6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

Lecture 9: Magnetic Diffusion Phenomena

I. Nonuniqueness of Voltage in an MQS System

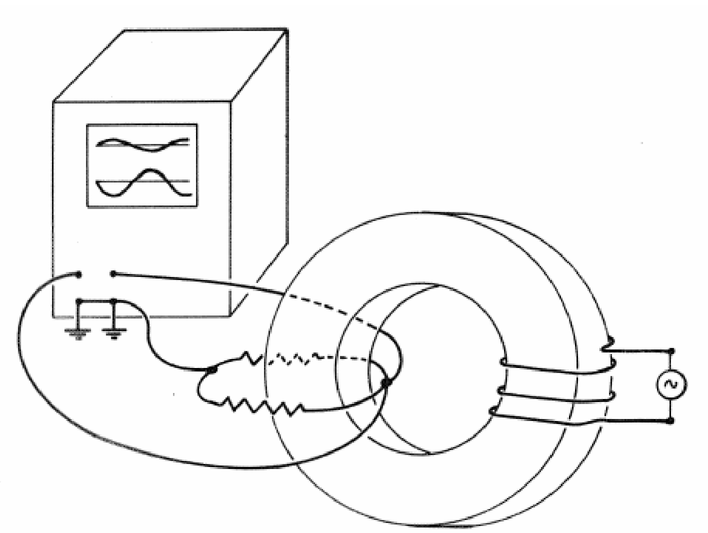


Figure 10.0.1 A pair of unequal resistors are connected in series around a magnetic circuit. Voltages measured between the terminals of the resistors by connecting the nodes to the dual-trace oscilloscope, as shown, differ in magnitude and are 180 degrees out of phase.

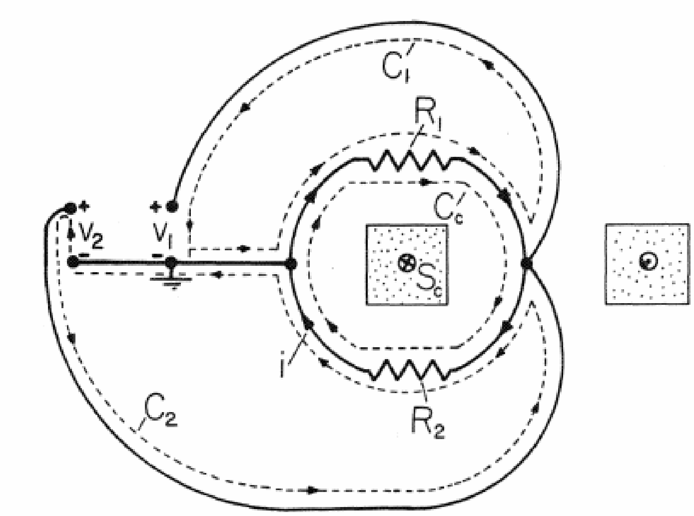


Figure 10.0.2 Schematic of circuit for experiment of Figure 10.0.1, showing contours used with Faraday's law to predict the differing voltages v_1 and v_2.

$$\Phi_\lambda = \int_{S_c} \overline{B} \cdot \overline{da}$$

$$\oint_{C_1} \overline{E} \cdot \overline{ds} = v_1 + iR_1 = 0$$

$$\oint_{C_2} \overline{E} \cdot \overline{ds} = -v_2 + iR_2 = 0$$

$$\oint_{C_c} \overline{E} \cdot ds = -\frac{d\Phi_\lambda}{dt} = i(R_1 + R_2)$$

$$i = -\frac{1}{(R_1 + R_2)} \frac{d\Phi_\lambda}{dt}$$

$$v_1 = -iR_1 = \frac{+R_1}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

$$v_2 = iR_2 = \frac{-R_2}{R_1 + R_2} \frac{d\Phi_\lambda}{dt}$$

$$\frac{v_1}{v_2} = -\frac{R_1}{R_2}$$

II. Diffusion of Axial Magnetic Fields into a Circular Tube

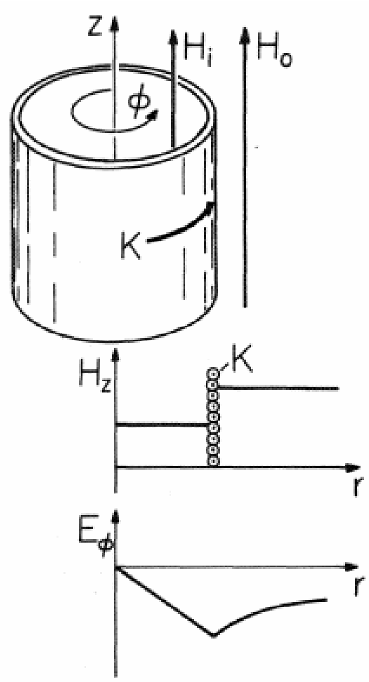


Figure 10.3.2 Circular cylindrical conducting shell with external axial field intensity $H_o(t)$ imposed. The response to a step in applied field is a current density that initially shields the field from the inner region. As this current decays, the field penetrates into the interior and is finally uniform throughout.

$$K_\phi = H_i - H_0 = J_\phi \Delta = \Delta \sigma E_\phi \Rightarrow E_\phi = \frac{K_\phi}{\sigma \Delta}$$

$$\oint_C \bar{E} \bullet \overline{ds} = E_\phi\, 2\pi a = -\frac{d}{dt}\left[\mu_0\, \pi a^2\, H_i\right] = \frac{2\pi a K_\phi}{\sigma \Delta} = \frac{2\pi a}{\sigma \Delta}\left(H_i - H_0\right)$$

$$\frac{dH_i}{dt} + \frac{2\cancel{\pi}\cancel{a}}{\sigma \Delta \mu_o \cancel{\pi} a^{\cancel{2}}}\left(H_i - H_0\right) = 0$$

$$\frac{dH_i}{dt} + \frac{H_i}{\tau_m} = \frac{H_0}{\tau_m} \quad ; \quad \tau_m = \frac{\mu_0\, \sigma\, \Delta\, a}{2} \quad \text{[Magnetic Diffusion Time]}$$

$$H_i = H_0\left[1 - e^{-t/\tau_m}\right]$$

$$K_\phi = H_i - H_0 = -H_0\, e^{-t/\tau_m}$$

Note: $L = \dfrac{\Phi}{K_\phi\, l} = \dfrac{\mu_0\, H_i\, \pi\, a^2}{H_i\, l} = \dfrac{\mu_0\, \pi\, a^2}{l}$

$$R = \frac{2\pi a}{\sigma\, l\, \Delta} \Rightarrow \tau_m = \frac{L}{R} = \frac{\mu_0\, \cancel{\pi}\, a^{\cancel{2}}}{\cancel{l}\, 2\, \cancel{\pi}\, \cancel{a}}\, \sigma\, \cancel{l}\, \Delta = \frac{\mu_0\, \sigma\, \Delta\, a}{2}$$

III. Edgerton's Boomer

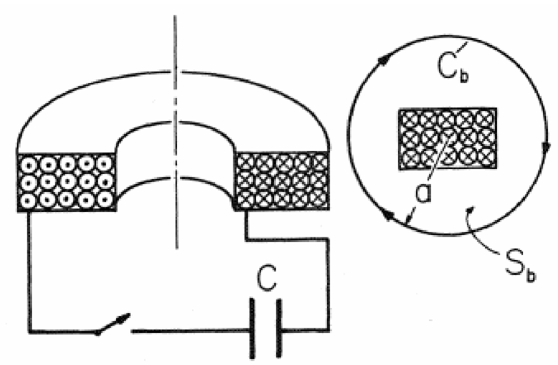


Figure 10.2.2 When the spark gap switch is closed, the capacitor discharges into the coil. The contour C_b is used to estimate the average magnetic field intensity that results.

$$\oint_{C_b} H \bullet ds \approx H_1 2\, \pi\, a = N_1\, i_1 \Rightarrow H_1 \approx \frac{N_1\, i_1}{2\pi a}$$

$$\lambda \approx N_1\left(\pi\, a^2\right)\mu_0\, H_1 = \frac{N_1^2\, \cancel{\pi}\, a^{\cancel{2}}\, \mu_0}{2\, \pi\, \cancel{a}}\, i_1 \approx \frac{N_1^2\, a\, \mu_0}{2}\, i_1$$

$$L = \frac{\lambda}{i_1} \approx \frac{N_1^2 a \mu_0}{2}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\frac{1}{2} L i_p^2 \approx \frac{1}{2} C v_p^2 \Rightarrow i_p \approx v_p \sqrt{C/L}$$

$C = 25\ \mu f,\ v_p = 4\ kV,\ N_1 = 50,\ a \approx 7\ cm$

$$L_1 \approx 0.1\ mH$$

$$i_p \approx 2000\,A,\ \omega \approx 20 \times 10^3 / s \Rightarrow f = \frac{\omega}{2\pi} \approx 3\,kHz$$

$$H_p \approx 2.3 \times 10^5\ A/m \Rightarrow B_p = \mu_0 H_p \approx 0.3\,\text{Teslas} \approx 3000\,\text{Gauss}$$

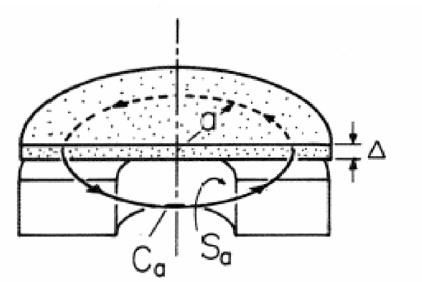


Figure 10.2.3 Metal disk placed on top of coil shown in Figure 10.2.2.

Courtesy of Hermann A. Haus and James R. Melcher. Used with permission.

$$\oint_{C_a} \overline{E} \bullet \overline{ds} \approx 2\pi a E_\phi = -\frac{d}{dt} \int_{S_a} \overline{B} \bullet \overline{da} \approx -\frac{d}{dt}\left(\mu_0 H_1 \pi a^2\right)$$

$$J_\phi = \sigma E_\phi = -\frac{\sigma \mu_0 a}{2} \frac{dH_1}{dt}$$

$$H_{ind} \approx \frac{i_2}{2\pi a} \approx \frac{\Delta \cancel{a} J_\phi}{2\pi \cancel{a}}$$

$$\left|\frac{H_{ind}}{H_1}\right| \approx \frac{\Delta \sigma \mu_0 a}{4\pi} \frac{1}{|H_1|}\left|\frac{dH_1}{dt}\right| \approx \frac{\omega \tau_m}{4\pi}\ ;\ \tau_m = \mu_0 \sigma \Delta a$$

$$\Delta = 2\,mm,\ a = 7\,cm,\ \tau_m \approx 6\,ms \Rightarrow \frac{\omega \tau_m}{4\pi} \approx 10$$

$$\bar{F} = \bar{J} \times \mu_0 \bar{H}\ , \quad \bar{f} = \int_V \bar{F}\, dV \quad = \quad \int_V \bar{J} \times \mu_0 \bar{H}\, dV \approx \frac{1}{2} K B \pi a^2\, \bar{i}_z$$

$$\approx \frac{1}{4} \mu_0 H^2 \pi a^2$$

Force per unit volume (for $\bar{F}$); total force (for $\bar{f}$)

$$M \frac{dv}{dt} = f_0\, T\, \delta(t)$$

$$M v\left(t = O_+\right) = f_0\, T$$

M = 80 grams, $H \approx 5 \times 10^5\ A/m$

$T \approx 1\,ms$

$$v\left(t = O_+\right) = \frac{f_0\, T}{M} = \frac{\frac{1}{4} \mu_0 H^2 \left(\pi_0{}^2\right) T}{M}$$

$$\approx 10\ m/s$$

$$\frac{1}{2} M v^2 \left(t = O_+\right) = M g h \Rightarrow h \approx \frac{1}{2} \frac{v^2\left(t = O_+\right)}{g} \approx 5\,m$$

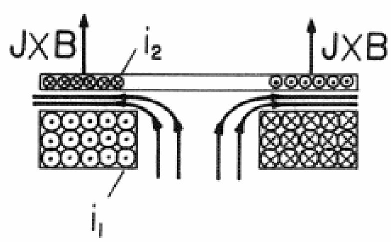


Figure 10.2.4 Currents induced in the metal disk tend to induce a field that bucks out that imposed by the driving coil. These currents result in a force on the disk that tends to propel it upward.

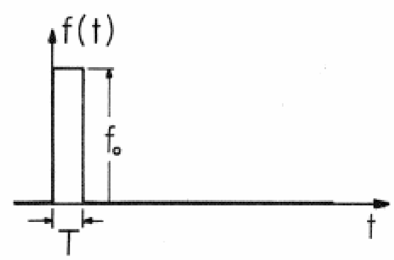


Figure 10.2.5 Because the magnetic force on the disk is always positive and lasts for a time T shorter than the time it takes the disk to leave the vicinity of the coil, it is represented by an impulse of magnitude $f_o T$.

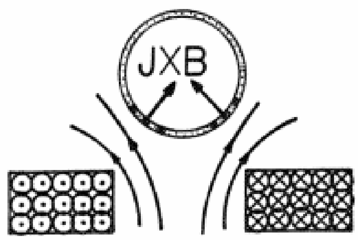


Figure 10.4.4 In an experiment giving evidence of the currents induced when a field is suddenly applied transverse to a conducting cylinder, an aluminum foil cylinder, subjected to the field produced by the experiment of Figure 10.2.2, is crushed.

IV. Magnetic Diffusion Equation

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t}\left(\mu \bar{H}\right)$$

$$\nabla \times \bar{H} = \bar{J} = \sigma \bar{E}$$

$$\nabla \cdot \left(\mu \bar{H}\right) = 0 \Rightarrow \nabla \cdot \bar{H} = 0$$

$$\nabla \times \left(\nabla \times \bar{H}\right) = \nabla\left(\cancelto{0}{\nabla \cdot \bar{H}}\right) - \nabla^2 \bar{H} = \sigma\left(\nabla \times \bar{E}\right) = -\sigma \mu \frac{\partial \bar{H}}{\partial t}$$

$$\nabla^2 \bar{H} = \mu \sigma \frac{\partial \bar{H}}{\partial t}$$ Magnetic Diffusion Equation

V. Magnetic Diffusion Transient Response

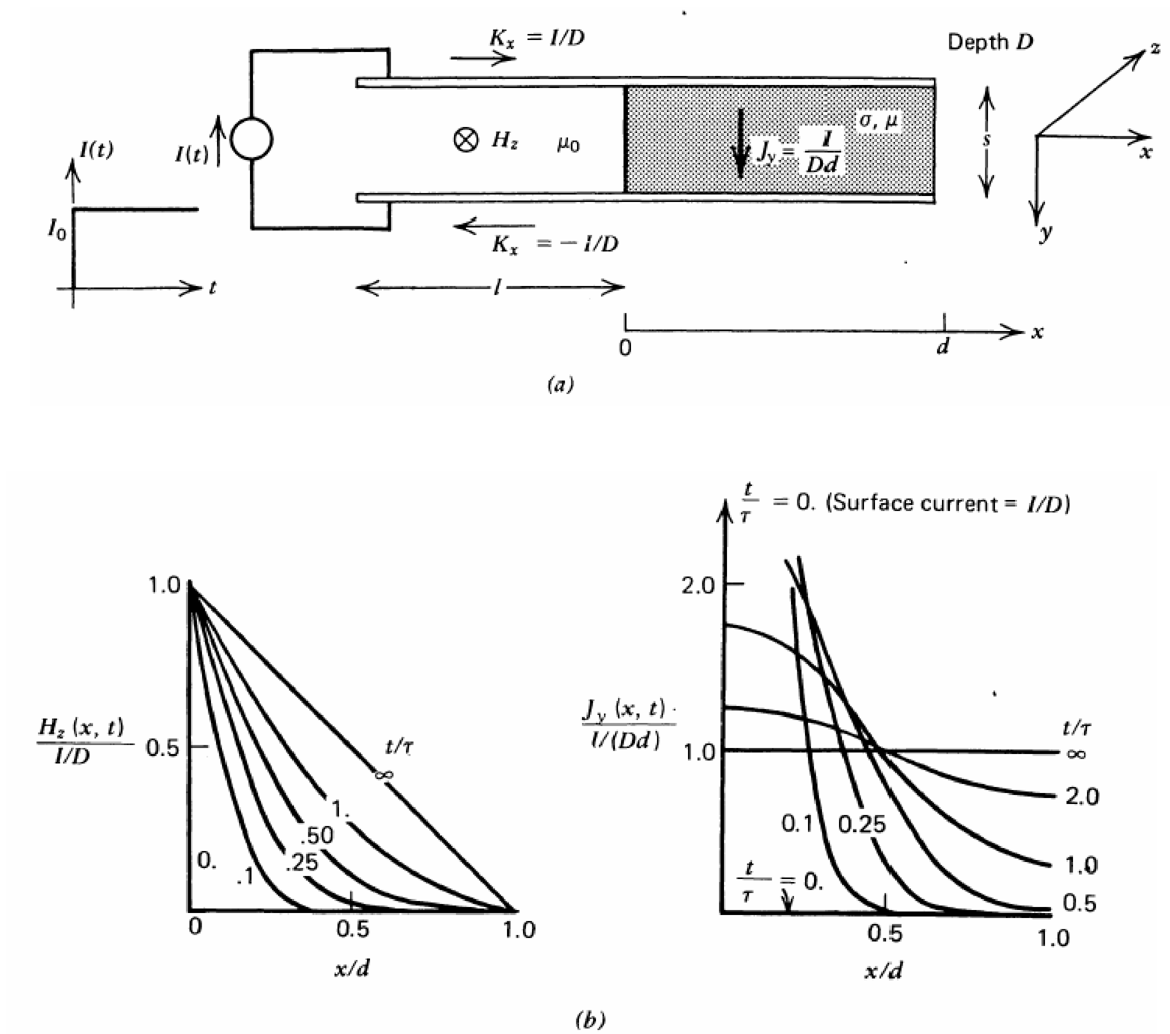


Figure 6-26 (*a*) A current source is instantaneously turned on at $t = 0$. The resulting magnetic field within the Ohmic conductor remains continuous and is thus zero at $t = 0$ requiring a surface current at $x = 0$. (*b*) For later times the magnetic field and current diffuse into the conductor with longest time constant $\tau = \sigma\mu d^2/\pi^2$ towards a steady state of uniform current with a linear magnetic field.

$$\frac{1}{\mu\sigma}\frac{\partial^2 H_z}{\partial x^2} = \frac{\partial H_z}{\partial t}$$

$$J_y = -\frac{\partial H_z}{\partial x}$$

Steady State Solution: $\frac{\partial^2 H_z}{\partial x^2} = 0 \Rightarrow H_z = ax + b \Rightarrow H_z(x) = \begin{cases} I/D & -l \le x \le o \\ \frac{I}{Dd}(d-x) & 0 \le x \le d \end{cases}$

$$J_y(x) = \begin{cases} 0 & -l \le x \le 0 \\ \frac{I}{Dd} & 0 \le x \le d \end{cases}$$

Total Solution for $0 \le x \le d$

$$H_z(x,t) = \frac{I}{Dd}(d-x) + \hat{H}(x)e^{-\alpha t}$$

$$\frac{d^2\hat{H}(x)}{dx^2} + \sigma\mu\alpha\hat{H}(x) = 0$$

$$\hat{H}(x) = A_1 \sin\sqrt{\sigma\mu\alpha}\ x \quad + \quad A_2 \cos\sqrt{\sigma\mu\alpha}\ x$$

$$\hat{H}_z(x=0) = I/D \qquad\qquad \hat{H}(x=0) = 0 \Rightarrow A_2 = 0$$

$$\Rightarrow$$

$$\hat{H}_z(x=d) = 0 \qquad\qquad \hat{H}(x=d) = 0 \Rightarrow A_1 \sin\sqrt{\sigma\mu\alpha}\, d = 0$$

$$\sqrt{\sigma\mu\alpha}\, d = n\pi \Rightarrow \alpha_n = \frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2, \qquad n = 1,2,3,\ldots$$

$$H_z(x,t) = \frac{I}{Dd}(d-x) + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d}\ e^{-\alpha_n t}$$

$$H_z(x,t=0) = 0 = \frac{I}{Dd}(d-x) + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d}$$

$$0 = \frac{I}{Dd}\underbrace{\int_0^d (d-x)\sin\frac{m\pi x}{d}dx}_{\frac{d^2}{m\pi}} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^d \sin\frac{n\pi x}{d}\ \sin\frac{m\pi x}{d}dx}_{=\begin{cases} 0 & m \neq n \\ \frac{d}{2} & m = n \end{cases}}$$

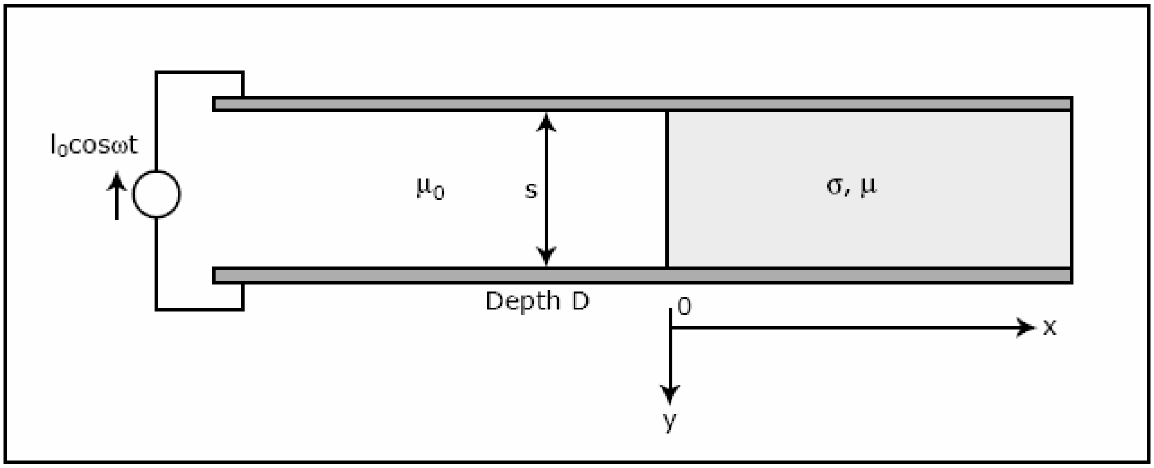
$$-\frac{I}{Dd}\ \frac{d^2}{m\pi} = \frac{A_m d}{2} \Rightarrow A_m = \frac{-2I}{m\pi D}$$

$$H_z(x,t) = \frac{I}{D}\left[1 - \frac{x}{d} - 2\sum_{n=1}^{\infty}\frac{\sin n\pi x/d}{n\pi} e^{-n^2 t/\tau}\right]$$

$$\tau = \frac{1}{\alpha_1} = \frac{\mu\sigma d^2}{\pi^2}$$

$$J_y = \frac{-\partial H_z}{\partial x} = \frac{I}{Dd}\left[1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi x}{d}\ e^{-n^2 t/\tau}\right]$$

VI. Sinusoidal Steady State Magnetic Diffusion



$$H_z(x,t) = \mathrm{Re}\left[\hat{H}_z(x)\, e^{j\omega t}\right]$$

$$\frac{\partial^2 H_z}{\partial x^2} = \sigma\mu \frac{\partial H_z}{\partial t} \Rightarrow \frac{d^2 \hat{H}_z}{dx^2} - j\omega\mu\sigma \hat{H}_z(x) = 0$$

$$\hat{H}_z(x) = H_0\, e^{-(1+j)x/\delta} \quad ; \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad x > 0$$

$$\hat{H}_z(x) = H_0 = \frac{I_0}{D} \quad x < 0$$

$$\hat{J}_y(x) = -\frac{d\hat{H}_z}{dx} = \frac{(1+j)}{\delta} H_o\, e^{-(1+j)x/\delta} \quad x > 0$$