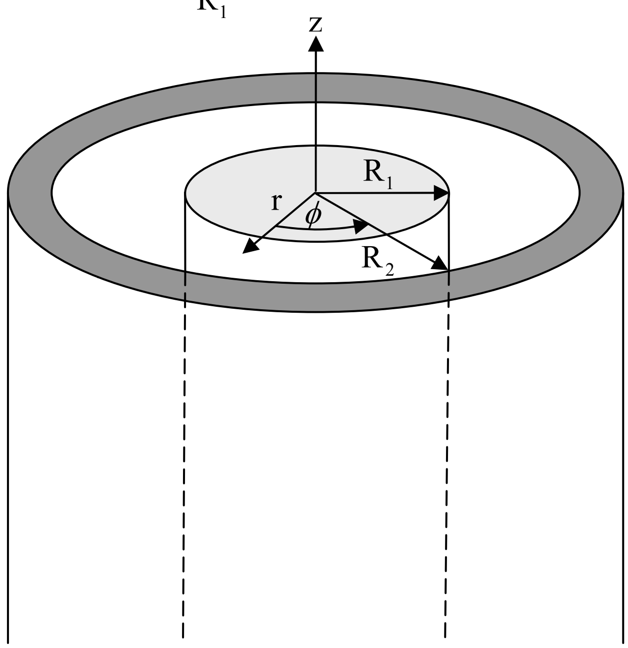
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.013 Electromagnetics and Applications
Quiz 1, October 20, 2005

6.013 Formula Sheet attached.

Problem 1 (35 Points)

$$\bar{J} = J_z(r)\,\bar{i}_z = J_0 \frac{r}{R_1}\,\bar{i}_z \quad \text{amperes/meter}^2 \quad 0 < r < R_1$$



A coaxial cable of very long length carries a z directed current density that varies with radial position on the inner cylinder as:

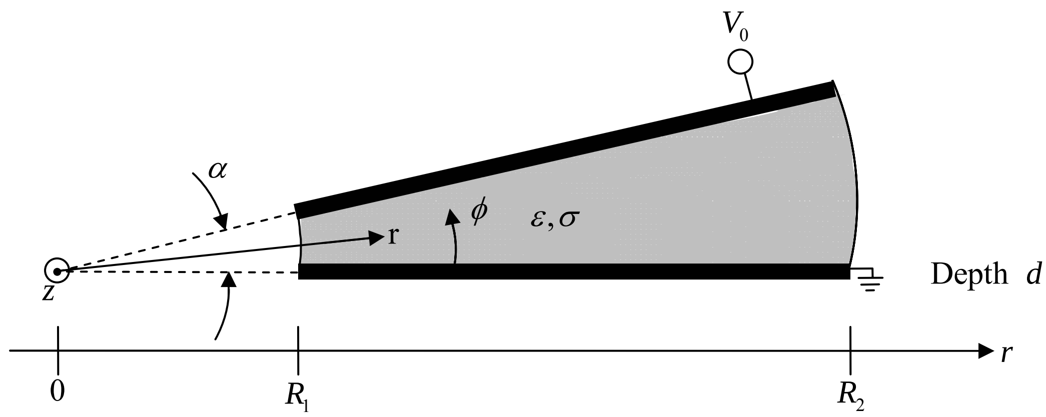
$$\bar{J} = J_z(r)\,\bar{i}_z = J_0 \frac{r}{R_1}\,\bar{i}_z \quad \text{amperes/meter}^2 \quad 0 < r < R_1$$

A perfectly conducting outer cylinder of radius R_2 carries all the return current so that $\bar{H} = 0$ for $r > R_2$.

a) Find the $\bar{H}$ field for $0 < r < R_1$.

b) What are the magnitude and direction of the surface current density (amperes/meter) on the $r = R_2$ surface?

Problem 2 (35 Points)



Two flat electrodes at angle α extend from radius R_1 to R_2 and have a depth d in the z direction (out of the paper). The electrodes enclose a lossy dielectric medium with permittivity ε and conductivity σ. There is no free volume charge within the lossy dielectric. The electric potentials on the electrodes are $\Phi(\phi = 0) = 0$ and $\Phi(\phi = \alpha) = V_0$.

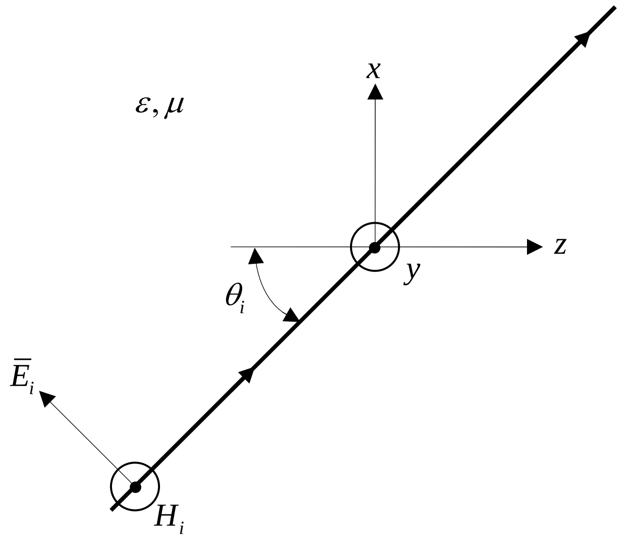
a) The scalar electric potential Φ is of the form $\Phi(\phi) = \mathrm{A}\phi + \mathrm{B}$. What values of A and B satisfy the boundary conditions?

b) Find the electric field $\overline{\mathrm{E}}(\mathrm{r}, \phi)$ within the lossy dielectric.

c) What is the free surface charge density, σ_{sf}, on the electrode at $\phi = \alpha$?

d) What is the capacitance of this device? You may neglect fringing fields.

Problem 3 (30 Points)



An electromagnetic wave is traveling at an angle θ_i with respect to the z axis within a medium with dielectric permittivity ε and magnetic permeability μ. The magnetic field is given as:

$$\bar{H}_i = H_0 \operatorname{Re}\left[e^{j\left(2\pi\times10^8 t - \pi\left(x+\sqrt{3}z\right)\right)} \right] \bar{i}_y \qquad \text{amperes/meter} .$$

a) Find the frequency in Hertz.

b) Find the wavelength in meters.

c) Find the numerical value of the speed of light in the medium in meters/second.

d) Find the angle θ_i.

6.013 Quiz 1 Formula Sheet
October 20, 2005

Cartesian Coordinates (x,y,z):

$$\nabla\Psi = \hat{x}\frac{\partial\Psi}{\partial x} + \hat{y}\frac{\partial\Psi}{\partial y} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\bullet\overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\overline{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

Cylindrical coordinates (r,φ,z):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial\Psi}{\partial\phi} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\bullet\overline{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\overline{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right) = \frac{1}{r}\det\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2}$$

Spherical coordinates (r,θ,φ):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

$$\nabla\bullet\overline{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\overline{A} = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$= \frac{1}{r^2\sin\theta}\det\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

Gauss' Divergence Theorem:

$$\int_V \nabla\bullet\overline{G}\,dv = \oint_A \overline{G}\bullet\hat{n}\,da$$

Stokes' Theorem:

$$\int_A (\nabla\times\overline{G})\bullet\hat{n}\,da = \oint_C \overline{G}\bullet d\overline{\ell}$$

Vector Algebra:

$$\nabla = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z$$

$$\overline{A}\bullet\overline{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\nabla\bullet(\nabla\times\overline{A}) = 0$$

$$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\bullet\overline{A}) - \nabla^2\overline{A}$$

<u>Basic Equations for Electromagnetics and Applications</u>

Fundamentals

$\bar{f} = q\left(\bar{E} + \bar{v} \times \mu_o \bar{H}\right)[N]$

$\nabla \times \bar{E} = -\partial \bar{B}/\partial t$

$\oint_c \bar{E} \bullet d\bar{s} = -\frac{d}{dt}\int_A \bar{B} \bullet d\bar{a}$

$\nabla \times \bar{H} = \bar{J} + \partial \bar{D}/\partial t$

$\oint_c \bar{H} \bullet d\bar{s} = \int_A \bar{J} \bullet d\bar{a} + \frac{d}{dt}\int_A \bar{D} \bullet d\bar{a}$

$\nabla \bullet \bar{D} = \rho \rightarrow \oint_A \bar{D} \bullet d\bar{a} = \int_V \rho dv$

$\nabla \bullet \bar{B} = 0 \rightarrow \oint_A \bar{B} \bullet d\bar{a} = 0$

$\nabla \bullet \bar{J} = -\partial \rho/\partial t$

$\bar{E}$ = electric field (Vm^{-1})
$\bar{H}$ = magnetic field (Am^{-1})
$\bar{D}$ = electric displacement (Cm^{-2})
$\bar{B}$ = magnetic flux density (T)
Tesla (T) = Weber m^{-2} = 10,000 gauss
ρ = charge density (Cm^{-3})
$\bar{J}$ = current density (Am^{-2})
σ = conductivity (Siemens m^{-1})
$\bar{J}_s$ = surface current density (Am^{-1})
ρ_s = surface charge density (Cm^{-2})
$\varepsilon_o \approx 8.854 \times 10^{-12}\ Fm^{-1}$
$\mu_o = 4\pi \times 10^{-7}\ Hm^{-1}$
$c = (\varepsilon_o\mu_o)^{-0.5} \cong 3 \times 10^8\ ms^{-1}$
$e = -1.60 \times 10^{-19}$ C
$\eta_o \cong 377$ ohms $= (\mu_o/\varepsilon_o)^{0.5}$

$\left(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2\right)\bar{E} = 0$ [Wave Eqn.]

$E_y(z,t) = E_+(z-ct) + E_-(z+ct) = \mathrm{Re}\{\underline{E}_y(z)e^{j\omega t}\}$

$H_x(z,t) = \eta_o^{-1}[E_+(z-ct)-E_-(z+ct)]$ [or(ωt-kz) or (t-z/c)]

$\oint_A \left(\bar{E} \times \bar{H}\right) \bullet d\bar{a} + (d/dt)\int_V \left(\varepsilon|\bar{E}|^2/2 + \mu|\bar{H}|^2/2\right)dv$

$= -\int_V \bar{E} \bullet \bar{J}\ dv$ (Poynting Theorem)

Media and Boundaries

$\bar{D} = \varepsilon_o \bar{E} + \bar{P}$

$\nabla \bullet \bar{D} = \rho_f,\ \tau = \varepsilon/\sigma$

$\nabla \bullet \varepsilon_o \bar{E} = \rho_f + \rho_p$

$\nabla \bullet \bar{P} = -\rho_p,\ \bar{J} = \sigma \bar{E}$

$\bar{B} = \mu \bar{H} = \mu_o\left(\bar{H} + \bar{M}\right)$

$\varepsilon = \varepsilon_o\left(1 - \omega_p^2/\omega^2\right),\ \ \omega_p = \left(Ne^2/m\varepsilon_o\right)^{0.5}$ **(Plasma)**

$\varepsilon_{eff} = \varepsilon(1 - j\sigma/\omega\varepsilon)$

skin depth $\delta = (2/\omega\mu\sigma)^{0.5}$ [m]

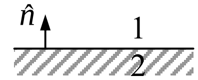
$\bar{E}_{1//} - \bar{E}_{2//} = 0$

$\bar{H}_{1//} - \bar{H}_{2//} = \bar{J}_s \times \hat{n}$

$B_{1\perp} - B_{2\perp} = 0$

$D_{1\perp} - D_{2\perp} = \rho_s$

↳ $0 =$ if $\sigma = \infty$



Electromagnetic Quasistatics

$\bar{E} = -\nabla\Phi(r),\quad \Phi(r) = \int_{V'}\left(\rho(\bar{r})/4\pi\varepsilon|\bar{r}' - \bar{r}|\right)dv'$

$\nabla^2\Phi = \frac{-\rho_f}{\varepsilon}$

C = Q/V = Aε/d [F]
L = Λ/I
i(t) = C dv(t)/dt
v(t) = L di(t)/dt = dΛ/dt
$w_e = Cv^2(t)/2;\ w_m = Li^2(t)/2$
$L_{solenoid} = N^2\mu A/W$
$\tau = RC,\ \tau = L/R$

$\Lambda = \int_A \bar{B} \bullet d\bar{a}$ (per turn)

$\bar{F} = \bar{I} \times \mu_o \bar{H}\ \left[Nm^{-1}\right]$

Electromagnetic Waves

$\left(\nabla^2 - \mu\varepsilon\partial^2/\partial t^2\right)\bar{E} = 0$ [Wave Eqn.]

$\left(\nabla^2 + k^2\right)\underline{\bar{E}} = 0,\ \ \underline{\bar{E}} = \underline{\bar{E}}_o e^{-j\bar{k}\bullet\bar{r}}$

$k = \omega(\mu\varepsilon)^{0.5} = \omega/c = 2\pi/\lambda$

$k_x^2 + k_y^2 + k_z^2 = k_o^2 = \omega^2\mu\varepsilon$

$v_p = \omega/k,\ \ v_g = (\partial k/\partial\omega)^{-1}$

$\theta_r = \theta_i$

$\sin\theta_t/\sin\theta_i = k_i/k_t = n_i/n_t$

$\theta_c = \sin^{-1}\left(n_t/n_i\right)$

$\theta_B = \tan^{-1}\left(\varepsilon_t/\varepsilon_i\right)^{0.5}$ for TM

$\theta > \theta_c \Rightarrow \underline{\bar{E}}_t = \underline{\bar{E}}_i \underline{T} e^{+\alpha x - jk_z z}$

$\underline{k} = \bar{k}' - j\bar{k}''$

$\underline{\Gamma} = \underline{T} - 1$

$\underline{T}_{TE} = 2/\left(1 + \left[\eta_i\cos\theta_t/\eta_t\cos\theta_i\right]\right)$

$\underline{T}_{TM} = 2/\left(1 + \left[\eta_t\cos\theta_t/\eta_i\cos\theta_i\right]\right)$