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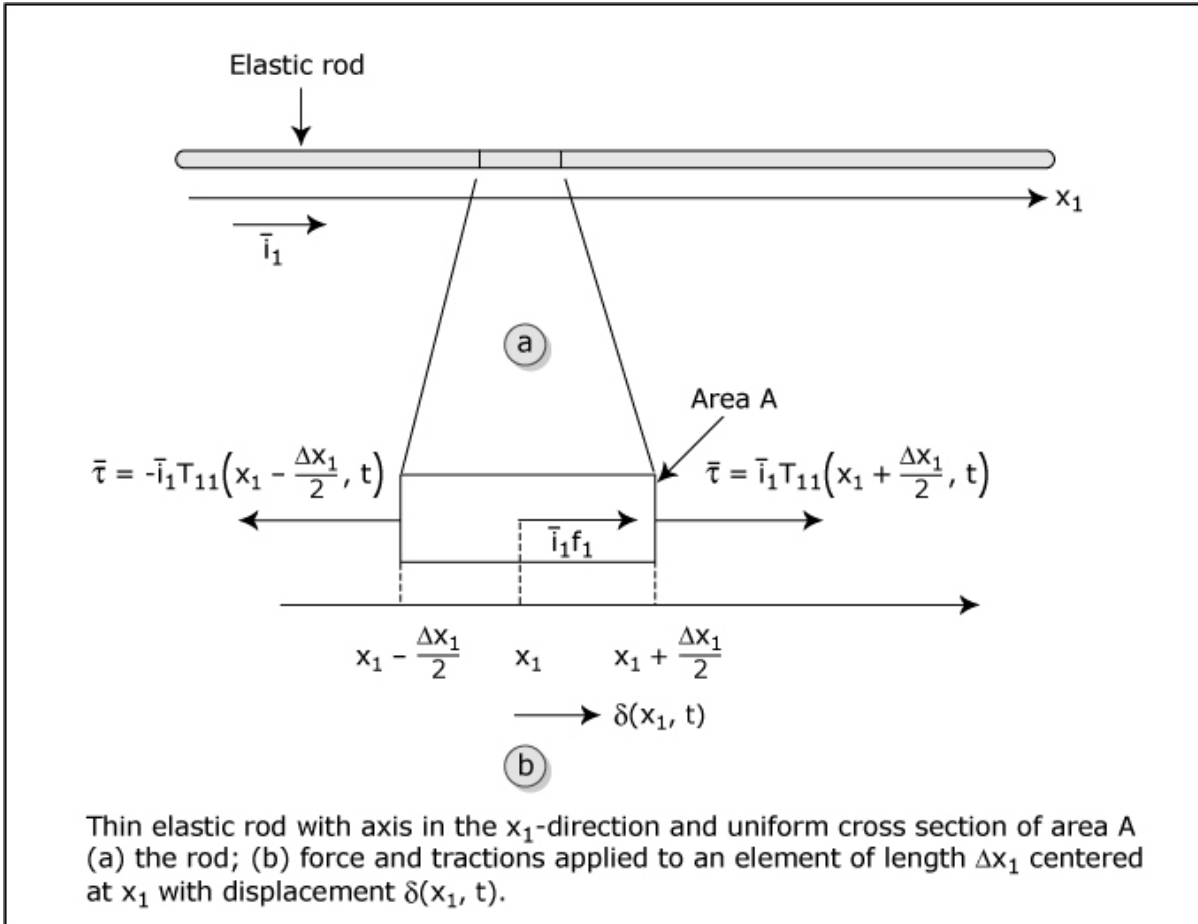
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6.641, Electromagnetic Fields, Forces, and Motion
 Prof. Markus Zahn
Lecture 16: Elastic Waves On a Thin Rod

I. Governing Equations



$$A \Delta x_1 \rho \frac{\partial^2 \delta}{\partial t^2} = \left[T_{11} \left(x_1 + \frac{\Delta x_1}{2} \right) - T_{11} \left(x_1 - \frac{\Delta x_1}{2} \right) \right] A + F A \Delta x_1$$

$$\rho \frac{\partial^2 \delta}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} + F \quad (F \text{ is body force density})$$

$$T_{11} = E \frac{\partial \delta}{\partial x_1} \quad \text{Hooke's Law (stress-strain relation)}$$

↑
 Young's modulus (modulus of elasticity)

$$\rho \frac{\partial^2 \delta}{\partial t^2} = E \frac{\partial^2 \delta}{\partial x_1^2} + F$$

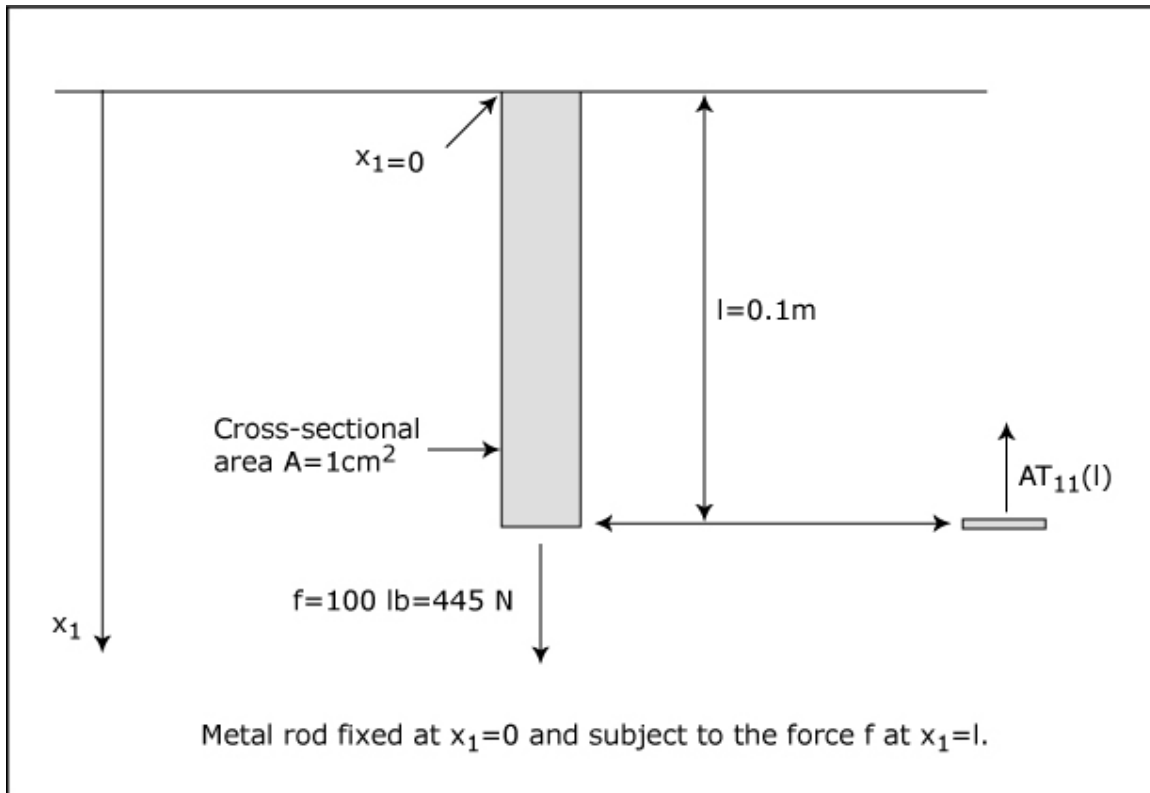
Modulus of Elasticity E and Density ρ for Representative Materials

Material	E-units of 10^{11}N/m^2	ρ -units of 10^3kg/m^3	v_p -units [†] m/sec
Aluminum (pure and alloy)	0.68-0.79	2.66-2.89	5100
Brass (60-70% Cu, 40-30% Zn)	1.0-1.1	8.36-8.51	3500
Copper	1.17-1.24	8.95-8.98	3700
Iron, cast (2.7-3.6% C)	0.89-1.45	6.96-7.35	4000
Steel (carbon and low alloy)	1.93-2.20	7.73-7.87	5100
Stainless steel (18% Cr, 8% Ni)	1.93-2.06	7.65-7.93	5100
Titanium (pure and alloy)	1.06-1.14	4.52	4900
Glass	0.49-0.79	2.38-3.88	4500
Methyl methacrylate	0.024-0.034	1.16	1600
Polyethylene	$1.38-3.8 \times 10^{-3}$	0.915	530
Rubber	$0.79-4.1 \times 10^{-5}$	0.99-1.245	46

* See S.H Crandall, and N.C. Dahl, *An Introduction to the Mechanics of Solids*, McGraw Hill, New York, 1959, for a list of references for these constants and a list of these constants in English units.

† Computed from average values of E and ρ .

II. Example Case Study - Change in Rod Length



Steady State: $\frac{\partial \delta}{\partial t} = 0, \quad F = 0$

$$\frac{\partial^2 \delta}{\partial x_1^2} = 0 \Rightarrow \delta = ax_1 + b$$

Boundary Condition at $x_1 = l$: $A T_{11}(x_1 = l) = A E \left. \frac{\partial \delta}{\partial x_1} \right|_{x_1=l} = AEa = f$

$$\delta(x_1 = 0) = 0 = b$$

$$\delta(x_1) = \frac{f x}{AE}$$

$$\delta(x_1 = l) = \frac{f l}{AE}$$

$$E = 0.7 \times 10^{11} \text{ N/m}^2 \text{ (Aluminum)}$$

$$A = 0.0001 \text{ m}^2, \quad f = 445 \text{ N}, \quad l = 0.1 \text{ m}$$

$$\delta(x_1 = l) = \frac{445(0.1)}{10^{-4} (0.7 \times 10^{11})} = 6.36 \times 10^{-6} \text{ m} = 6.36 \text{ } \mu\text{m}$$

III. Elastic Waves : $F = 0$

A. Wave Equation $\frac{\partial^2 \delta}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 \delta}{\partial x^2}, \quad v_p^2 = \frac{E}{\rho} \Rightarrow v_p = \sqrt{\frac{E}{\rho}}$

$$\frac{\partial^2 \delta}{\partial t^2} = v_p^2 \frac{\partial^2 \delta}{\partial x^2}$$

$$\delta = \delta_+(x - v_p t) + \delta_-(x + v_p t)$$

Superposition: $\delta = \delta_+ (x - v_p t) = \delta_+ (\alpha)$, $\alpha = x - v_p t$

$$\frac{\partial \delta}{\partial t} = \frac{\partial \delta}{\partial \alpha} \frac{\partial \alpha}{\partial t} = -v_p \frac{\partial \delta}{\partial \alpha}$$

$$\frac{\partial^2 \delta}{\partial t^2} = \frac{\partial}{\partial \alpha} \left(-v_p \frac{\partial \delta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial t} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2}$$

$$\frac{\partial \delta}{\partial x} = \frac{\partial \delta}{\partial \alpha} \frac{\partial \alpha}{\partial x} = \frac{\partial \delta}{\partial \alpha}$$

$$\frac{\partial^2 \delta}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial \delta}{\partial \alpha} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial \delta}{\partial \alpha} \right) \frac{\partial \alpha}{\partial x} = \frac{\partial^2 \delta}{\partial \alpha^2}$$

$$\frac{\partial^2 \delta}{\partial t^2} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2} = v_p^2 \frac{\partial^2 \delta}{\partial x^2} = v_p^2 \frac{\partial^2 \delta}{\partial \alpha^2}$$

Q.E.D

For $\delta = \delta_- (x + v_p t)$ let $v_p \rightarrow -v_p$

B. Velocity and Stress Variables

$$v = \frac{\partial \delta}{\partial t}, \quad T = E \frac{\partial \delta}{\partial x}$$

$$\rho \frac{\partial^2 \delta}{\partial t^2} = \rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}$$

$$\frac{\partial T}{\partial t} = E \frac{\partial^2 \delta}{\partial x \partial t} = E \frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial t} \left| \rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x} \right.$$

$$\Rightarrow \frac{\partial^2 T}{\partial x \partial t} = \rho \frac{\partial^2 v}{\partial t^2} = E \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial}{\partial x} \left| \frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x} \right.$$

$$\frac{\partial^2 v}{\partial t^2} = v_p^2 \frac{\partial^2 v}{\partial x^2}, \quad v_p^2 = E/\rho$$

$$\frac{\partial}{\partial x} \left| \frac{\partial v}{\partial t} = \frac{1}{\rho} \frac{\partial T}{\partial x} \right.$$

$$\Rightarrow \frac{1}{\rho} \frac{\partial^2 T}{\partial x^2} = \frac{1}{E} \frac{\partial^2 T}{\partial t^2}$$

$$\frac{\partial}{\partial t} \left| \frac{\partial v}{\partial x} = \frac{1}{E} \frac{\partial T}{\partial t} \right.$$

$$\frac{\partial^2 T}{\partial t^2} = v_p^2 \frac{\partial^2 T}{\partial x^2}, \quad v_p^2 = E/\rho$$

$$v = v_+(x - v_p t) + v_-(x + v_p t)$$

$$T = T_+(x - v_p t) + T_-(x + v_p t)$$

C. Method of Characteristics

Lagrange Multipliers

$$\begin{aligned} \lambda_1 \left| \rho \frac{\partial v}{\partial t} + 0 \frac{\partial v}{\partial x} - 0 \frac{\partial T}{\partial t} - \frac{\partial T}{\partial x} = 0 \right. \\ \lambda_2 \left| 0 \frac{\partial v}{\partial t} - E \frac{\partial v}{\partial x} + \frac{\partial T}{\partial t} + 0 \frac{\partial T}{\partial x} = 0 \right. \end{aligned}$$

$$\cancel{\rho \lambda_1} \frac{\partial v}{\partial t} - \frac{\lambda_2 E}{\rho \lambda_1} \frac{\partial v}{\partial x} + \frac{\lambda_2}{\rho \lambda_1} \frac{\partial T}{\partial t} - \frac{\lambda_1}{\rho \lambda_1} \frac{\partial T}{\partial x} = 0$$

$$\underbrace{\frac{dv}{dt}}_{\frac{\lambda_2}{\rho \lambda_1} \left(\frac{dT}{dt} \right)} = 0$$

$$dv(x, t) = \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx$$

$$\Rightarrow \frac{dx}{dt} = \frac{-\lambda_2 E}{\rho \lambda_1} = \frac{-\lambda_1}{\lambda_2}$$

$$dT(x, t) = \frac{\partial T}{\partial t} dt + \frac{\partial T}{\partial x} dx$$

$$\left(\frac{\lambda_2}{\lambda_1} \right)^2 = \frac{\rho}{E} \Rightarrow \frac{\lambda_2}{\lambda_1} = \pm \sqrt{\frac{\rho}{E}}$$

$$\frac{dv}{dt} + \frac{\lambda_2}{\rho \lambda_1} \frac{dT}{dt} = 0 \Rightarrow dv \pm \frac{1}{\sqrt{\rho E}} dT = 0$$

$$dv + \frac{1}{\sqrt{\rho E}} dT = 0 \quad \text{on} \quad \frac{dx}{dt} = -\sqrt{E/\rho} = -v_p$$

$$dv - \frac{1}{\sqrt{\rho E}} dT = 0 \quad \text{on} \quad \frac{dx}{dt} = +\sqrt{E/\rho} = +v_p$$

$$v + \frac{1}{\sqrt{\rho E}} T = C_- \quad \text{on} \quad \frac{dx}{dt} = -v_p$$

$$v - \frac{1}{\sqrt{\rho E}} T = C_+ \quad \text{on} \quad \frac{dx}{dt} = +v_p$$

$$v = \frac{C_+ + C_-}{2}$$

$$\frac{T}{\sqrt{\rho E}} = \frac{C_- - C_+}{2}$$

D. "Cook-Book" Method

$$dv(x, t) = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial t} dt, \quad \rho \frac{\partial v}{\partial t} = \frac{\partial T}{\partial x}$$

$$dT(x, t) = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt, \quad \frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x}$$

$$\begin{bmatrix} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} \partial v / \partial t \\ \partial v / \partial x \\ \partial T / \partial t \\ \partial T / \partial x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ dv \\ dT \end{bmatrix}$$

$$\frac{\partial v}{\partial t} = \begin{array}{c} \det \\ \left[\begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dv & dx & 0 & 0 \\ dT & 0 & dt & dx \end{array} \right] \\ \hline \det \\ \left[\begin{array}{cccc} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{array} \right] \end{array}$$

← set each determinant to zero for general solution

1st Characteristic equation

$$\det \begin{bmatrix} \rho & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} = 0$$

$$\rho \begin{bmatrix} -E & 1 & 0 \\ dx & 0 & 0 \\ 0 & dt & dx \end{bmatrix} + dt \begin{bmatrix} 0 & 0 & -1 \\ -E & 1 & 0 \\ 0 & dt & dx \end{bmatrix}$$

$$= \rho dx \begin{bmatrix} -E & 1 \\ dx & 0 \end{bmatrix} - dt \begin{bmatrix} -E & 1 \\ 0 & dt \end{bmatrix}$$

$$= -(dx)^2 \rho + E(dt)^2$$

$$\left(\frac{dx}{dt} \right)^2 = \frac{E}{\rho} \Rightarrow \frac{dx}{dt} = \pm \sqrt{E/\rho} = \pm v_p$$

2nd Characteristic equation

$$\det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -E & 1 & 0 \\ dv & dx & 0 & 0 \\ dT & 0 & dt & dx \end{bmatrix} = \begin{bmatrix} 0 & -E & 1 \\ dv & dx & 0 \\ dT & 0 & dt \end{bmatrix}$$

$$= E \begin{bmatrix} dv & 0 \\ dT & dt \end{bmatrix} + \begin{bmatrix} dv & dx \\ dT & 0 \end{bmatrix}$$

$$= E dv dt - dx dT = 0$$

$$E dv - dT \frac{dx}{dt} = 0 \Rightarrow E dv \mp \sqrt{\frac{E}{\rho}} dT = 0$$

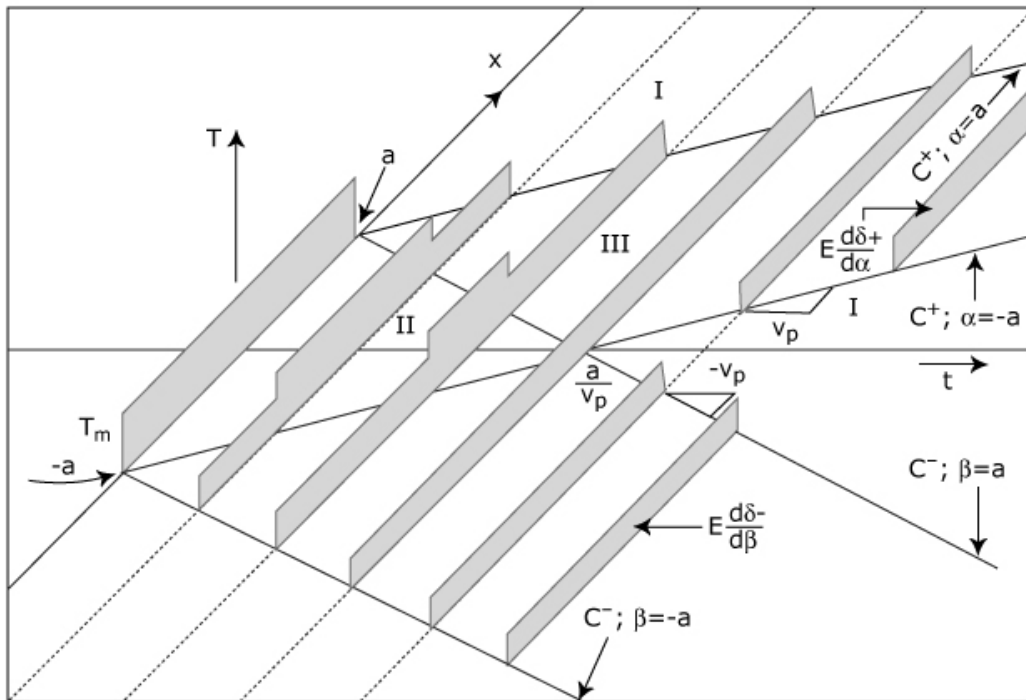
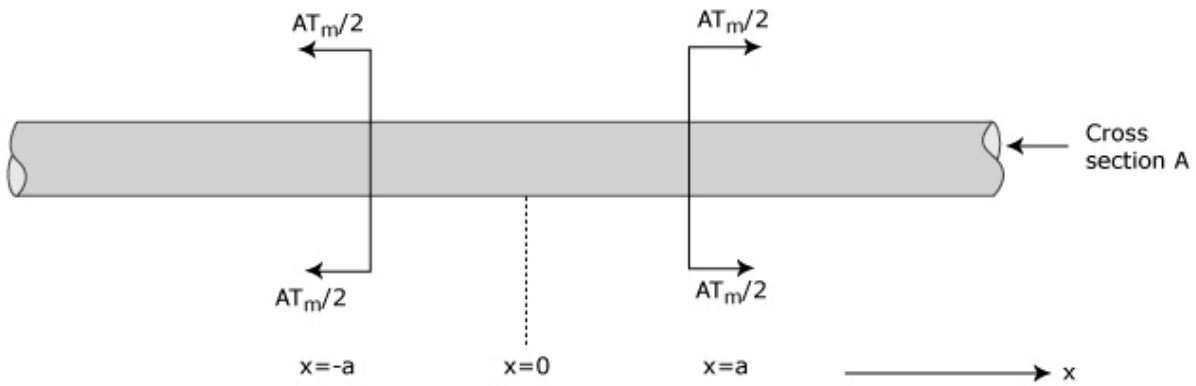
$$dv \mp \frac{dT}{\sqrt{\rho E}} = 0 \Rightarrow v \mp \frac{T}{\sqrt{\rho E}} = \text{constant}$$

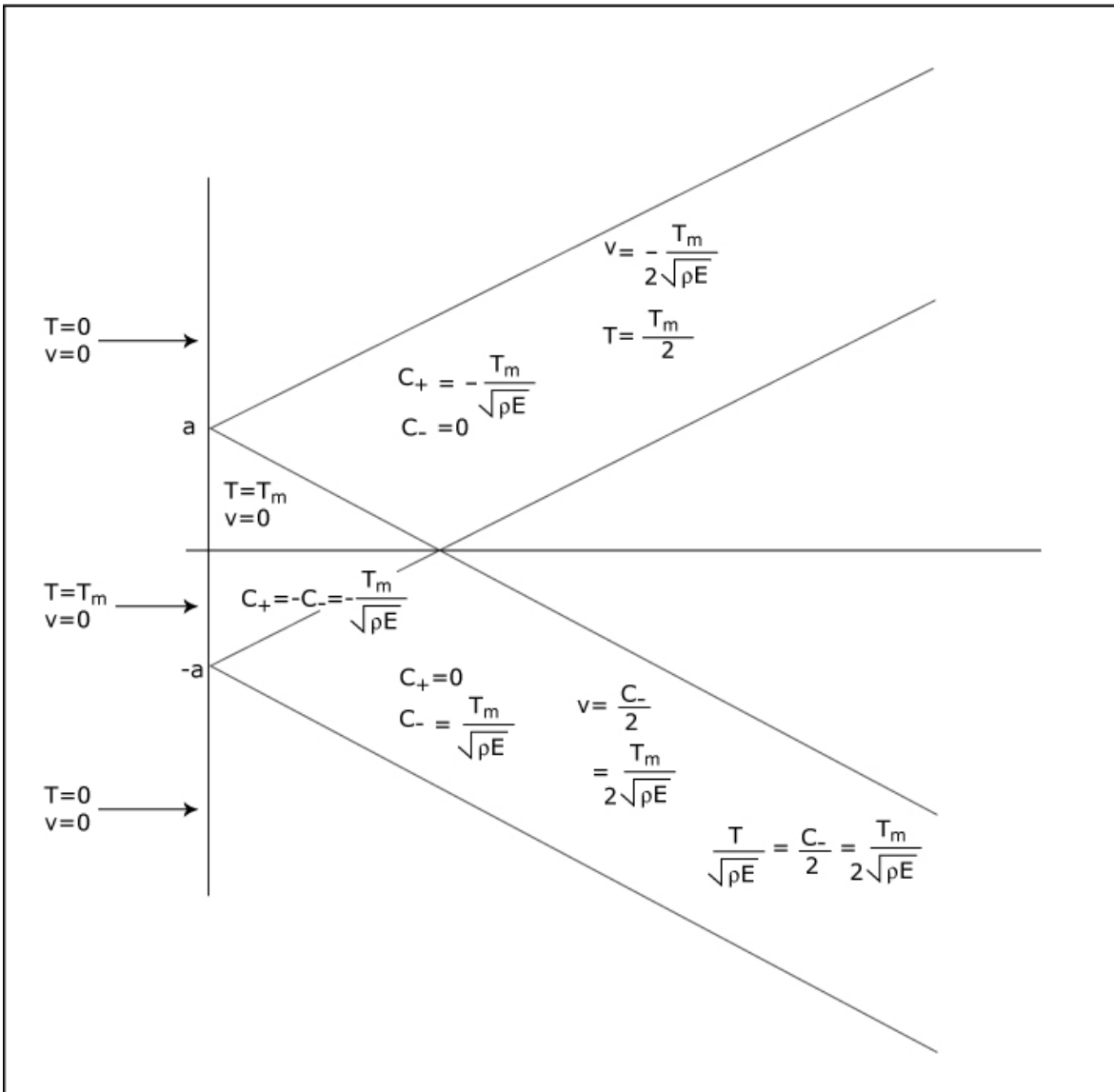
$$v - \frac{T}{\sqrt{\rho E}} = C_+ \quad \text{on} \quad \frac{dx}{dt} = +v_p$$

$$v + \frac{T}{\sqrt{\rho E}} = C_- \quad \text{on} \quad \frac{dx}{dt} = -v_p$$

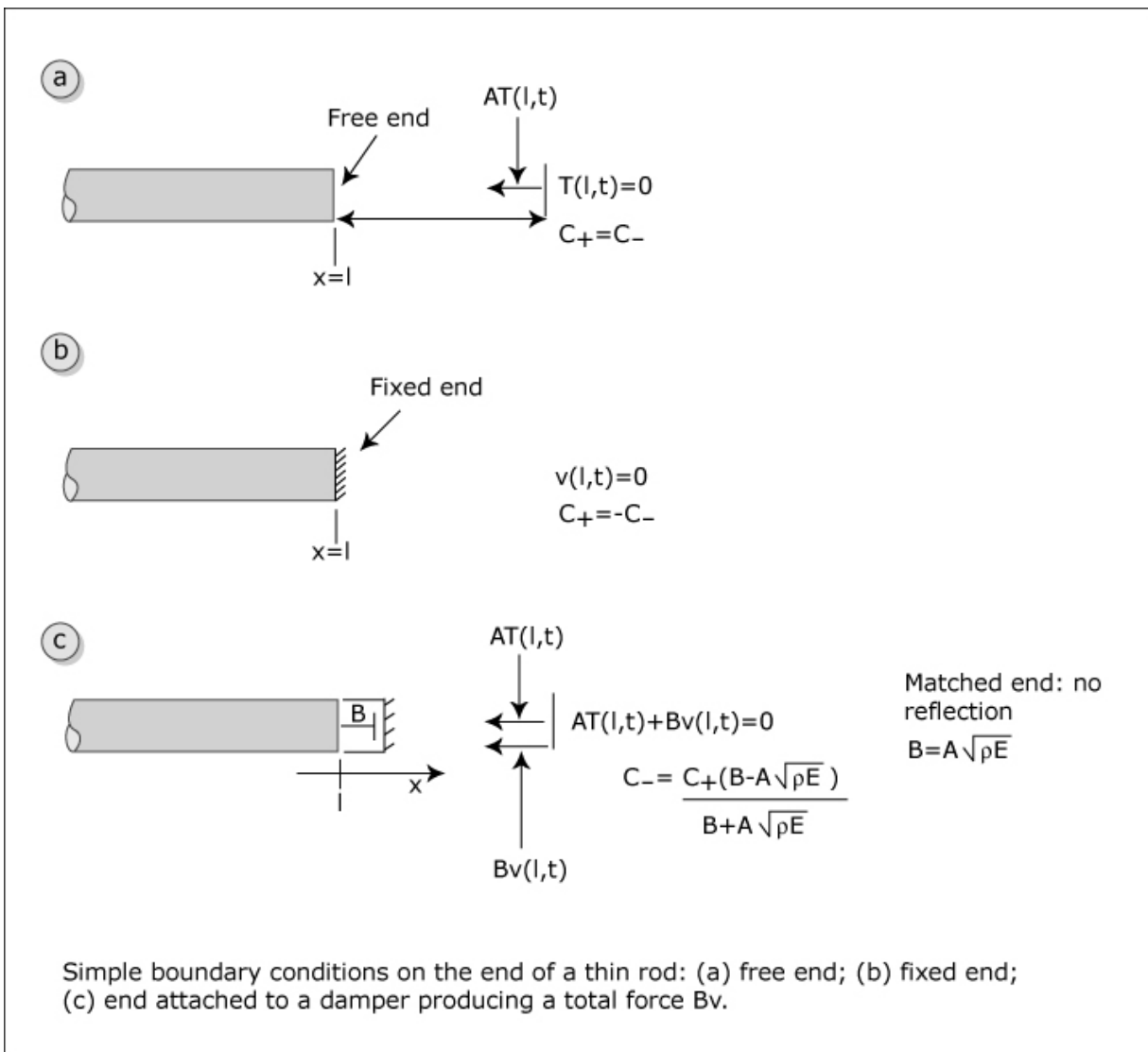
IV. Case Study : Region of Initial Uniform Stress

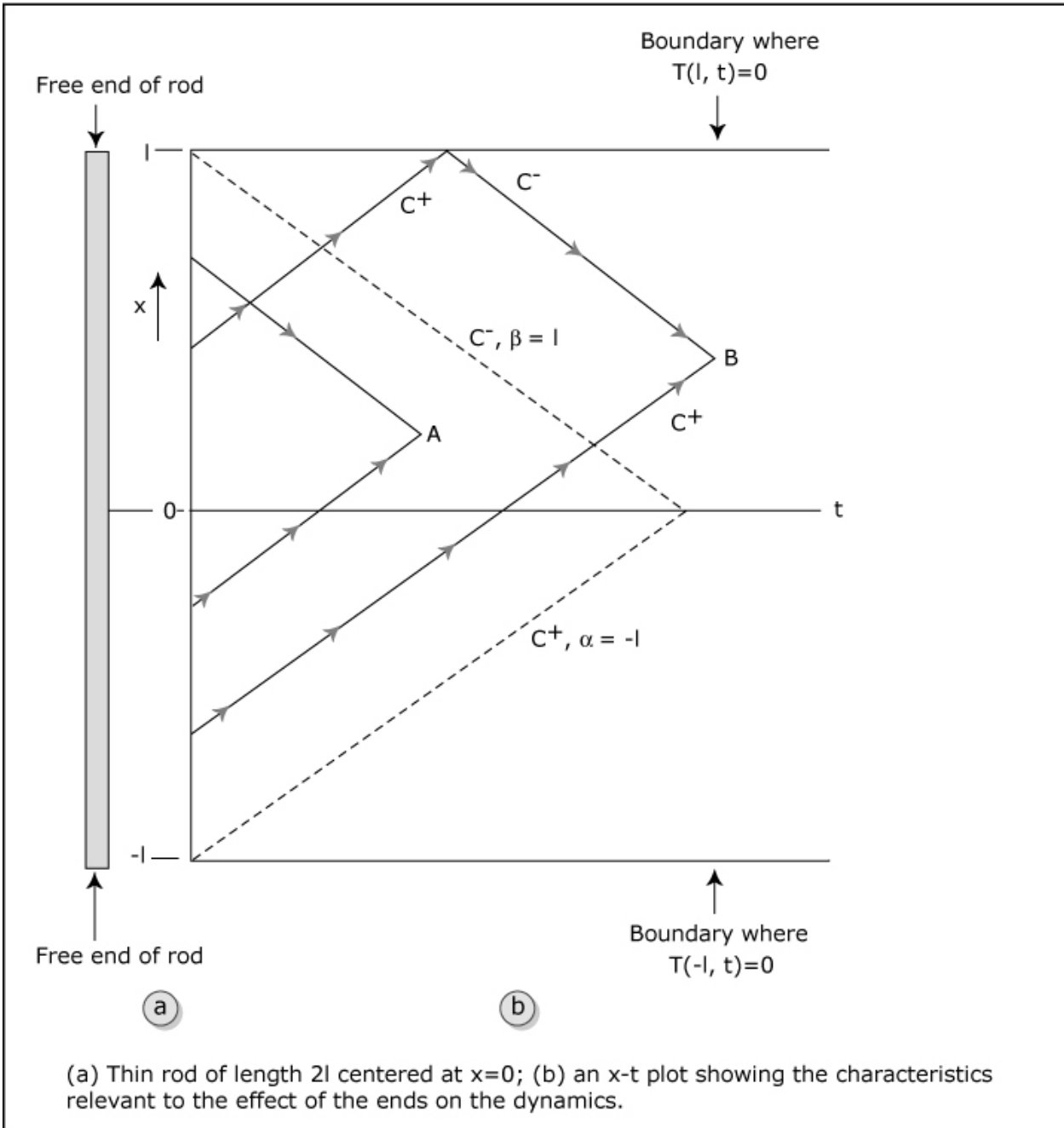
$$t = 0: \quad T(x, t = 0) = \begin{cases} T_m & |x| < a \\ 0 & |x| > a \end{cases}$$

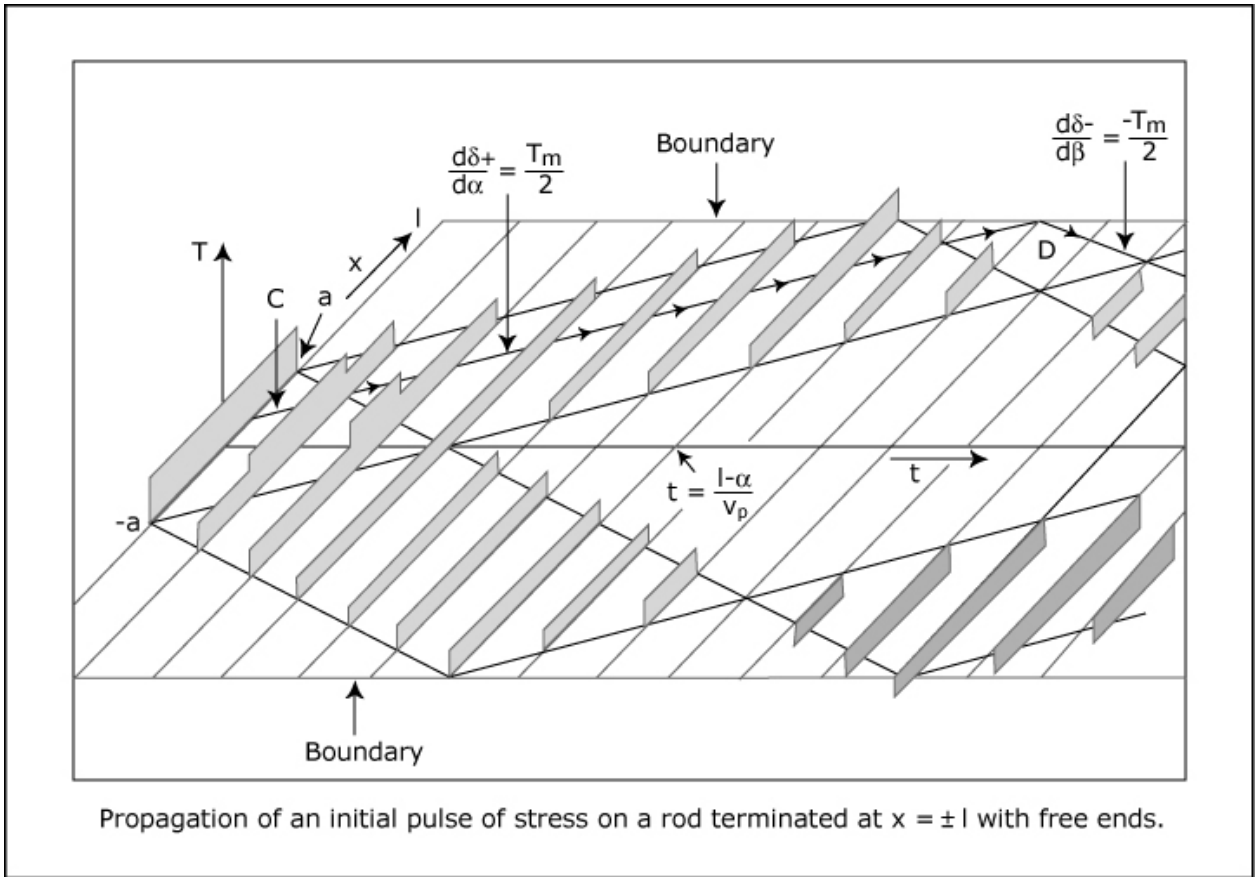




VIII. Boundary Conditions

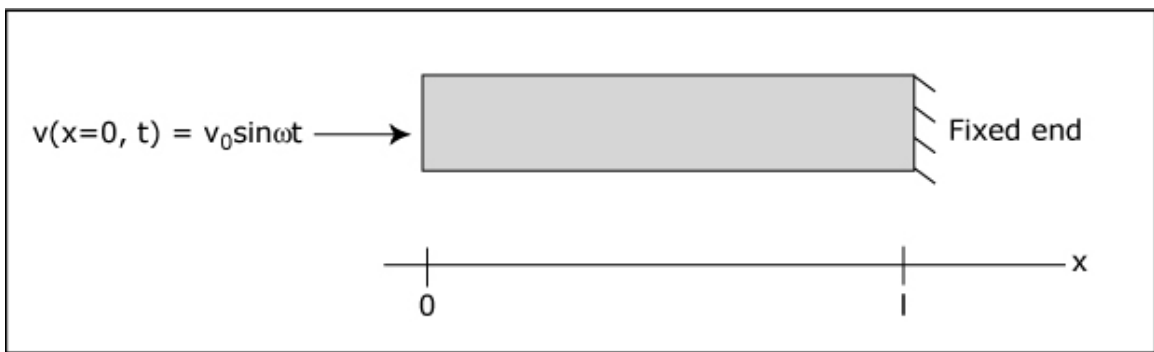






IX. Sinusoidal Steady State

A. Velocity Driven Thin Rod



$$v(x, t) = \text{Re}[\hat{v}(x) e^{j\omega t}]$$

B.C: $\hat{v}(x=l) = -j v_0$

$$\hat{v}(x=l) = 0$$

$$\frac{\partial^2 v}{\partial t^2} = v_p^2 \frac{\partial^2 v}{\partial x^2} \Rightarrow -\omega^2 \hat{v}(x) = \frac{d^2 \hat{v}}{dx^2} v_p^2$$

$$\frac{d^2 \hat{v}}{dx^2} + k^2 \hat{v}(x) = 0, \quad k^2 = \omega^2 / v_p^2 = \frac{\rho \omega^2}{E}$$

$$\hat{v}(x) = A \sin kx + B \cos kx$$

$$\hat{v}(x=0) = B = -j v_0$$

$$\hat{v}(x=l) = 0 = A \sin kl - j v_0 \cos kl \Rightarrow A = j v_0 \cot kl$$

$$\hat{v}(x) = j v_0 \left(\frac{\cos kl}{\sin kl} \sin kx - \cos kx \right)$$

$$= \frac{j v_0}{\sin kl} (\cos kl \sin kx - \cos kx \sin kl)$$

$$= \frac{j v_0 \sin k(x-l)}{\sin kl}$$

$$\frac{\partial T}{\partial t} = E \frac{\partial v}{\partial x} \Rightarrow j \omega \hat{T}(x) = j v_0 E k \frac{\cos k(x-l)}{\sin kl}$$

$$\hat{T}(x) = v_0 E \frac{k}{\omega} \frac{\cos k(x-l)}{\sin kl}$$

$$= v_0 E \sqrt{\rho/E} \frac{\cos k(x-l)}{\sin kl}$$

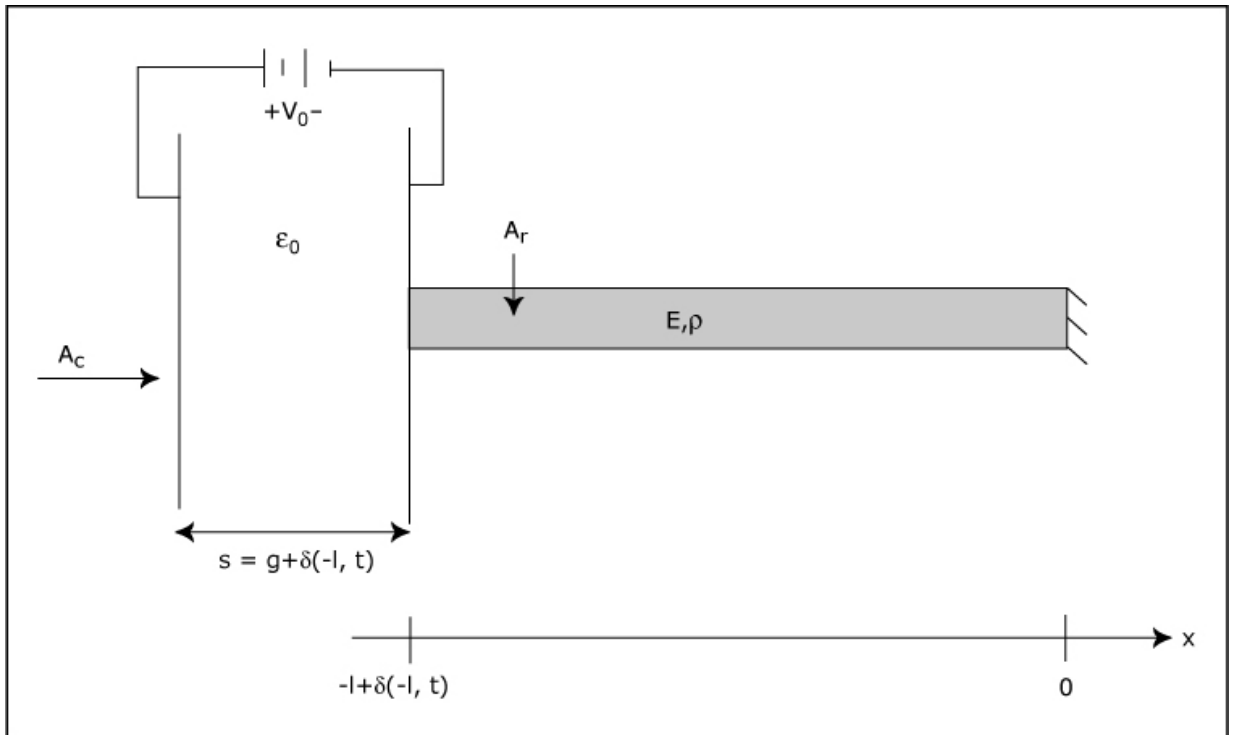
$$= v_0 \sqrt{\rho E} \frac{\cos k(x-l)}{\sin kl}$$

$$v(x, t) = \text{Re} \left[\hat{v}(x) e^{j\omega t} \right] = -v_0 \frac{\sin k(x-l)}{\sin kl} \sin \omega t$$

$$T(x, t) = \text{Re} \left[\hat{T}(x) e^{j\omega t} \right] = \frac{V_0 \sqrt{\rho E}}{\sin kl} \cos k(x-l) \cos \omega t$$

$$\text{Resonance: } \sin kl = 0 \Rightarrow kl = n\pi \Rightarrow k = \frac{\omega}{v_p} = \frac{n\pi}{l}$$

X. Electromechanical Coupling



$$C = \frac{\epsilon_0 A_c}{s}, \quad f_x = \frac{1}{2} v^2 \frac{dC}{ds} = -\frac{1}{2} \frac{v^2 \epsilon_0 A_c}{s^2}$$

$$f_x \text{ has steady part \& transient part} \quad = -\frac{1}{2} \frac{V_0^2 \epsilon_0 A_c}{(g + \delta(-l, t))^2}$$

$$\delta = \delta_0(x) + \delta'(x, t) \quad = \frac{-\frac{1}{2} \epsilon_0 A_c V_0^2}{g^2 \left(1 + \frac{\delta(-l, t)}{g} \right)^2}$$

$$\text{Steady part } \left(\frac{\partial}{\partial t} \delta_0 = 0 \right) \quad \approx \frac{-\frac{1}{2} \epsilon_0 A_c V_0^2}{g^2 \left(1 + \frac{2\delta(-l, t)}{g} \right)}$$

$$\approx \frac{-1}{2} \varepsilon_0 A_c V_0^2 \left(1 - \frac{2\delta(-l, t)}{g} \right)$$

$$\frac{\partial^2 \delta_0}{\partial x^2} = 0 \Rightarrow \delta_0 = ax + b$$

$$\delta_0(x=0) = b = 0$$

$$\delta_0 = ax$$

$$\begin{aligned} EA_r \left. \frac{\partial \delta_0}{\partial x} \right|_{x=-l} &= + \frac{1}{2} \frac{\varepsilon_0 A_c V_0^2}{g^2} \left[1 + \frac{2al}{g} \right] \\ &= EA_r a \end{aligned}$$

$$a \left[EA_r - \frac{\varepsilon_0 A_c V_0^2 l}{g^3} \right] = + \frac{1}{2} \frac{\varepsilon_0 A_c V_0^2}{g^2} \Rightarrow a = \frac{+\frac{1}{2} \varepsilon_0 A_c V_0^2}{g^2 \left[EA_r - \frac{\varepsilon_0 A_c V_0^2 l}{g^3} \right]}$$

$$\text{Transient Part: } \delta'(x, t) = \text{Re} \left[\hat{\delta}'(x) e^{j\omega t} \right]$$

$$\rho \frac{\partial^2 \delta'}{\partial t^2} = \frac{E \partial^2 \delta'}{\partial x^2} \Rightarrow \hat{\delta}'(x) = A_1 \sin kx + A_2 \cos kx, k = \omega/v_p$$

$$\hat{\delta}'(x=0) = 0 = A_2$$

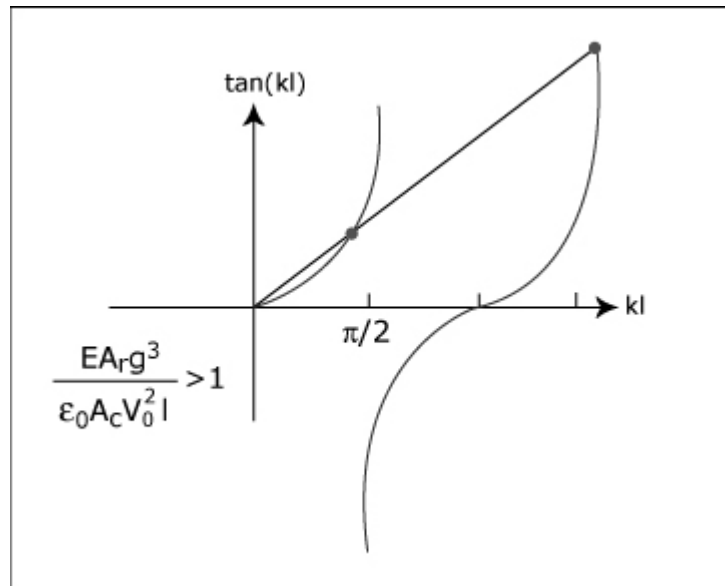
$$\hat{\delta}'(x) = A_1 \sin kx$$

B.C. at $x = -l$

$$EA_r \left. \frac{\partial \delta'}{\partial x} \right|_{x=-l} = - \frac{\varepsilon_0 A_c V_0^2}{g^3} \delta'(-l, t)$$

$$EA_r k \cos kl = + \frac{\epsilon_0 A_c V_0^2}{g^3} \sin kl$$

$$\tan kl = \frac{EA_r g^3}{\epsilon_0 A_c V_0^2 l} (kl)$$



$$\frac{EA_r g^3}{\epsilon_0 A_c V_0^2 l} < 1 \Rightarrow k = jk_i$$

$$\tan kl = \tan(jk_i l) = j \tanh(k_i l) = jk_i l \left(\frac{EA_r g^3}{\epsilon_0 A_c V_0^2 l} \right)$$

$$\omega_i = k_i v_p$$

