Lecture 15: Force Densities, Stress Tensors, and Forces

I. Maxwell Stress Tensor

A. Notation

$$F_x = \nabla \bullet \bar{\tau}_x, \quad \bar{\tau}_x = T_{xx}\,\bar{i}_x + T_{xy}\,\bar{i}_y + T_{xz}\,\bar{i}_z$$

$$F_y = \nabla \bullet \bar{\tau}_y, \quad \bar{\tau}_y = T_{yx}\,\bar{i}_x + T_{yy}\,\bar{i}_y + T_{yz}\,\bar{i}_z$$

$$F_z = \nabla \bullet \bar{\tau}_z, \quad \bar{\tau}_z = T_{zx}\,\bar{i}_x + T_{zy}\,\bar{i}_y + T_{zz}\,\bar{i}_z$$

$$\bar{\bar{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$f_x = \int_V F_x dV = \int_V \nabla \bullet \bar{\tau}_x dV = \oint_S \bar{\tau}_x \bullet \bar{n}\, da = \oint_S \left[T_{xx}n_x + T_{xy}n_y + T_{xz}n_z\right] da$$

$$\bar{\tau}_x \bullet \bar{n} = T_{xx}n_x + T_{xy}n_y + T_{xz}n_z = T_{xn}n_n$$

$$\bar{\tau}_y \bullet \bar{n} = T_{yx}n_x + T_{yy}n_y + T_{yz}n_z = T_{yn}n_n$$

$$\bar{\tau}_z \bullet \bar{n} = T_{zx}n_x + T_{zy}n_y + T_{zz}n_z = T_{zn}n_n$$

$$f_i = \int_V \nabla \bullet \bar{\tau}_i dV = \oint_S \bar{\tau}_i \bullet \bar{n}\, dV = \oint_S T_{ij}n_j dS = \int_V F_i dV$$

$$\bar{F}_i = \nabla \bullet \bar{\tau}_i = \frac{\partial}{\partial x} T_{ix} + \frac{\partial}{\partial y} T_{iy} + \frac{\partial}{\partial z} T_{iz}$$

$$= \frac{\partial}{\partial x_j} T_{ij}$$

B. EQS Stress Tensor

$$\bar{F} = \rho_f \bar{E} - \frac{1}{2}\bar{E} \bullet \bar{E}\, \nabla \varepsilon + \nabla\left(\frac{1}{2}\bar{E} \bullet \bar{E}\frac{\partial \varepsilon}{\partial \rho}\rho\right)$$

$$= \nabla \bullet \left(\varepsilon \bar{E}\right)\bar{E} - \frac{1}{2}\left(\bar{E} \bullet \bar{E}\right)\nabla \varepsilon + \nabla\left(\frac{1}{2}\bar{E} \bullet \bar{E}\frac{\partial \varepsilon}{\partial \rho}\rho\right)$$

$$F_i = \frac{\partial\left(\varepsilon E_j\right)}{\partial x_j} E_i - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$\nabla \times \bar{E} = 0 \Rightarrow \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_j E_i\right) - \varepsilon E_j \frac{\partial E_i}{\partial x_j} - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_i E_j\right) - \underbrace{\varepsilon E_j \frac{\partial E_j}{\partial x_i}}_{\varepsilon \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_j E_j\right)} - \frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2} E_K E_K \frac{\partial \varepsilon}{\partial \rho} \rho\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\varepsilon E_i E_j\right) - \frac{\partial}{\partial x_i}\left[\frac{1}{2} \varepsilon E_K E_K - \frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} E_K E_K\right]$$

$$\frac{\partial}{\partial x_j} = \delta_{ij} \frac{\partial}{\partial x_i}$$

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{Kronecker Delta}$$

$$F_i = \frac{\partial}{\partial x_j}\left[\varepsilon E_i E_j - \frac{1}{2} \delta_{ij} E_K E_K \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right)\right] = \frac{\partial}{\partial x_j} T_{ij}$$

$$T_{ij} = \varepsilon E_i E_j - \frac{1}{2} \delta_{ij} E_K E_K \left(\varepsilon - \rho \frac{\partial \varepsilon}{\partial \rho}\right)$$

C. MQS Stress Tensor

$$\bar{F} = \bar{J}_f \times \bar{B} - \frac{1}{2} \bar{H} \cdot \bar{H} \, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$= \left(\nabla \times \bar{H}\right) \times \left(\mu \bar{H}\right) - \frac{1}{2} \bar{H} \cdot \bar{H} \, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$\left(\nabla \times \bar{H}\right) \times \bar{H} = \left(\bar{H} \cdot \nabla\right) \bar{H} - \frac{1}{2} \nabla\left(\bar{H} \cdot \bar{H}\right)$$

$$\bar{F} = \mu\left[\left(\bar{H} \cdot \nabla\right) \bar{H} - \frac{1}{2} \nabla\left(\bar{H} \cdot \bar{H}\right)\right] - \frac{1}{2} \bar{H} \cdot \bar{H} \, \nabla \mu + \nabla\left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \bar{H} \cdot \bar{H}\right)$$

$$F_i = \mu\left[H_j \frac{\partial}{\partial x_j} H_i - \frac{1}{2}\frac{\partial}{\partial x_i}\left(H_K H_K\right)\right] - \frac{1}{2} H_K H_K \frac{\partial \mu}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho \frac{\partial \mu}{\partial \rho} H_K H_K\right)$$

$$= \frac{\partial}{\partial x_j}\left(\mu H_i H_j\right) - \underbrace{H_i \frac{\partial}{\partial x_j}\left(\mu H_j\right)}_{\nabla \cdot \overline{B} = 0} - \underbrace{\frac{\mu}{2}\frac{\partial}{\partial x_i} H_K H_K - \frac{1}{2} H_K H_K \frac{\partial \mu}{\partial x_i}}_{-\frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu H_K H_K\right)} + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho \frac{\partial \mu}{\partial \rho} H_K H_K\right)$$

$$F_i = \frac{\partial}{\partial x_j}\left(\mu H_i H_j\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu H_K H_K - \rho \frac{\partial \mu}{\partial \rho} H_K H_K\right)$$

$$= \frac{\partial}{\partial x_j}\left[\mu H_i H_j - \frac{1}{2}\delta_{ij} H_K H_K \left(\mu - \rho \frac{\partial \mu}{\partial \rho}\right)\right] = \frac{\partial}{\partial x_j} T_{ij}$$

$$T_{ij} = \mu H_i H_j - \frac{1}{2}\delta_{ij} H_K H_K \left(\mu - \rho \frac{\partial \mu}{\partial \rho}\right)$$

II. Air-Gap Magnetic Machines

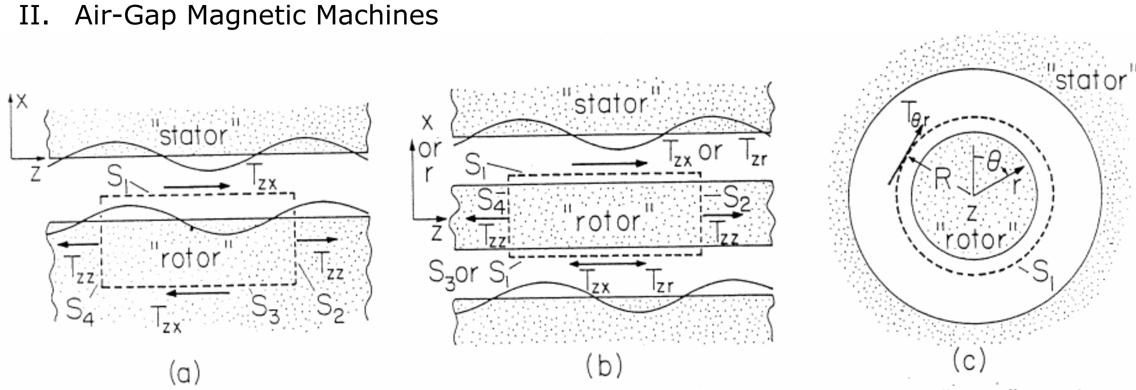
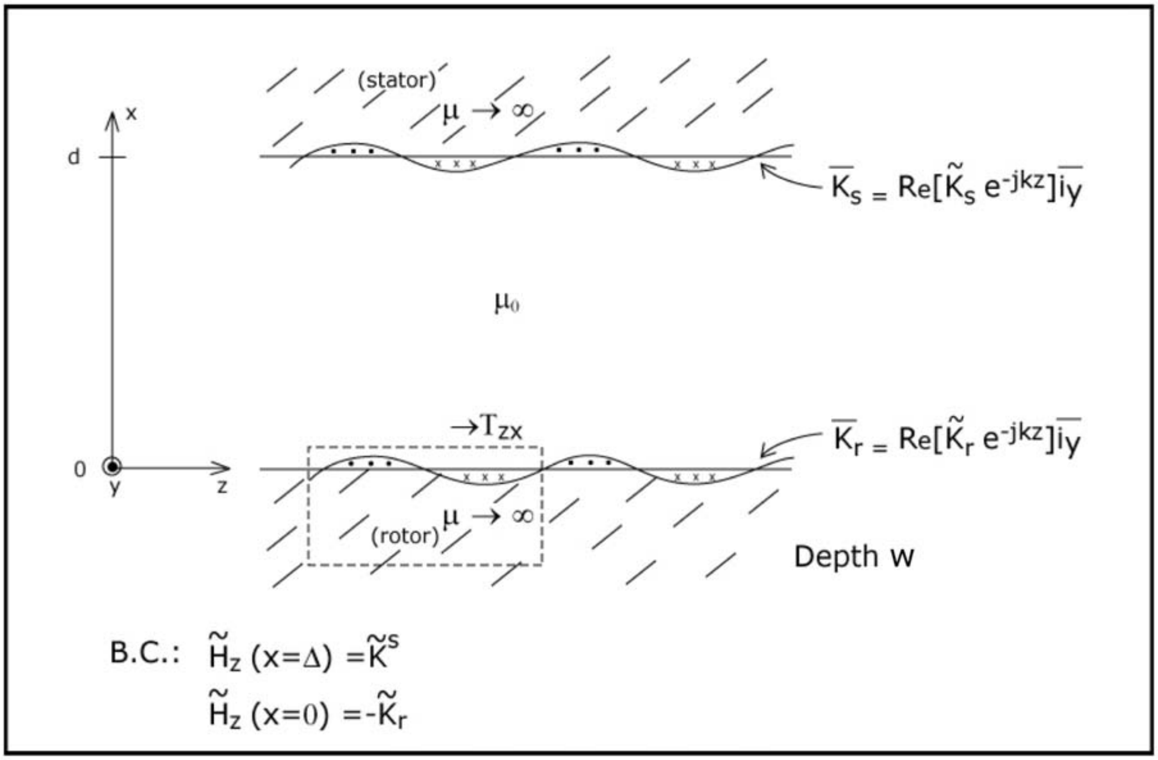


Fig. 4.2.1. Typical "air-gap" configurations in which a force or torque on a rigid "rotor" results from spatially periodic sources interacting with spatially periodic excitations on a rigid "stator." Because of the periodicity, the force or torque can be represented in terms of the electric or magnetic stress acting at the air-gap surfaces S_1: (a) planar geometry or developed model; (b) planar or cylindrical beam; (c) cylindrical rotor.

A. Generalized Description



$$f_z = \oint_S T_{zx} n_x dz dy = w \int_0^{2\pi/k} \mu_0 H_z H_x \Big|_{x=0} dz = w \int_0^{2\pi/k} \mu_0 H_z^r H_x^r dz$$

force on a wavelength

$$a(z,t) = \text{Re}\left[\tilde{A} e^{-jkz}\right],\ b(z,t) = \text{Re}\left[\tilde{B} e^{-jkz}\right]$$

$$\frac{k}{2\pi} \int_0^{2\pi/k} a(z,t)\, b(z,t)\, dz = \frac{1}{2} \text{Re}\left[\tilde{A}\tilde{B}^*\right] = \frac{1}{2} \text{Re}\left[\tilde{A}^* \tilde{B}\right]$$

$$f_z = \frac{2\pi w}{k} \frac{\mu_0}{2} \text{Re}\left[\tilde{H}_z^r \tilde{H}_x^{r\,*}\right]$$

$$= \frac{\cancel{2}\pi w \mu_0}{k \cancel{2}} \text{Re}\left[-\tilde{K}_r \tilde{H}_x^{r\,*}\right]$$

$$\begin{bmatrix} \tilde{B}_x^s \\ \tilde{B}_x^r \end{bmatrix} = \mu_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\chi}_s \\ \tilde{\chi}_r \end{bmatrix}$$

$$\tilde{H}_z = +jk\tilde{\chi} \Rightarrow \tilde{\chi}^s = \frac{1}{jk}\tilde{H}_z^s = \frac{\tilde{K}^s}{jk}$$

$$\tilde{\chi}^r = \frac{\tilde{H}_z^r}{jk} = -\frac{\tilde{K}_r}{jk}$$

$$\mu_0 \tilde{H}_x^r = \mu_0 k \left[\frac{-\tilde{\chi}^s}{\sinh kd} + \tilde{\chi}^r \coth kd \right]$$

$$= \mu_0 k \left[\frac{-\tilde{K}^s}{jk \sinh kd} - \frac{\tilde{K}^r}{jk} \coth kd \right]$$

$$\text{Re}\left[-\tilde{K}_r^* \tilde{H}_x^r \mu_0 \right] = -\text{Re}\left[\frac{+j\mu_0 \cancel{k}}{\cancel{k}} \left(\frac{\tilde{K}_r^* \tilde{K}^s}{\sinh kd} + \tilde{K}_r^* \tilde{K}_r \coth kd \right) \right]$$

$$= \text{Re}\left[-\mu_0 j \tilde{K}_r^* \tilde{K}_s^s \big/ \sinh kd \right]$$

$$f_z = -\frac{\pi w}{k} \frac{\mu_0}{\sinh kd} \text{Re}\left[j\tilde{K}_r^* \tilde{K}_s \right] \text{ (force on each wavelength)}$$

B. Synchronous Interaction

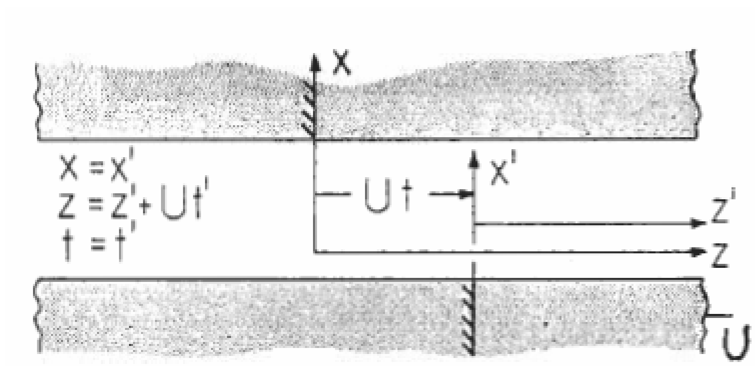


Fig. 4.3.1. Rotor and stator reference frames z' and z.

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$$K^s = K_0^s \sin\left[\omega_s t - kz\right] = \mathrm{Re}\left[-jK_0^s e^{j(\omega_s t - kz)}\right]$$

$$K^r = K_0^r \sin\left[\omega_r t - k\left(z' - \delta\right)\right]; \quad z' = z - Ut$$

$$= K_0^r \sin\left[\left(\omega_r + kU\right)t - k\left(z - \delta\right)\right]$$

$$= \mathrm{Re}\left[-jK_0^r\, e^{j(\omega_r + kU)t} e^{jk\delta}\right]$$

$$\tilde{K}^s = -jK_o^s\, e^{j\omega_s t}$$

$$\tilde{K}^r = -jK_o^r e^{jk\delta}\, e^{j(\omega_r + kU)t}$$

$$f_z = -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}\mathrm{Re}\left[j\left(-jK_0^s\right)e^{j\omega_s t}\left(jK_0^r e^{-jk\delta}\right)e^{-j(\omega_r + kU)t}\right]$$

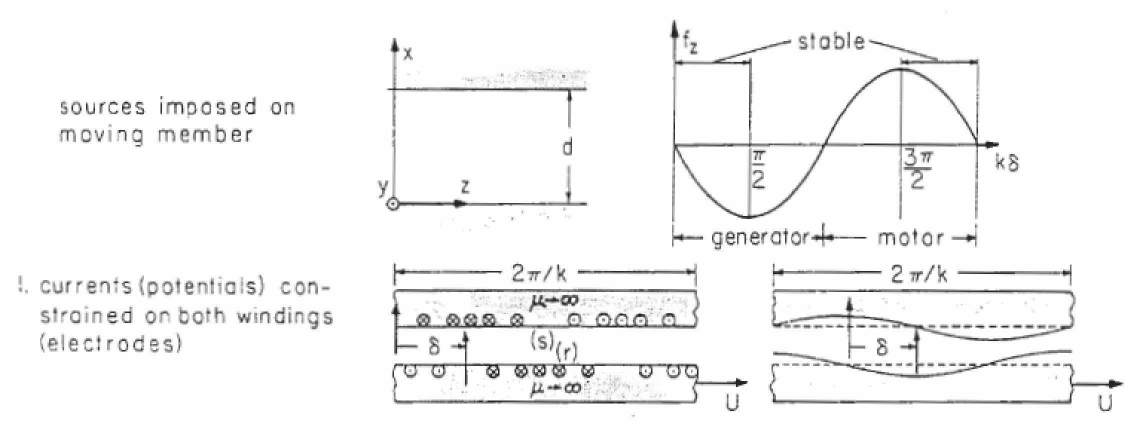
$$= -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}K_0^s K_0^r \mathrm{Re}\left[je^{-jk\delta}e^{j(\omega_s - \omega_r - kU)t}\right]$$

For time average force $\Rightarrow \omega_s = \omega_r + kU$ (synchronous condition)

Usually $\omega_r = 0 \Rightarrow \omega_s = kU$

$$\left\langle f_z \right\rangle = -\frac{\pi W}{k}\frac{\mu_0}{\sinh kd}K_0^s K_0^r \sin k\delta$$

Table 4.3.1. Basic configurations illustrating classes of electromechanical interactions and devices. MQS and EQS systems respectively in left and right columns.



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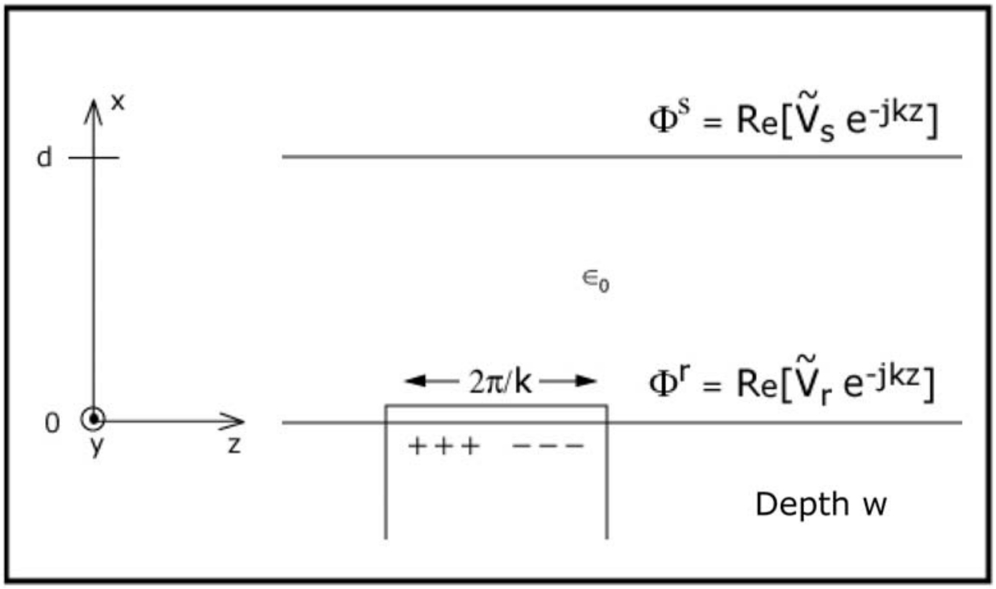
III. Electrostatic Machine



$$f_z = \frac{w2\pi}{k}\int_0^{2\pi/k} T_{zx}\Big|_{x=0} dz = \frac{2\pi w}{k}\int_0^{2\pi/k} \varepsilon_0 E_z E_x\Big|_{x=0}$$

$$\tilde{E}_z^r = jk\tilde{V}_r$$

$$f_z = \frac{1}{\cancel{2}}\frac{\cancel{2}\pi w}{k}\mathrm{Re}\left[\varepsilon_0 \tilde{E}_z^{r\,*}\tilde{E}_x^r\right]$$

$$= \frac{\pi w}{k}\mathrm{Re}\left[\varepsilon_0\left(-jk\tilde{V}_r^*\right)\tilde{E}_x^r\right]$$

$$\begin{bmatrix}\tilde{D}_x^s \\ \tilde{D}_x^r\end{bmatrix} = \varepsilon_0 k \begin{bmatrix} -\coth kd & \dfrac{1}{\sinh kd} \\ -\dfrac{1}{\sinh kd} & \coth kd \end{bmatrix}\begin{bmatrix}\tilde{V}_s \\ \tilde{V}_r\end{bmatrix}$$

$$\varepsilon_0 \tilde{E}_x^r = \varepsilon_0 k\left[\frac{-\tilde{V}_s}{\sinh kd} + \tilde{V}_r \coth kd\right]$$

$$\mathrm{Re}\left[-jk\varepsilon_0 \tilde{V}_r^* \tilde{E}_x^r\right] = \mathrm{Re}\left[-jk^2\varepsilon_0 \tilde{V}_r^*\left(\frac{-\tilde{V}_s}{\sinh kd} + \tilde{V}_r \coth kd\right)\right]$$

$$= \mathrm{Re}\left[+jk^2\varepsilon_0 \tilde{V}_s \tilde{V}_r^* \big/ \sinh kd\right]$$

$$f_z = \frac{\pi w}{\cancel{k}} \frac{\cancel{k^2}\varepsilon_0}{\sinh kd} \mathrm{Re}\left[j\tilde{V}_s \tilde{V}_r^* \right]$$

$$V_s = V_0^s \cos\left(\omega_s t - kz\right)$$
$$V_r = -V_0^r \cos\left(\omega_r t - k\left(z' - \delta\right)\right);\ z' = z - Ut$$
$$\tilde{V}^r = -V_0^r e^{j(\omega_r + kU)t} e^{jk\delta}$$
$$\tilde{V}^s = V_0^r e^{j\omega_s t}$$

$$\langle f_z \rangle = \frac{\pi w k \varepsilon_0}{\sinh kd} \mathrm{Re}\left[-j\, V_0^s\ V_0^r\ e^{-jk\delta}\ e^{j(\omega_s - \omega_r - kU)t} \right]$$

$$\omega_s = \omega_r + kU$$

$$\langle f_z \rangle = -\frac{\pi w k \varepsilon_0}{\sinh kd} V_0^s\ V_0^r \sin\ k\delta$$

IV. Derivation of the Korteweg-Helmholtz Force Density for Incompressible Media from the Quasistatic Poynting's Theorem

A. Poynting's Theorem

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}_f + \frac{\partial \bar{D}}{\partial t}$$

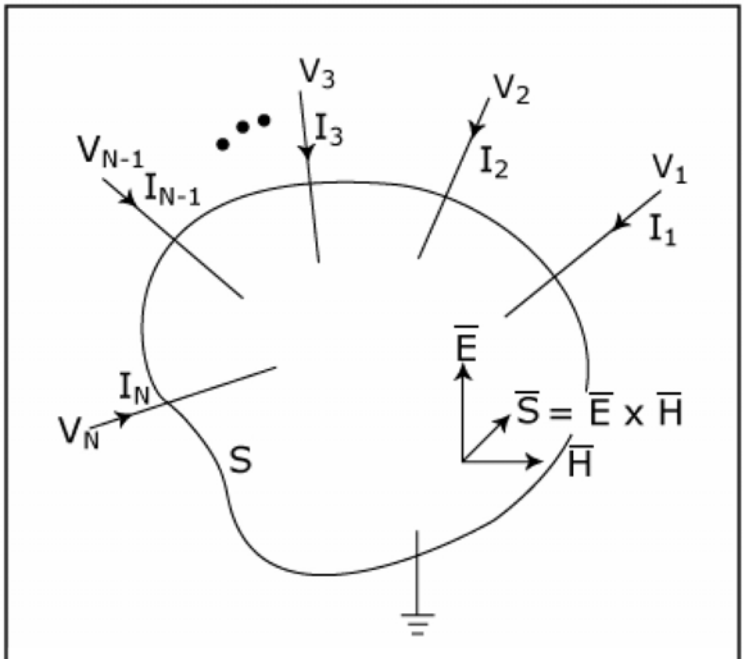
$$\nabla \bullet \bar{D} = \rho_f$$

$$\nabla \bullet \bar{B} = 0$$

$$\nabla \bullet \left(\bar{E} \times \bar{H}\right) = \bar{H} \bullet \left(\nabla \times \bar{E}\right) - \bar{E} \bullet \left(\nabla \times \bar{H}\right)$$

$$= -\bar{H} \bullet \frac{\partial \bar{B}}{\partial t} - \bar{E} \bullet \frac{\partial \bar{D}}{\partial t} - \bar{E} \bullet \bar{J}_f$$

B. Power In Quasistatic Electric Circuits



The circuit power into an N terminal network $\sum_{k=1}^{N} V_k I_k$ equals the electromagnetic power flow into the surface surrounding the network, $-\oint_S \overline{E}\times\overline{H}\bullet\overline{dS}$

Far away from the circuit elements

$$\nabla\times\overline{E} = 0 \Rightarrow \overline{E} = -\nabla\,\Phi$$

$$\nabla\times\overline{H} = \overline{J}_f \Rightarrow \nabla\bullet\overline{J}_f = 0$$

$$P_{in} = -\oint_S \left(\overline{E}\times\overline{H}\right)\bullet\overline{da}$$

$$= +\oint_S \left(\nabla\,\Phi\times\overline{H}\right)\bullet\overline{da}$$

$$= \int_V \nabla\bullet\left(\nabla\,\Phi\times\overline{H}\right)\,dV$$

$$\nabla\bullet\left(\nabla\,\Phi\times\overline{H}\right) = \overline{H}\bullet\overbrace{\nabla\times\left(\nabla\,\Phi\right)}^{0} - \nabla\,\Phi\bullet\left(\nabla\times\overline{H}\right)$$

$$= -\overline{J}_f\bullet\nabla\,\Phi = -\nabla\bullet\left(\overline{J}_f\,\Phi\right)$$

$$P_{in} = -\int_V \nabla \bullet \left(\bar{J}_f\, \Phi\right) dV$$

$$= -\oint_S \bar{J}_f\, \Phi \bullet \overline{da}$$

$$= -\sum_{k=1}^{N} V_k \underbrace{\oint_S \bar{J}_f \bullet \overline{da}}_{-I_k}$$

$$= \sum_{k=1}^{N} V_k\, I_k$$

C. Electroquasistatics (EQS)

Ohmic Media: $\bar{J}_f{}' = \sigma \bar{E}' = \bar{J}_f - \rho_f\, \bar{v} \Rightarrow \bar{J}_f = \sigma \bar{E} + \rho_f\, \bar{v}$

$$\bar{D} = \varepsilon(x,y,z)\bar{E}$$

$$\int_V \nabla \bullet \left(\bar{E} \times \bar{H}\right) dV = \oint_S \bar{E} \times \bar{H} \bullet \overline{da} = -\sum_k V_k\, I_k = -\int_V \bar{E} \bullet \frac{\partial}{\partial t}\left(\varepsilon(x,y,z)\bar{E}\right) dV - \int_V \bar{E} \bullet \left(\sigma \bar{E} + \rho_f\, \bar{v}\right) dV$$

$$\sum_k V_k\, I_k = \int_V \frac{\varepsilon(x,y,z)\bar{E}}{\varepsilon(x,y,z)} \bullet \frac{\partial}{\partial t}\left(\varepsilon(x,y,z)\bar{E}\right) dV + \int_V \sigma \left|\bar{E}\right|^2 dV + \int_V \rho_f\, \bar{E} \bullet \bar{v}\, dV$$

$$= \int_V \frac{1}{2} \frac{1}{\varepsilon(x,y,z)} \frac{\partial}{\partial t}\left[\varepsilon^2(x,y,z)\left|\bar{E}\right|^2\right] dV + \int_V \sigma \left|\bar{E}\right|^2 dV + \int_V \rho_f\, \bar{E} \bullet \bar{v}\, dV$$

$$\int_V \frac{1}{2\varepsilon(x,y,z)} \frac{\partial}{\partial t}\left[\varepsilon^2(x,y,z)\left|\bar{E}\right|^2\right] dV = \int_V \frac{\partial}{\partial t}\left[\frac{\varepsilon^{\cancel{2}}(x,y,z)\left|\bar{E}\right|^2}{2\,\varepsilon\cancel{(x,y,z)}}\right] dV - \int_V \frac{\varepsilon^2(x,y,z)\left|\bar{E}\right|^2}{2} \frac{\partial}{\partial t}\left(\frac{1}{\varepsilon(x,y,z)}\right) dV$$

$$= \int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(x,y,z)\left|\bar{E}\right|^2\right] dV + \int_V \frac{\left|\bar{E}\right|^2}{2} \frac{\partial}{\partial t}\left(\varepsilon(x,y,z)\right) dV$$

Theorem: $\dfrac{d}{dt}\int_V \alpha \, dV = \int_V \dfrac{\partial \alpha}{\partial t} dV + \int_V \nabla \cdot \left(\alpha \, \bar{v}\right) dV$

Conservation of mass: $\alpha = \rho$ mass density

$$\frac{d}{dt}\int_V \rho \, dV = 0 = \int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot \left(\rho \, \bar{v}\right) dV$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \, \bar{v}\right) = 0 = \frac{\partial \rho}{\partial t} + \left(\bar{v} \cdot \nabla\right)\rho + \rho\left(\nabla \cdot \bar{v}\right) = 0$$

Incompressible: $\boxed{\dfrac{d\rho}{dt} = \dfrac{\partial \rho}{\partial t} + \left(\bar{v} \cdot \nabla\right)\rho = 0}$ $\Rightarrow$ $\boxed{\nabla \cdot \bar{v} = 0}$

$$\frac{d}{dt}\int_V \varepsilon \, dV = \int_V \frac{\partial \varepsilon}{\partial t} dV + \int_V \nabla \cdot \left(\varepsilon \, \bar{v}\right) dV = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \left(\varepsilon \, \bar{v}\right) = \frac{\partial \varepsilon}{\partial t} + \left(\bar{v} \cdot \nabla\right)\varepsilon + \varepsilon \; \overbrace{\nabla \cdot \bar{v}}^{0} = 0$$

$$\boxed{\frac{\partial \varepsilon}{\partial t} = -\left(\bar{v} \cdot \nabla\right)\varepsilon}$$

$$\int_V \frac{1}{2\varepsilon(x,y,z)} \frac{\partial}{\partial t}\left[\varepsilon^2 (x,y,z) \left|\bar{E}\right|^2\right] dV = \int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(x,y,z)\left|\bar{E}\right|^2\right] dV$$

$$+ \int_V \frac{\left|\bar{E}\right|^2}{2}\left(-\bar{v} \cdot \nabla\right)\varepsilon(x,y,z) \, dV$$

$$\sum_k V_k \, I_k = \underbrace{\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(x,y,z)\left|\bar{E}\right|^2\right] dV}_{\text{Energy Stored } (W_E) \text{ Rate}} + \underbrace{\int_V \sigma \left|\bar{E}\right|^2 dV}_{\substack{\text{Power} \\ \text{Dissipated } P_E}} + \underbrace{\int_V \underbrace{\left[\rho_f \bar{E} - \frac{1}{2}\left|\bar{E}\right|^2 \nabla \varepsilon\right]}_{\text{Force Density}} \cdot \bar{v} \, dV}_{\text{Work Rate = Mechanical Power}}$$

$$\bar{F} = \rho_f \, \bar{E} - \frac{1}{2}\left|\bar{E}\right|^2 \nabla \varepsilon \qquad \text{(force per unit volume)}$$

nt/m^3

$$\overline{f} = \int_V \overline{F}\, dV$$

force (nts)

D. Magnetoquasistatics

$$\overline{J}_f' = \overline{J}_f,\ \overline{E}' = \overline{E} + \overline{v} \times \overline{B} \quad \Rightarrow \overline{J}_f' = \overline{J}_f = \sigma \overline{E}' = \sigma\left(\overline{E} + \overline{v} \times \overline{B}\right)$$

$$\overline{B} = \mu(x,y,z)\overline{H}$$

$$\int_V \nabla \bullet \left(\overline{E} \times \overline{H}\right) dV = \oint_S \overline{E} \times \overline{H} \bullet \overline{da} = -\sum_k V_k I_k = -\int_V \overline{H} \bullet \frac{\partial}{\partial t}\left(\mu(x,y,z)\overline{H}\right) dV$$

$$-\int_V \left[\overline{E}' - \overline{v} \times \overline{B}\right] \bullet \overline{J}_f\, dV$$

$$P_{\text{dissipated}} = \int_V \overline{E}' \bullet \overline{J}_f'\, dV = \int_V \overline{E}' \bullet \overline{J}_f\, dV$$

$$\overline{J}_f \bullet \left(\overline{v} \times \overline{B}\right) = -\overline{J}_f \bullet \left(\overline{B} \times \overline{v}\right) = -\left(\overline{J}_f \times \overline{B}\right) \bullet \overline{v}$$

$$\overline{H} \bullet \frac{\partial}{\partial t}\left[\mu(x,y,z)\overline{H}\right] = \frac{\mu(x,y,z)\overline{H}}{\mu(x,y,z)} \frac{\partial}{\partial t}\left[\mu(x,y,z)\overline{H}\right]$$

$$= \frac{1}{\mu(x,y,z)} \frac{\partial}{\partial t}\left[\frac{1}{2}\mu^2(x,y,z)\left|\overline{H}\right|^2\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{\frac{1}{2}\mu^{\cancel{2}}(x,y,z)\left|\overline{H}\right|^2}{\cancel{\mu(x,y,z)}}\right] - \frac{1}{2}\mu^2(x,y,z)\left|\overline{H}\right|^2 \frac{\partial}{\partial t}\left[\frac{1}{\mu(x,y,z)}\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(x,y,z)\left|\overline{H}\right|^2\right] + \frac{1}{2}\frac{\cancel{\mu^2(x,y,z)}\left|\overline{H}\right|^2}{\cancel{\mu^2(x,y,z)}} \frac{\partial}{\partial t}\left[\mu(x,y,z)\right]$$

$$\frac{d}{dt}\int_V \mu\, dV = 0 \Rightarrow \frac{\partial \mu}{\partial t} + \left(\overline{v} \bullet \nabla\right)\mu = 0 \qquad \left(\nabla \bullet \overline{v} = 0\right)$$

$$\overline{H} \bullet \frac{\partial}{\partial t}\left[\mu(x,y,z)\overline{H}\right] = \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(x,y,z)\left|\overline{H}\right|^2\right] - \frac{1}{2}\left|\overline{H}\right|^2 \nabla\mu \bullet \overline{v}$$

$$\sum_k V_k\, I_k = \int_V \underbrace{\frac{\partial}{\partial t}\left[\frac{1}{2}\mu(x,y,z)\left|\overline{H}\right|^2\right]}_{\text{Energy density } W_M} dV + P_{dissipated}$$

$$+\int_V \underbrace{\overline{v}\cdot\Big[\underbrace{\overline{J}_f\times\overline{B}-\frac{1}{2}\left|\overline{H}\right|^2\nabla\mu}_{\overline{F}_M=\text{ force density}}\Big]\, dV}_{\text{Mechanical Power}}$$

$$W_M = \underbrace{\int_V \frac{1}{2}\mu(x,y,z)\left|\overline{H}\right|^2 dV}_{\text{Total Magnetic Energy}}, \qquad P_{dissipated} = \int_V \overline{E}'\cdot\overline{J}_f\, dV = \int_V \overline{E}'\cdot\overline{J}_f{}'\, dV = \int_V \sigma\left|\overline{E}'\right|^2 dV$$

$$F_M = \overline{J}_f\times\overline{B}-\frac{1}{2}\left|\overline{H}\right|^2\nabla\mu \qquad \text{force density}$$

V. Compressible Media

A. Electroquasistatics (EQS)

Ohmic media: $\overline{J}' = \sigma\overline{E}'$

Polarization dependent on mass density (ρ) alone, electrically linear

$\overline{D} = \varepsilon(\rho)\overline{E}$

EQS Galilean Transformation: $\overline{J} = \sigma\overline{E} + \rho_f\overline{v}$

$$\int_V \nabla\cdot\left(\overline{E}\times\overline{H}\right) dV = \oint_S \overline{E}\times\overline{H}\cdot\overline{da} = -\sum_k V_k\, I_k = -\int_V \overline{E}\cdot\frac{\partial}{\partial t}\left[\varepsilon(\rho)\overline{E}\right] dV$$

$$-\int_V \overline{E}\cdot\left(\sigma\overline{E}+\rho_f\,\overline{v}\right) dV$$

$$\overline{E}\cdot\frac{\partial}{\partial t}\left[\varepsilon(\rho)\overline{E}\right] = \frac{\varepsilon(\rho)\overline{E}}{\varepsilon(\rho)}\cdot\frac{\partial}{\partial t}\left[\varepsilon(\rho)\overline{E}\right] = \frac{1}{\varepsilon(\rho)}\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon^2(\rho)\left|\overline{E}\right|^2\right]$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\frac{\varepsilon^2(\rho)}{\varepsilon(\rho)}\left|\bar{E}\right|^2\right] - \frac{\varepsilon^2(\rho)\left|\bar{E}\right|^2}{2}\frac{\partial}{\partial t}\left(\frac{1}{\varepsilon(\rho)}\right)$$

$$\bar{E}\bullet\frac{\partial}{\partial t}\left[\varepsilon(\rho)\bar{E}\right] = \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(\rho)\left|\bar{E}\right|^2\right] + \frac{\cancel{\varepsilon^2(\rho)}\left|\bar{E}\right|^2}{2}\left(\frac{+1}{\cancel{\varepsilon^2(\rho)}}\frac{\partial\varepsilon(\rho)}{\partial t}\right)$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(\rho)\left|\bar{E}\right|^2\right] + \frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon(\rho)}{\partial t}$$

$$\frac{\partial\varepsilon(\rho)}{\partial t} = \frac{\partial\varepsilon(\rho)}{\partial\rho}\frac{\partial\rho}{\partial t} \quad ; \qquad \frac{\partial\rho}{\partial t} + \nabla\bullet\left(\rho\bar{v}\right) = 0 \qquad \text{(Conservation of mass)}$$

$$\frac{\partial\varepsilon(\rho)}{\partial t} = \frac{\partial\varepsilon(\rho)}{\partial\rho}\left(-\nabla\bullet\left(\rho\bar{v}\right)\right)$$

$$-\sum_k V_k I_k = -\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(\rho)\left|\bar{E}\right|^2\right] dV - \int_V \frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon(\rho)}{\partial t}\, dV$$

$$-\int_V \sigma\left|\bar{E}\right|^2 dV - \int_V \rho_f\,\bar{E}\bullet\bar{v}\, dV$$

$$\int_V \frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon(\rho)}{\partial t}\, dV = -\int_V \frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon}{\partial\rho}\nabla\bullet\left(\rho\bar{v}\right) dV$$

$$= -\int_V \nabla\bullet\left[\frac{1}{2}\frac{\partial\varepsilon}{\partial\rho}\left|\bar{E}\right|^2\rho\bar{v}\right] dV + \int_V \rho\bar{v}\bullet\nabla\left[\frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon}{\partial\rho}\right] dV$$

$$= -\oint_S \frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\left|\bar{E}\right|^2\bar{v}\bullet\bar{n}\, da + \int_V \bar{v}\bullet\left\{\nabla\left[\frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\left|\bar{E}\right|^2\right] - \frac{1}{2}\left|\bar{E}\right|^2\frac{\partial\varepsilon}{\partial\rho}\nabla\rho\right\} dV$$

$$\sum_k V_k I_k = \int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon(\rho)\left|\bar{E}\right|^2\right] dV + \int_V \sigma\left|\bar{E}\right|^2 dV$$

$$-\oint_S \frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\left|\bar{E}\right|^2\bar{v}\bullet\bar{n}\, da$$

$$+\int_V \bar{v}\bullet\left[\rho_f\bar{E} - \frac{1}{2}\left|\bar{E}\right|^2\nabla\varepsilon + \nabla\left[\frac{1}{2}\rho\frac{\partial\varepsilon}{\partial\rho}\left|\bar{E}\right|^2\right]\right] dV$$

where

$$\frac{\partial \varepsilon}{\partial \rho} \nabla \rho = \nabla \varepsilon$$

$$W_E = \int_V \underbrace{\frac{1}{2} \varepsilon(\rho) \left|\bar{E}\right|^2 dV}_{\text{electric energy}} , \qquad P_{\text{dissipated}} = \int_V \sigma \left|\bar{E}\right|^2 dV \qquad \text{(power dissipated)}$$

$$\bar{F}_E = \rho_f \bar{E} - \frac{1}{2} \left|\bar{E}\right|^2 \nabla \varepsilon + \nabla \left[\frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} \left|\bar{E}\right|^2 \right] \qquad \text{force density}$$

$$\oint_S \frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} \left|\bar{E}\right|^2 \bar{v} \cdot \bar{n} \, da = 0 \qquad \text{because as } S \to \infty, \ \left|\bar{E}\right|^2 da \to 0$$

$$\sum_k V_k I_k = \frac{\partial W_E}{\partial t} + P_{\text{dissipated}} + \underbrace{\int_V \bar{F}_E \cdot \bar{v} \, dV}_{\text{Mechanical Power}}$$

B. Magnetoquasistatics (MQS)

MQS Galilean Transformation: $\bar{J}_f{}' = \bar{J}_f, \ \bar{E}' = \bar{E} + \bar{v} \times \bar{B}$

$$\bar{B} = \mu(\rho) \bar{H}$$

$$\int_V \nabla \cdot \left(\bar{E} \times \bar{H} \right) dV = \oint_S \bar{E} \times \bar{H} \cdot \overline{da} = -\sum_k V_k I_k = -\int_V \bar{H} \cdot \frac{\partial}{\partial t} \left[\mu(\rho) \bar{H} \right] dV$$

$$-\int_V \left[\bar{E}' - \bar{v} \times \bar{B} \right] \cdot \bar{J}_f \, dV$$

$$P_{\text{dissipated}} = \int_V \bar{E}' \cdot \bar{J}_f' \, dV = \int_V \bar{E}' \cdot \bar{J}_f \, dV$$

$$\bar{J}_f \cdot \left(\bar{v} \times \bar{B} \right) = -\bar{J}_f \cdot \left(\bar{B} \times \bar{v} \right) = -\left(\bar{J}_f \times \bar{B} \right) \cdot \bar{v}$$

$$\bar{H} \cdot \frac{\partial}{\partial t} \left[\mu(\rho) \bar{H} \right] = \frac{\mu(\rho) \bar{H}}{\mu(\rho)} \cdot \frac{\partial}{\partial t} \left[\mu(\rho) \bar{H} \right] = \frac{1}{\mu(\rho)} \frac{\partial}{\partial t} \left[\frac{1}{2} \mu^2(\rho) \left|\bar{H}\right|^2 \right]$$

$$= \frac{\partial}{\partial t} \left[\frac{1}{2} \frac{\mu^2(\rho)}{\mu(\rho)} \left|\bar{H}\right|^2 \right] - \frac{1}{2} \mu^2(\rho) \left|\bar{H}\right|^2 \frac{\partial}{\partial t} \left[\frac{1}{\mu(\rho)} \right]$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(\rho)\left|\bar{H}\right|^2\right] + \frac{1}{2}\frac{\cancel{\mu^2(\rho)}}{\cancel{\mu^2(\rho)}}\left|\bar{H}\right|^2 \frac{\partial \mu(\rho)}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(\rho)\left|\bar{H}\right|^2\right] + \frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu(\rho)}{\partial t}$$

$$\frac{d}{dt}\int_V \mu(\rho)\, dV = 0 \quad \Rightarrow \quad \frac{\partial \mu(\rho)}{\partial t} + \nabla \cdot \left[\mu(\rho)\, \bar{v}\right] = 0$$

$$\frac{\partial \mu(\rho)}{\partial t} = \frac{\partial \mu(\rho)}{\partial \rho}\frac{\partial \rho}{\partial t} \quad ; \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\, \bar{v}\right) = 0$$

$$\frac{\partial \mu(\rho)}{\partial t} = \frac{\partial \mu(\rho)}{\partial \rho}\left(-\nabla \cdot \left(\rho\, \bar{v}\right)\right)$$

$$-\sum_k V_k\, I_k = -\int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(\rho)\left|\bar{H}\right|^2\right] dV - \int_V \frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu(\rho)}{\partial t}\, dV - P_{diss}$$

$$-\int_V \left(\bar{J}_f \times \bar{B}\right) \cdot \bar{v}\, dV$$

$$\int_V \frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu(\rho)}{\partial t}\, dV = -\int_V \frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu}{\partial \rho}\nabla \cdot \left(\rho\, \bar{v}\right) dV$$

$$= -\int_V \nabla \cdot \left[\frac{1}{2}\frac{\partial \mu}{\partial \rho}\left|\bar{H}\right|^2 \rho\, \bar{v}\right] dV + \int_V \rho\, \bar{v} \cdot \nabla\left[\frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu}{\partial \rho}\right] dV$$

$$= -\oint_S \frac{1}{2}\rho \frac{\partial \mu}{\partial \rho}\left|\bar{H}\right|^2 \bar{v} \cdot \bar{n}\, da + \int_V \bar{v} \cdot \left\{\nabla\left[\frac{1}{2}\rho\frac{\partial \mu}{\partial \rho}\left|\bar{H}\right|^2\right] - \frac{1}{2}\left|\bar{H}\right|^2 \frac{\partial \mu}{\partial \rho}\nabla \rho\right\} dV$$

$$\sum_k V_k\, I_k = \int_V \frac{\partial}{\partial t}\left[\frac{1}{2}\mu(\rho)\left|\bar{H}\right|^2\right] dV + P_{diss} - \oint_S \frac{1}{2}\rho\frac{\partial \mu}{\partial \rho}\left|\bar{H}\right|^2 \bar{v} \cdot \bar{n}\, da$$

$$+\int_V \bar{v} \cdot \left[\bar{J}_f \times \bar{B} - \frac{1}{2}\left|\bar{H}\right|^2 \nabla \mu + \nabla\left(\frac{1}{2}\rho\frac{\partial \mu}{\partial \rho}\left|\bar{H}\right|^2\right)\right] dV$$

where

$$\frac{\partial \mu}{\partial \rho} \nabla \rho = \nabla \mu$$

magnetic energy

$$W_M = \int_V \frac{1}{2} \mu(\rho) \left|\bar{H}\right|^2 \, dV, \quad P_{dissipated} = \int_V \bar{E}' \cdot \bar{J}_f' \, dV = \int_V \bar{E}' \cdot \bar{J}_f \, dV \quad \text{Power dissipated}$$

$$\bar{F}_M = \bar{J}_f \times \bar{B} - \frac{1}{2} \left|\bar{H}\right|^2 \nabla \mu + \nabla \left(\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \left|\bar{H}\right|^2 \right) \text{ force density}$$

$$\oint_S \frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \left|\bar{H}\right|^2 \bar{v} \cdot \bar{n} \, da = 0 \qquad \text{because as } S \to \infty, \left|\bar{H}\right|^2 \, da \to 0$$

$$\sum_k V_k I_k = \frac{\partial W_M}{\partial t} + P_{dissipated} + \underbrace{\int_V \bar{F}_M \cdot \bar{v} \, dV}_{\text{Mechanical Power}}$$

C. Conclusions

Force densities

$$\text{EQS: } \bar{F}_E = \rho_f \bar{E} - \frac{1}{2} \left|\bar{E}\right|^2 \nabla \varepsilon + \nabla \left[\frac{1}{2} \rho \frac{\partial \varepsilon}{\partial \rho} \left|\bar{E}\right|^2 \right]$$

$$\text{MQS: } \bar{F}_M = \bar{J}_f \times \bar{B} - \frac{1}{2} \left|\bar{H}\right|^2 \nabla \mu + \nabla \left[\frac{1}{2} \rho \frac{\partial \mu}{\partial \rho} \left|\bar{E}\right|^2 \right]$$