{k} = k_x\bar{i}_x + k_y\bar{i}_y + k_z\bar{i}_z \text{ is the wave vector}$$
$$\bar{r} = x\bar{i}_x + y\bar{i}_y + z\bar{i}_z \text{ is a position vector}$$
$$e^{-j\bar{k}\cdot\bar{r}} = e^{-j(k_x x + k_y y + k_z z)}$$
$$\nabla\left(e^{-j\bar{k}\cdot\bar{r}}\right) = -j\left(k_x\bar{i}_x + k_y\bar{i}_y + k_z\bar{i}_z\right)e^{-j\bar{k}\cdot\bar{r}} = -j\bar{k}e^{-j\bar{k}\cdot\bar{r}}$$
$$\nabla \rightarrow -j\bar{k}$$

$$
\begin{aligned}
\nabla \times \hat{\bar{E}} = -j\omega\mu\hat{\bar{H}} \quad &\Rightarrow \quad -j\bar{k} \times \hat{\bar{E}} = -j\omega\mu\hat{\bar{H}} \\
&\qquad\quad \bar{k} \times \hat{\bar{E}} = \omega\mu\hat{\bar{H}} \\
\nabla \times \hat{\bar{H}} = j\omega\epsilon\hat{\bar{E}} \quad &\Rightarrow \quad -j\bar{k} \times \hat{\bar{H}} = j\omega\epsilon\hat{\bar{E}} \\
&\qquad\quad \bar{k} \times \hat{\bar{H}} = -\omega\epsilon\hat{\bar{E}} \\
\nabla \cdot \hat{\bar{E}} = 0 \quad &\Rightarrow \quad -j\bar{k} \cdot \hat{\bar{E}} = 0 \qquad (\bar{k} \perp \hat{\bar{E}}) \\
\nabla \cdot \hat{\bar{H}} = 0 \quad &\Rightarrow \quad -j\bar{k} \cdot \hat{\bar{H}} = 0 \qquad (\bar{k} \perp \hat{\bar{H}})
\end{aligned}
$$

$$
\bar{k} \times \left(\bar{k} \times \hat{\bar{E}}\right) = \bar{k}\left(\cancelto{0}{\bar{k} \cdot \hat{\bar{E}}}\right) - \hat{\bar{E}}(\bar{k} \cdot \bar{k}) = \omega\mu\left(\bar{k} \times \hat{\bar{H}}\right) = -\omega^2\epsilon\mu\hat{\bar{E}}
$$

$$
|\bar{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\epsilon\mu
$$

$$
\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})
$$

$$
\hat{\bar{S}} = \frac{1}{2}\hat{\bar{E}} \times \hat{\bar{H}}^*, \hat{\bar{H}} = \frac{1}{\omega\mu}(\bar{k} \times \hat{\bar{E}})
$$

$$
\hat{\bar{S}} = \frac{1}{2}\hat{\bar{E}} \times \left(\frac{1}{\omega\mu}\bar{k} \times \hat{\bar{E}}^*\right) = \frac{1}{2\omega\mu}\left(\bar{k}(\hat{\bar{E}} \cdot \hat{\bar{E}}^*) - \hat{\bar{E}}^*\cancelto{0}{(\hat{\bar{E}} \cdot \bar{k})}\right)
$$

$$
\hat{\bar{S}} = \frac{\bar{k}|\hat{\bar{E}}|^2}{2\omega\mu} \qquad (\hat{\bar{S}} \text{ in the direction of } \bar{k})
$$

II. Oblique Incidence Onto a Perfect Conductor

A. $\bar{E}$ Field Parallel to Interface (TE - Transverse Electric)

$$
\bar{E}_i = \mathrm{Re}\left[\hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)}\bar{i}_y\right]
$$

$$
\bar{H}_i = \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}(-\cos(\theta_i)\bar{i}_x + \sin(\theta_i)\bar{i}_z)e^{j(\omega t - k_{xi}x - k_{zi}z)}\right]
$$

$$
k_{xi} = k\sin(\theta_i), k_{zi} = k\cos(\theta_i), k = \omega\sqrt{\epsilon\mu}, \eta = \sqrt{\frac{\mu}{\epsilon}}
$$

$$
\bar{E}_r = \mathrm{Re}\left[\hat{E}_r e^{j(\omega t - k_{xr}x + k_{zr}z)}\bar{i}_y\right]
$$

$$
\bar{H}_r = \mathrm{Re}\left[\frac{\hat{E}_r}{\eta}(\cos(\theta_r)\bar{i}_x + \sin(\theta_r)\bar{i}_z)e^{j(\omega t - k_{xr}x + k_{zr}z)}\right]
$$

$$
k_{xr} = k\sin(\theta_r), k_{zr} = k\cos(\theta_r)
$$

Boundary conditions require that

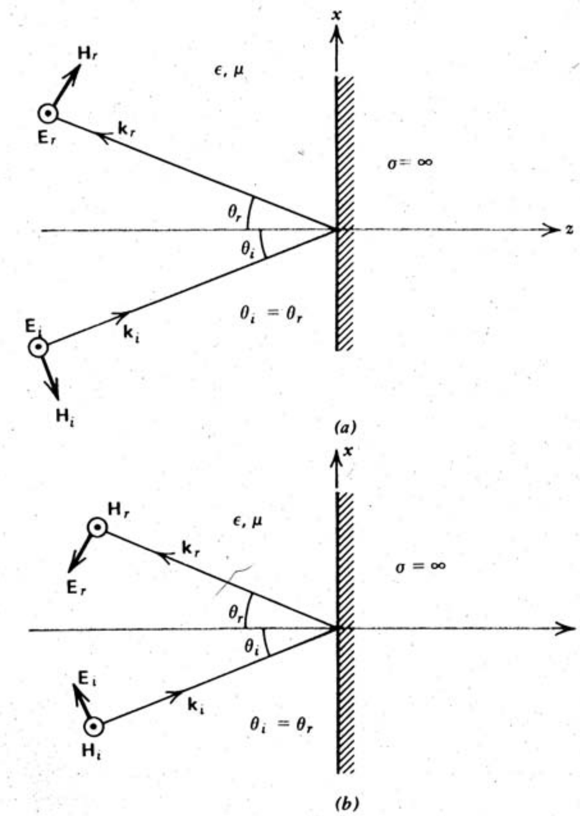

Figure 7-17 A uniform plane wave obliquely incident upon a perfect conductor has its angle of incidence equal to the angle of reflection. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

$$\begin{aligned}
\hat{E}_y(x, z=0) = 0 &= \hat{E}_{yi}(x, z=0) + \hat{E}_{yr}(x, z=0) \\
&= \hat{E}_i e^{-jk_{xi}x} + \hat{E}_r e^{-jk_{xr}x} = 0 \\
\hat{H}_z(x, z=0) = 0 &= \hat{H}_{zi}(x, z=0) + \hat{H}_{zr}(x, z=0) \\
&\frac{1}{\eta}\left(\hat{E}_i e^{-jk_{xi}x}\sin(\theta_i) + \hat{E}_r e^{-jk_{xr}x}\sin(\theta_r)\right) = 0
\end{aligned}$$

$$k_{xi} = k_{xr} \Rightarrow \sin(\theta_i) = \sin(\theta_r) \Rightarrow \theta_i = \theta_r \qquad \left(\begin{array}{c}\text{angle of incidence =} \\ \text{angle of reflection}\end{array}\right)$$

$$\hat{E}_r = -\hat{E}_i$$

$$\begin{aligned}
\hat{E}_i = E_i(\text{real}) \Rightarrow E_y(x, z, t) &= \text{Re}\left[\hat{E}_i\left(e^{-jk_z z} - e^{+jk_z z}\right)e^{j(\omega t - k_x x)}\right] \\
&= 2E_i \sin(k_z z)\sin(\omega t - k_x x) \\
\bar{H}(x, z, t) &= \text{Re}\left[\frac{\hat{E}_i}{\eta}\left[\cos(\theta)\left(-e^{-jk_z z} - e^{+jk_z z}\right)\bar{i}_x + \sin(\theta)\left(e^{-jk_z z} - e^{+jk_z z}\right)\bar{i}_z\right]\right. \\
&\qquad \left. \cdot\, e^{j(\omega t - k_x x)}\right] \\
&= \frac{2E_i}{\eta}\Big[-\cos(\theta)\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_x \\
&\qquad + \sin(\theta)\sin(k_z z)\sin(\omega t - k_x x)\bar{i}_z\Big]
\end{aligned}$$

$$K_y(x, z=0, t) = -H_x(x, z=0, t) = \frac{2E_i}{\eta}\cos(\theta)\cos(\omega t - k_x x)$$

$$\langle \bar{S} \rangle = \frac{1}{2}\,\mathrm{Re}\left[\hat{\bar{E}} \times \hat{\bar{H}}^*\right] = \frac{2E_i^2}{\eta}\sin(\theta)\sin^2(k_z z)\bar{i}_x$$

B. $\bar{H}$ Field Parallel to Interface (TM - Transverse Magnetic)

$$\begin{aligned}
\bar{E}_i &= \mathrm{Re}\left[\hat{E}_i\left(\cos(\theta_i)\bar{i}_x - \sin(\theta_i)\bar{i}_z\right)e^{j(\omega t - k_{xi}x - k_{zi}z)}\right] \\
\bar{H}_i &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}e^{j(\omega t - k_{xi}x - k_{zi}z)}\bar{i}_y\right] \\
\bar{E}_r &= \mathrm{Re}\left[\hat{E}_r\left(-\cos(\theta_r)\bar{i}_x - \sin(\theta_r)\bar{i}_z\right)e^{j(\omega t - k_{xr}x + k_{zr}z)}\right] \\
\bar{H}_r &= \mathrm{Re}\left[\frac{\hat{E}_r}{\eta}e^{j(\omega t - k_{xr}x - k_{zr}z)}\bar{i}_y\right]
\end{aligned}$$

$$\begin{aligned}
E_x(x, z=0, t) = 0 &\Rightarrow \hat{E}_i\cos(\theta_i)e^{-jk_{xi}x} - \hat{E}_r\cos(\theta_r)e^{-jk_{xr}x} = 0 \\
k_{xi} = k_{xr} &\Rightarrow \sin(\theta_i) = \sin(\theta_r) \Rightarrow \theta_i = \theta_r \\
\hat{E}_i &= \hat{E}_r \\
\hat{E}_i = E_i\,(\text{real}) \Rightarrow \bar{E} &= \mathrm{Re}\left[\hat{E}_i\left[\cos(\theta)\left(e^{-jk_z z} - e^{+jk_z z}\right)\bar{i}_x - \sin(\theta)\left(e^{-jk_z z} + e^{+jk_z z}\right)\bar{i}_z\right]e^{j(\omega t - k_x x)}\right] \\
&= 2E_i\left[\cos(\theta)\sin(k_z z)\sin(\omega t - k_x x)\bar{i}_x - \sin(\theta)\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_z\right] \\
\bar{H} &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta}\left(e^{-jk_z z} + e^{+jk_z z}\right)e^{j(\omega t - k_x x)}\bar{i}_y\right] \\
&= \frac{2E_i}{\eta}\cos(k_z z)\cos(\omega t - k_x x)\bar{i}_y
\end{aligned}$$

$$K_x(x, z=0) = H_y(x, z=0) = \frac{2E_i}{\eta}\cos(\omega t - k_x x)$$

$$\sigma_s(x, z=0) = -\epsilon E_z(x, z=0) = 2\epsilon E_i\sin(\theta)\cos(\omega t - k_x x)$$

Check: Conservation of Charge

$$\underbrace{\nabla_\Sigma \cdot \bar{K}}_{\substack{\text{surface}\\\text{divergence}}} + \frac{\partial \sigma_s}{\partial t} = 0 \Rightarrow \frac{\partial K_x}{\partial x} + \frac{\partial \sigma_s}{\partial t} = 0$$

$$\langle \bar{S} \rangle = \frac{1}{2}\,\mathrm{Re}\left(\hat{\bar{E}} \times \hat{\bar{H}}^*\right) = \frac{2E_i^2}{\eta}\sin(\theta)\cos^2(k_z z)\bar{i}_x$$

III. Oblique Incidence Onto a Dielectric

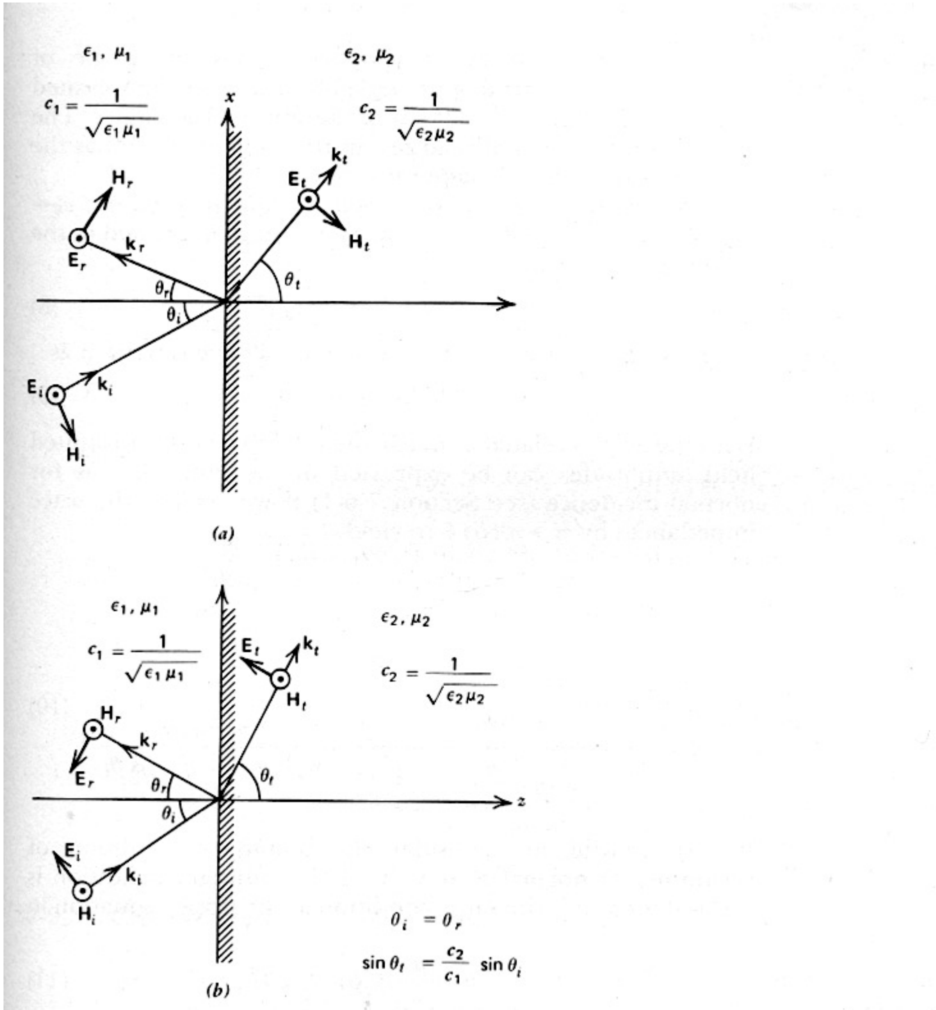

Figure 7-18 A uniform plane wave obliquely incident upon a dielectric interface also has its angle of incidence equal to the angle of reflection while the transmitted angle is given by Snell's law. ***(a)*** Electric field polarized parallel to the interface. ***(b)*** Magnetic field parallel to the interface.

A. TE ($\bar{E} \parallel$ Interface) Waves

$$\bar{E}_i = \mathrm{Re}\left[\hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)} \bar{i}_y\right]$$

$$\bar{H}_i = \mathrm{Re}\left[\frac{\hat{E}_i}{\eta_1}\left(-\cos(\theta_i)\bar{i}_x + \sin(\theta_i)\bar{i}_z\right) e^{j(\omega t - k_{xi}x - k_{zi}z)}\right]$$

$$\bar{E}_r = \mathrm{Re}\left[\hat{E}_r e^{j(\omega t - k_{xr}x + k_{zr}z)} \bar{i}_y\right]$$

$$\bar{H}_r = \mathrm{Re}\left[\frac{\hat{E}_r}{\eta_1}\left(\cos(\theta_r)\bar{i}_x + \sin(\theta_r)\bar{i}_z\right) e^{j(\omega t - k_{xr}x + k_{zr}z)}\right]$$

$$\bar{E}_t = \mathrm{Re}\left[\hat{E}_t e^{j(\omega t - k_{xt}x - k_{zt}z)} \bar{i}_y\right]$$

$$\bar{H}_t = \mathrm{Re}\left[\frac{\hat{E}_r}{\eta_2}\left(-\cos(\theta_t)\bar{i}_x + \sin(\theta_t)\bar{i}_z\right) e^{j(\omega t - k_{xt}x - k_{zt}z)}\right]$$

$$k_{xi} = k_1 \sin(\theta_i) \quad k_{xr} = k_1 \sin(\theta_r) \quad k_{xt} = k_2 \sin(\theta_t)$$
$$k_{zi} = k_1 \cos(\theta_i) \quad k_{zr} = k_1 \cos(\theta_r) \quad k_{zt} = k_2 \cos(\theta_t)$$
$$k_1 = \tfrac{\omega}{c_1} = \omega\sqrt{\epsilon_1\mu_1} \quad k_2 = \tfrac{\omega}{c_2} = \omega\sqrt{\epsilon_2\mu_2}$$
$$c_1 = \tfrac{1}{\sqrt{\epsilon_1\mu_1}} \qquad c_2 = \tfrac{1}{\sqrt{\epsilon_2\mu_2}}$$
$$\eta_1 = \sqrt{\tfrac{\mu_1}{\epsilon_1}} \qquad \eta_2 = \sqrt{\tfrac{\mu_2}{\epsilon_2}}$$

$$E_y(z = 0_-) = E_y(z = 0_+) \Rightarrow \hat{E}_i e^{-jk_{xi}x} + \hat{E}_r e^{-jk_{xr}x} = \hat{E}_t e^{-jk_{xt}x}$$
$$H_x(z = 0_-) = H_x(z = 0_+) \Rightarrow \frac{1}{\eta_1}\left(-\hat{E}_i \cos(\theta_i) e^{-jk_{xi}x} + \hat{E}_r \cos(\theta_r) e^{-jk_{xr}x}\right)$$
$$= -\frac{1}{\eta_2}\hat{E}_t \cos(\theta_t) e^{-jk_{xt}x}$$
$$k_{xi} = k_{xr} = k_{xt} \Rightarrow k_1 \sin(\theta_i) = k_1 \sin(\theta_r) = k_2 \sin(\theta_t)$$
$$\theta_i = \theta_r$$
$$\sin(\theta_t) = \frac{k_1}{k_2}\sin(\theta_i) = \frac{\not\omega c_2}{\not\omega c_1}\sin(\theta_i) = \frac{c_2}{c_1}\sin(\theta_i) \qquad \text{(Snell's Law)}$$

Index of refraction:
$$n = \frac{c_0}{c} = \frac{\sqrt{\epsilon\mu}}{\sqrt{\epsilon_0\mu_0}} = \sqrt{\epsilon_r\mu_r}$$
$$\sin(\theta_t) = \frac{n_1}{n_2}\sin(\theta_i)$$

Reflection Coefficent:
$$R = \frac{\hat{E}_r}{\hat{E}_i} = \frac{\frac{\eta_2}{\cos(\theta_t)} - \frac{\eta_1}{\cos(\theta_i)}}{\frac{\eta_2}{\cos(\theta_t)} + \frac{\eta_1}{\cos(\theta_i)}} = \frac{\eta_2\cos(\theta_i) - \eta_1\cos(\theta_t)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

Transmission Coefficent:
$$T = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2}{\cos(\theta_t)\left(\frac{\eta_2}{\cos(\theta_t)} + \frac{\eta_1}{\cos(\theta_i)}\right)} = \frac{2\eta_2\cos(\theta_i)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

B. Brewster's Angle of No Reflection

$$R = 0 \Rightarrow \eta_2\cos(\theta_i) = \eta_1\cos(\theta_t)$$
$$\eta_2^2\cos^2(\theta_i) = \eta_2^2(1 - \sin^2(\theta_i)) = \eta_1^2\cos^2(\theta_t) = \eta_1^2(1 - \sin^2(\theta_t)) = \eta_1^2\left(1 - \frac{c_2^2}{c_1^2}\sin^2(\theta_i)\right)$$
$$\sin^2(\theta_i)\left(\frac{\eta_1^2 c_2^2}{c_1^2} - \eta_2^2\right) = \eta_1^2 - \eta_2^2$$
$$\sin^2(\theta_i)\left[\frac{\mu_1}{\not\epsilon_1}\frac{\not\epsilon_1\mu_1}{\epsilon_2\mu_2} - \frac{\mu_2}{\epsilon_2}\right] = \frac{\mu_1}{\epsilon_1} - \frac{\mu_2}{\epsilon_2}$$
$$\sin^2(\theta_i) = \sin^2(\theta_B) = \frac{1 - \frac{\epsilon_2\mu_1}{\epsilon_1\mu_2}}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}$$

θ_B is called the Brewster angle. There is no Brewster angle for TE polarization if $\mu_1 = \mu_2$.

C. Critical Angle of No Power Transmission

If $c_2 > c_1$, $\sin(\theta_t)$ can be greater than 1:

$$\sin(\theta_t) = \frac{c_2}{c_1}\sin(\theta_i)$$
$$\theta_i = \theta_c \Rightarrow \sin(\theta_i) = \frac{c_1}{c_2} \qquad \text{(Real solution for } \theta_i \text{ if } c_1 < c_2)$$

θ_c is called the critical angle. At the critical angle, $\theta_t = \frac{\pi}{2} \Rightarrow k_{zt} = k_2\cos(\theta_t) = 0$.

$$\text{For } \theta_i > \theta_c, \quad \sin(\theta_t) > 1 \Rightarrow \cos(\theta_t) = \sqrt{1-\sin^2(\theta_t)} \Rightarrow -j\alpha = k_{zt}$$

$$\bar{E}_t = \text{Re}\left[\hat{E}_t e^{j(\omega t - k_{xt}x)} e^{-\alpha z}\bar{i}_y\right]$$

$$\bar{H}_t = \text{Re}\left[\frac{\hat{E}_t}{\eta_2}\left(-\cos(\theta_t)\bar{i}_x + \sin(\theta_t)\bar{i}_z\right) e^{j(\omega t - k_{xt})} e^{-\alpha z}\right]$$

These are non-uniform plane waves.

$$\langle S_z \rangle = -\frac{1}{2}\text{Re}\left[\hat{E}_y \hat{H}_x^*\right] = -\frac{1}{2}\text{Re}\left[\frac{\hat{E}_t\hat{E}_t^*}{\eta_2}\left(-\cos(\theta_t)\right)^* e^{-2\alpha z}\right]$$

$$= 0 \qquad \left(\cos(\theta_t) = -\frac{j\alpha}{k_2}\right)$$

D. TM ($\bar{H} \parallel$ interface) Waves

$$\bar{E}_i = \text{Re}\left[\hat{E}_i\left(\cos(\theta_i)\bar{i}_x - \sin(\theta_i)\bar{i}_z\right) e^{j(\omega t - k_{xi}x - k_{zi}z)}\right]$$

$$\bar{H}_i = \text{Re}\left[\frac{\hat{E}_i}{\eta_1} e^{j(\omega t - k_{xi}x - k_{zi}z)}\bar{i}_y\right]$$

$$\bar{E}_r = \text{Re}\left[\hat{E}_r\left(-\cos(\theta_r)\bar{i}_x - \sin(\theta_r)\bar{i}_z\right) e^{j(\omega t - k_{xr}x + k_{zr}z)}\right]$$

$$\bar{H}_r = \text{Re}\left[\frac{\hat{E}_r}{\eta_1} e^{j(\omega t - k_{xr}x + k_{zr}z)}\bar{i}_y\right]$$

$$\bar{E}_t = \text{Re}\left[\hat{E}_t\left(\cos(\theta_t)\bar{i}_x - \sin(\theta_t)\bar{i}_z\right) e^{j(\omega t - k_{xt}x - k_{zt}z)}\right]$$

$$\bar{H}_t = \text{Re}\left[\frac{\hat{E}_t}{\eta_2} e^{j(\omega t - k_{xt}x - k_{zt}z)}\bar{i}_y\right]$$

$$E_x(x, z=0_-, t) = E_x(x, z=0_+, t) \Rightarrow \hat{E}_i\cos(\theta_i)e^{-jk_{xi}x} - \hat{E}_r\cos(\theta_r)e^{-jk_{xr}x} = \hat{E}_t\cos(\theta_t)e^{-jk_{xt}x}$$

$$H_y(x, z=0_-, t) = H_y(x, z=0_+, t) \Rightarrow \frac{1}{\eta_1}\left(\hat{E}_1 e^{-jk_{xi}x} + \hat{E}_r e^{-jk_{xr}x}\right) = \frac{1}{\eta_2}\hat{E}_t e^{-jk_{xt}x}$$

$$k_{xi} = k_{xr} = k_{xt} \Rightarrow \theta_i = \theta_r$$

$$\sin(\theta_t) = \frac{c_2}{c_1}\sin(\theta_i) \qquad \text{(Snell's Law)}$$

$$R = \frac{\hat{E}_r}{\hat{E}_i} = \frac{\eta_1\cos(\theta_i) - \eta_2\cos(\theta_t)}{\eta_2\cos(\theta_t) + \eta_1\cos(\theta_1)}$$

$$T = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2\cos(\theta_i)}{\eta_2\cos(\theta_t) + \eta_1\cos(\theta_i)}$$

Brewster's Angle:

$$
\begin{aligned}
R = 0 \Rightarrow \eta_1 \cos(\theta_i) &= \eta_2 \cos(\theta_t) \\
\eta_1^2 \cos^2(\theta_i) = \eta_1^2(1 - \sin^2(\theta_i)) = \eta_2^2 \cos^2(\theta_t) &= \eta_2^2(1 - \sin^2(\theta_t)) \\
&= \eta_2^2 \left(1 - \frac{c_2^2}{c_1^2} \sin^2(\theta_i)\right) \\
\sin^2(\theta_i) \left[\frac{\eta_2^2 c_2^2}{c_1^2} - \eta_1^2\right] &= \eta_2^2 - \eta_1^2 \\
\sin^2(\theta_i) \left[\frac{\cancel{\mu_2}}{\epsilon_2} \frac{\epsilon_1 \mu_1}{\epsilon_2 \cancel{\mu_2}} - \frac{\mu_1}{\epsilon_1}\right] &= \frac{\mu_2}{\epsilon_2} - \frac{\mu_1}{\epsilon_1} \\
\sin^2(\theta_i) = \sin^2(\theta_B) &= \frac{1 - \frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}}{1 - \left(\frac{\epsilon_1}{\epsilon_2}\right)^2}
\end{aligned}
$$

If $\mu_1 = \mu_2$:

$$
\begin{aligned}
\sin^2(\theta_B) = \frac{1}{1 + \frac{\epsilon_1}{\epsilon_2}} &\Rightarrow \tan(\theta_B) = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \\
\theta_B + \theta_t &= \frac{\pi}{2} \\
\frac{1}{\sin^2(\theta_B)} = \frac{1}{\sin^2(\theta_C)} + 1 &\Rightarrow \theta_C > \theta_B
\end{aligned}
$$