6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2006

Final- Solutions

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Problem 1

A

<u>**Question:**</u> **What are the boundary conditions necessary to solve for the magnetic fields for $x < 0$ and for $0 < x < s$?**

<u>**Solution:**</u>

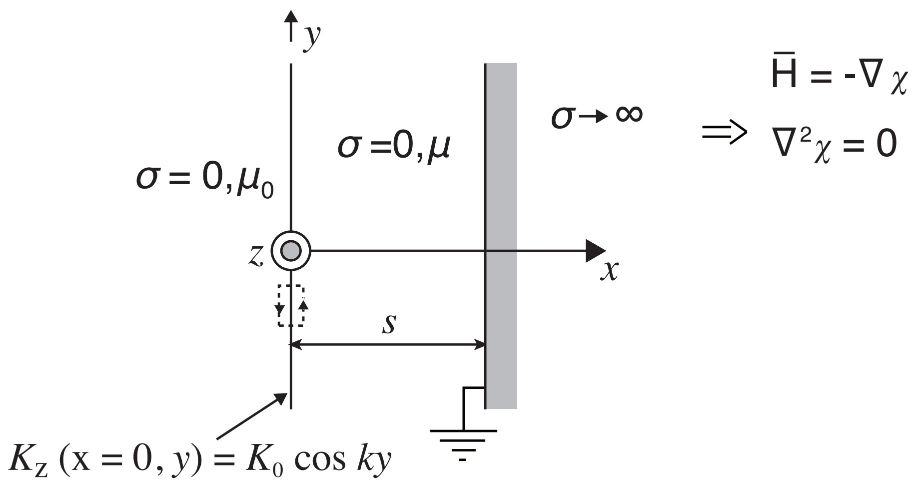

Figure 1: A sheet of surface current at $x = 0$. (Image by MIT OpenCourseWare.)

B.C. I $\quad \overline{n} \times \left[\overline{H}_{II} - \overline{H}_I\right] = \overline{K} \Rightarrow H_y(x = 0_+, y) - H_y(x = 0_-, y) = K_0 \cos ky$

B.C. II $\quad \chi \rightarrow 0$ as $x \rightarrow -\infty$

B.C. III $\quad \overline{n} \cdot \left[\mu_{II}\overline{H}_{II} - \mu_I\overline{H}_I\right] = 0 \Rightarrow H_x(x = s_-, y) = 0$

B.C. IV $\quad \overline{n} \cdot \left[\mu_{II}\overline{H}_{II} - \mu_I\overline{H}_I\right] = 0 \Rightarrow \mu_0 H_x\Big|_{x=0^-} = \mu H_x\Big|_{x=0^+}$

B

Question: What are the magnetic scalar potential and magnetic field distributions for $x < 0$ and $0 < x < s$?

Hint: The algebra will be greatly reduced if you use one of the following forms of the potential for the region $0 < x < s$

I. $\sin(ky)\cosh k(x-s)$

II. $\cos(ky)\cosh k(x-s)$

III. $\sin(ky)\sinh k(x-s)$

IV. $\cos(ky)\sinh k(x-s)$

Solution:

$$\nabla^2\chi = 0 \Rightarrow \frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = 0$$

General solution is of the form

$$\chi(x,y) = e^{kx}\left(C_1\sin ky + C_2\cos ky\right) + e^{-kx}\left(C_3\sin ky + C_4\cos ky\right)$$

For $x < 0$ only e^{kx} term is relevant from B.C. II $\lim_{x\to-\infty}\chi \to 0$; also from surface current boundary condition I only $\sin ky$ term is needed. Therefore, $\chi(x,y) = C_1 e^{kx}\sin ky$ for $x < 0$. For $x > 0$, we can work with the general solution in sinh, cosh, due to surface current condition in y, only $\sin ky$ term is needed; also due to boundary condition at $x = 0$, $x = s$ we can use the form

$$\chi(x,y) = A_1\sin ky\cosh(x-s) \qquad 0 < x < s$$

$$\Rightarrow \chi(x,y) = \begin{cases} C_1\sin ky e^{kx} & x < 0 \\ A_1\sin ky\cosh k(x-s) & 0 < x < s \end{cases}$$

$$\Rightarrow \overline{H} = -\nabla\chi = \begin{cases} -C_1 k e^{kx}(\sin ky\bar{i}_x + \cos ky\bar{i}_y) & x < 0 \\ -A_1 k(\sin ky\sinh k(x-s)\bar{i}_x + \cos ky\cosh k(x-s)\bar{i}_y) & 0 < x < s \end{cases}$$

$$\text{B.C. III} \Rightarrow H_x\Big|_{x=s} = 0 \Rightarrow \text{ satisfied}$$

$$\text{B.C. IV} \Rightarrow \mu_0 H_x\Big|_{x=0^-} = \mu H_x\Big|_{x=0^+} \Rightarrow -\mu_0 C_1 \cancel{k} = -\mu A_1 \cancel{k}\sinh(-ks) \Rightarrow C_1 = -\frac{\mu}{\mu_0}A_1\sinh ks$$

$$\text{B.C. I} \Rightarrow H_y\Big|_{x=0^+} - H_y\Big|_{x=0^-} = K_0\cos ky \Rightarrow -A_1 k\cosh(-ks) + C_1 k = K_0$$

$$\Rightarrow -A_1\cosh ks + C_1 = \frac{K_0}{k} \Rightarrow -A_1\cosh ks - \frac{\mu}{\mu_0}A_1\sinh ks = \frac{K_0}{k}$$

$$\Rightarrow A_1(\mu_0\cosh ks + \mu\sinh ks) = -\frac{\mu_0 K_0}{k}$$

$$\Rightarrow A_1 = -\frac{\mu_0 K_0}{k(\mu_0\cosh ks + \mu\sinh ks}$$

$$\Rightarrow C_1 = \frac{\mu K_0\sinh ks}{k(\mu_0\cosh ks + \mu\sinh ks)}$$

$$\Rightarrow \chi = \frac{K_0}{k(\mu_0 \cosh ks + \mu \sinh ks)} \begin{cases} \mu \sinh ks e^{kx} \sin ky & x < 0 \\ -\mu_0 \cosh k(x-s) \sin ky & 0 < x < s \end{cases}$$

$$\overline{H} = \frac{K_0}{(\mu_0 \cosh ks + \mu \sinh ks)} \begin{cases} -\mu \sinh ks e^{kx} (\sin ky \bar{i}_x + \cos ky \bar{i}_y) & x < 0 \\ +\mu_0 (\sin ky \sinh k(x-s) \bar{i}_x + \cos ky \cosh k(x-s) \bar{i}_y) & 0 < x < s \end{cases}$$

C

Question: What is the surface current distribution on the $x = s$ surface?

Solution:

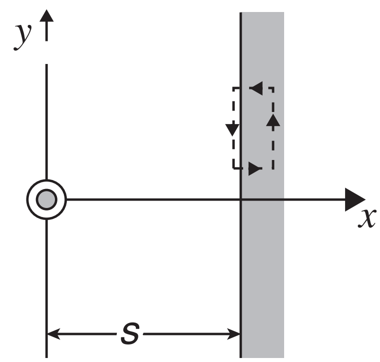

Figure 2: Surface current boundary condition at $x = s$. (Image by MIT OpenCourseWare.)

From magnetic field boundary condition $\overline{n} \times \left[\overline{H}_{II} - \overline{H}_I\right] = \overline{K}_S$. Therefore, $-H_y\Big|_{x=s_-} = K_{sz}\Big|_{x=s}$.

$$\overline{K}(x = s, y) = -\frac{K_0 \mu_0 \cos ky \bar{i}_z}{\mu_0 \cosh ks + \mu \sinh ks}$$

D

Question: Use the Maxwell Stress Tensor to find the total force, magnitude and direction, on a section of the perfect conductor at $x = s$ that extends over a wavelength $0 < y < \frac{2\pi}{k}$ and $0 < z < D$. Assume that μ in the region $0 < x < s$ does not depend on density so that $\frac{d\mu}{d\rho} = 0$.

Hint: $\int \cos^2(ky)dy = \frac{1}{2}y + \frac{1}{4k}\sin(2ky)$

Solution:

$$T_{xx} = \frac{1}{2}\mu \left[H_x^2 - H_y^2 - H_z^2\right]\Big|_{x=s_-}$$

$$T_{zx} = \mu H_z H_x\Big|_{x=s_-} = 0 \qquad (H_z(x, y) = 0)$$

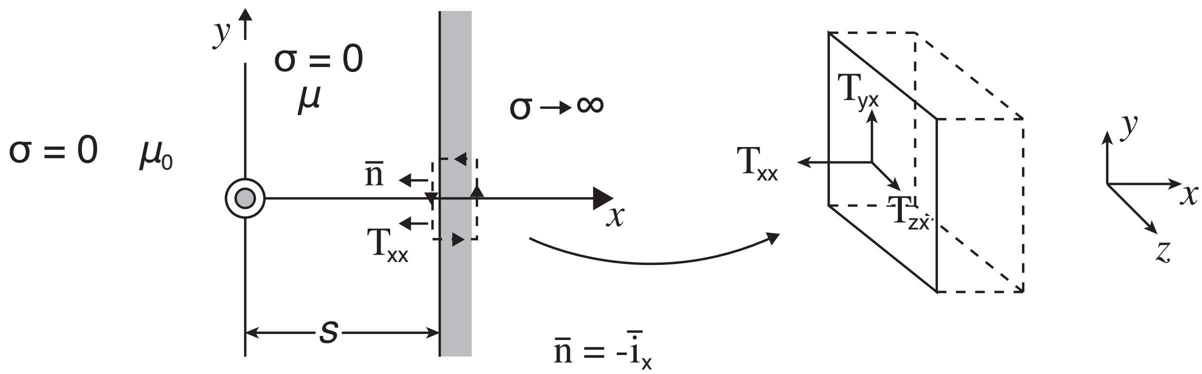

Figure 3: Maxwell Stress tensor used to find force on perfect conductor. (Image by MIT OpenCourseWare.)

$$T_{yx} = \mu H_y H_x\Big|_{x=s_-} = 0 \qquad \left(H_x\Big|_{x=s_-} = 0\right)$$

$$T_{xx} = \frac{1}{2}\mu\left[\underbrace{H_x^2\Big|_{x=s_-}}_{0} - H_y^2\Big|_{x=s_-}\right] = -\frac{1}{2}\mu\left[\frac{K_0\mu_0}{\mu_0\cosh ks + \mu\sinh ks}\right]^2 \cos^2 ky$$

$$T_{xy} = T_{yx} = \mu(H_y H_x)_{x=s_-} = 0 \qquad \text{because } H_x\Big|_{x=s_-} = 0$$

$$f_x = \oint_S T_{xj} n_j da \Rightarrow f_x = -D\int_0^{\frac{2\pi}{k}} T_{xx} dy$$

$$= \frac{D\mu}{2}\left[\frac{K_0\mu_0}{\mu_0\cosh ks + \mu\sinh ks}\right]^2 \underbrace{\int_0^{\frac{2\pi}{k}} \cos^2 ky dy}_{\left(\frac{y}{2} + \frac{\sin(2ky)}{4k}\right)\Big|_0^{\frac{2\pi}{k}} = \frac{\pi}{k}}$$

$$f_x = \frac{D\mu\pi}{2k}\left[\frac{K_0\mu_0}{\mu_0\cosh ks + \mu\sinh ks}\right]^2$$

Problem 2

A

Question: What is the electric field in the free space region, $0 < r < R_1$ **as a function of time?**

Solution: Charge relaxation.

$$\left.\begin{array}{rr} \text{Conservation of charge} & \nabla\cdot\overline{J}_f + \frac{\partial\rho_f}{\partial t} = 0 \\ \left.\begin{array}{rr} \text{Gauss' Law} & \nabla\cdot\overline{E} = \frac{\rho_f}{\epsilon} \\ \text{Linear Media} & \overline{J_f} = \sigma\overline{E} \end{array}\right\} & \nabla\cdot\overline{J}_f = \sigma\nabla\cdot\overline{E} = \sigma\frac{\rho_f}{\varepsilon} = 0 \end{array}\right\} \frac{\sigma}{\varepsilon}\rho_f + \frac{\partial\rho_f}{\partial t} = 0$$

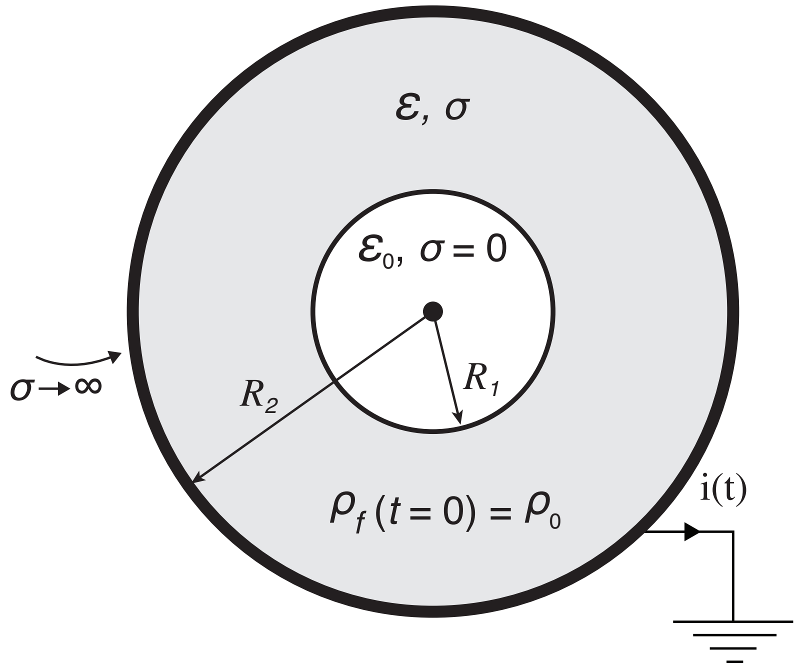

Figure 4: Cylindrical shell of uniform volume charge. (Image by MIT OpenCourseWare.)

$$\Rightarrow \rho_f(t) = \rho_f(t=0)e^{-\frac{t}{\tau}} \qquad \text{where } \tau = \frac{\varepsilon}{\sigma}$$

$$\Rightarrow \rho_f(r,t) = \begin{cases} 0 & 0 < r < R_1 \\ \rho_0 e^{-\frac{t}{\tau}} & R_1 < r < R_2 \end{cases}$$

$$\overline{E}(r,t) = 0 \text{ for } 0 \leq r \leq R_1$$

B

Question: What is the volume charge density and electric field within the cylindrical shell, $R_1 < r < R_2$ as a function of radius and time?

Solution: Using Gauss' Law

$$\oint_S \varepsilon \overline{E} \cdot d\overline{a} = \int \rho_f dV$$

$$\text{Charge enclosed} \begin{cases} 0 & 0 < r < R_1 \\ \pi(r^2 - R_1^2) d\rho_0 e^{-\frac{t}{\tau}} & R_1 < r < R_2 \end{cases}$$

$$\begin{cases} 2\pi r d\varepsilon_0 E_r = 0 & 0 < r < R_1 \\ 2\pi r d\varepsilon E_r = \pi(r^2 - R_1^2) d\rho_0 e^{-\frac{t}{\tau}} & R_1 < r < R_2 \end{cases}$$

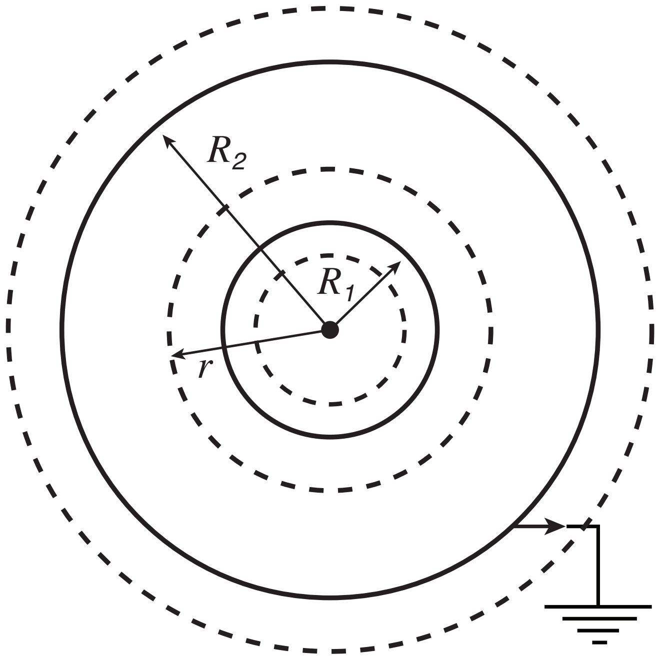

Figure 5: Using Gaussian surfaces (dashes) with Gauss' law to determine the electric field in each region. (Image by MIT OpenCourseWare.)

$$E_r = \begin{cases} 0 & 0 < r < R_1 \\ \frac{r^2 - R_1^2}{2\varepsilon r}\rho_0 e^{-\frac{t}{\tau}} & R_1 < r < R_2 \\ 0 & r > R_2 \end{cases}$$

The final case ($r > R_2$) is given by the problem statement.

C

Question: What is the surface charge density on the interface at $r = R_2$?

Solution:

$$\overline{n} \cdot \left[\overline{D}_{II} - \overline{D}_I\right] = \sigma_{sf}$$

$$\text{at } r = R_2 : \varepsilon_0 E_r\Big|_{r=R_2^+} - \varepsilon E_r|\Big|_{r=R_2^-} = \sigma_{sf}\Big|_{r=R_2} = -\frac{R_2^2 - R_1^2}{2R_2}\rho_0 e^{-t/\tau}$$

D

Question: What is the ground current, $i(t)$?

Solution: From Ampere's Law

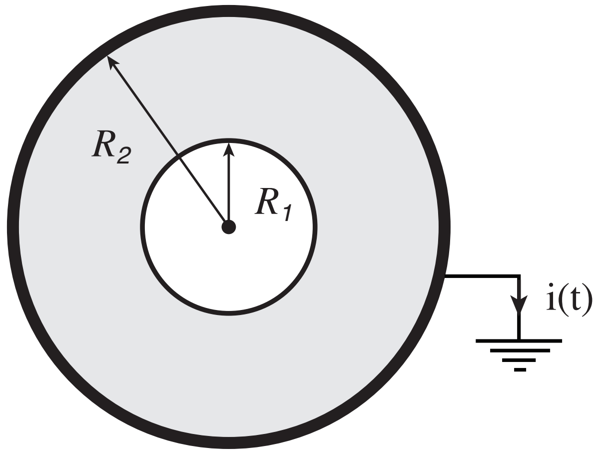

Figure 6: Using conservation of charge at $r = R_2$ to determine the terminal current $i(t)$. (Image by MIT OpenCourseWare.)

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \Rightarrow \nabla \cdot [\underbrace{\overline{J}}_{\substack{\text{conduction}\\ \text{current}}} + \underbrace{\frac{\partial \overline{D}}{\partial t}}_{\substack{\text{displacement}\\ \text{current}}}] = 0$$

$\Rightarrow$

$$\begin{aligned} i(t) &= 2\pi R_2 d(\sigma E_r + \varepsilon \frac{\partial E_r}{\partial t})\Big|_{r=R_2} \\ &= 2\pi R_2 d\left[\sigma \frac{R_2^2 - R_1^2}{2\varepsilon R_2}\rho_0 e^{-\frac{t}{\tau}} + \varepsilon\left(-\frac{R_2^2 - R_1^2}{2\varepsilon R_2}\rho_0 \frac{1}{\tau} e^{-\frac{t}{\tau}}\right)\right] \\ &= 2\pi R_2 d\left[\sigma \frac{R_2^2 - R_1^2}{2\varepsilon R_2}\rho_0 e^{-\frac{t}{\tau}} - \varepsilon \frac{R_2^2 - R_1^2}{2\varepsilon R_2}\rho_0 \frac{\sigma}{\varepsilon} e^{-\frac{t}{\tau}}\right] = 0 \end{aligned}$$

$$i(t) = 0$$

Problem 3

A

Question: Prove that the magnetic scalar potential χ obeys Laplace's equation for $0 < r < R_p$, and $R_p < r < R$ where $\overline{H} = -\nabla\chi$.

Solution: Since there are no free currents in the regions $0 < r < R_p$ and $R_p < r < R$, from Ampere's Law

$$\nabla \times \overline{H} = \overline{J}_f = 0 \Rightarrow \overline{H} = -\nabla\chi$$

for $0 < r < R_p$

$$\overline{B} = \mu_0(\overline{H} + \overline{M}), \nabla \cdot \overline{B} = 0 \Rightarrow \nabla \cdot \mu_0(\overline{H} + \overline{M}) = 0 \Rightarrow \nabla \cdot \overline{H} = -\nabla \cdot \overline{M}$$

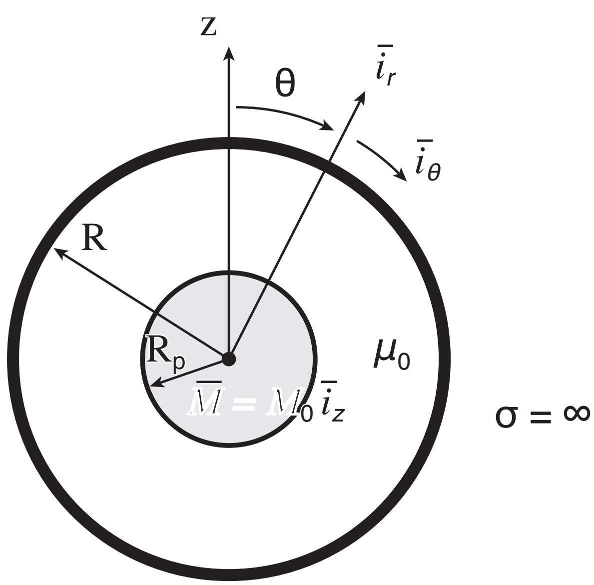

Figure 7: A magnetized sphere. (Image by MIT OpenCourseWare.)

$$\overline{H} = -\nabla\chi \Rightarrow \nabla^2\chi = \nabla\cdot\overline{M} = \frac{\partial M_z}{\partial z} = 0$$

Because M_z is constant, $\nabla^2\chi = 0$.

For $R_p < r < R$:

$$\overline{B} = \mu_0(\overline{H} + \overline{M}), \overline{M} = 0$$

From Gauss' Law

$$\nabla\cdot\overline{B} = 0 \Rightarrow \nabla\cdot(\mu_0\overline{H}) = 0 \Rightarrow \nabla\cdot(\mu_0(-\nabla\chi)) = 0$$

$$\Rightarrow \nabla^2\chi = 0$$

because μ_0 is constant.

B

Question: What are the boundary conditions required to determine the magnetic field in regions $0 < r < R_p$ and $R_p < r < R$?

Solution: B.C. I

$$\overline{n}\cdot\left[\overline{B}_{II} - \overline{B}_I\right] = 0 \Rightarrow H_r\Big|_{r=R_-} = 0$$

B.C. II

Since there are no surface currents on $r = R_p$ surface.

$$\overline{n}\times\left[\overline{H}_{II} - \overline{H}_I\right] = \overline{K} = 0 \Rightarrow H_\theta\Big|_{r=R_p^+} = H_\theta\Big|_{r=R_p^-}$$

$$\overline{B} = \mu_0(\overline{H} + \overline{M}), \nabla \cdot \overline{B} = 0 \Rightarrow \nabla \cdot (\mu_0(\overline{H} + \overline{M})) = 0$$

$$\Rightarrow \nabla \cdot (\mu_0 \overline{H}) = -\nabla \cdot (\mu_0 \overline{M}) \Rightarrow \overline{n} \cdot \left[\mu_0(\overline{H}_{II} - \overline{H}_I)\right] = -\overline{n} \cdot \left[\mu_0(\overline{M}_{II} - \overline{M}_I)\right]$$

$$\Rightarrow H_r\Big|_{r=R_p^+} - H_r\Big|_{r=R_p^-} = +M_r\Big|_{r=R_p^-}$$

B.C. III

$$H_r\Big|_{r=R_p^+} = H_r\Big|_{r=R_p^-} + M_0 \cos\theta$$

C

Question: Find the magnetic field $\overline{H}(r,\theta)$ in regions $0 < r < R_p$ and $R_p < r < R$.

Solution:

$$\chi = \begin{cases} Ar\cos\theta & r < R_p \\ Br\cos\theta + \frac{C}{r^2}\cos\theta & R_p < r < R \end{cases}$$

$$\overline{H} = -\nabla\chi = -\frac{\partial \chi}{\partial r}\bar{i}_r - \frac{1}{r}\frac{\partial \chi}{\partial \theta}\bar{i}_\theta$$

$$\Rightarrow \overline{H} = -\nabla\chi = \begin{cases} -A\cos\theta\bar{i}_r + A\sin\theta\bar{i}_\theta & r < R_p \\ -(B\cos\theta - \frac{2C}{r^3}\cos\theta)\bar{i}_r + (B\sin\theta + \frac{C}{r^3}\sin\theta)\bar{i}_\theta & R_p < r < R \end{cases}$$

B.C. I: $\Rightarrow H_r\Big|_{r=R_-} = 0 \Rightarrow B\cos\theta - \frac{2C}{R^3}\cos\theta = 0 \Rightarrow B = \frac{2C}{R^3} \Rightarrow C = \frac{BR^3}{2}$

B.C. II $\Rightarrow H_\theta\Big|_{r=R_p^-} = H_\theta\Big|_{r=R_p^+} \Rightarrow A\sin\theta = B\sin\theta + \frac{C}{R_p^3}\sin\theta$

$$A = B + \frac{1}{R_p^3}\frac{R^3}{2}B = B(1 + \frac{R^3}{2R_p^3}) = \frac{2R_p^3 + R^3}{2R_p^3}B = A$$

B.C. III

$$H_r\Big|_{r=R_p^+} = H_r\Big|_{r=R_p^-} + M_0\cos\theta$$

$$-B\cos\theta + \frac{2C}{R_p^3}\cos\theta = -A\cos\theta + M_0\cos\theta$$

$$-B + \frac{2}{R_p^3}\frac{BR^3}{2} = -B - \frac{R^3}{2R_p^3}B + M_0$$

$$B\left(-1 + \frac{R^3}{R_p^3} + 1 + \frac{R^3}{2R_p^3}\right) = M_0 \Rightarrow \frac{3}{2}\frac{R^3}{R_p^3}B = M_0$$

$$B = \frac{2}{3}M_0\left(\frac{R_p}{R}\right)^3 \Rightarrow C = \frac{BR^3}{2} \Rightarrow C = \frac{1}{3}M_0R_p^3 \Rightarrow A = M_0\frac{2R_p^3 + R^3}{3R^3}$$

$$\overline{H} = \begin{cases} M_0\frac{2R_p^3+R^3}{3R^3}\left(-\cos\theta\bar{i}_r + \sin\theta\bar{i}_\theta\right) & r < R_p \\ -\frac{2M_0}{3}\left[\left(\frac{R_p}{R}\right)^3 - \left(\frac{R_p}{r}\right)^3\right]\cos\theta\bar{i}_r + \frac{M_0}{3}\left[2\left(\frac{R_p}{R}\right)^3 + \left(\frac{R_p}{r}\right)^3\right]\sin\theta\bar{i}_\theta & R_p < r < R \end{cases}$$

D

Question: Find the free surface current density $\overline{K}$ on the $r = R$ surface.

Solution:

$$\overline{n} \times \left[\overline{H}_{II} - \overline{H}_I\right] = \overline{K}$$

$$\Rightarrow -H_\theta\Big|_{r=R^-} = K_\phi$$

$$\Rightarrow \overline{K} = -M_0\left(\frac{R_p}{R}\right)^3 \sin\theta \bar{i}_\theta$$

Problem 4

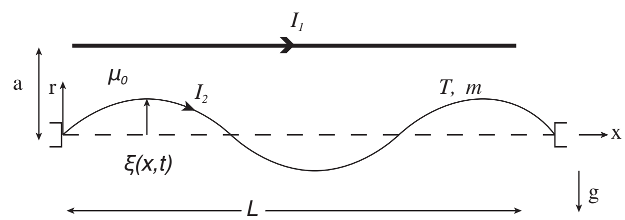

Figure 8: A current carrying string in a magnetic field from a line current. (Image by MIT OpenCourseWare.)

A

Question: To linear terms in membrane displacement $\xi(x, t)$, find the magnetic force per unit length on the string centered at $r = 0$.

Solution: From Ampere's Law, $\oint_C \overline{H} \cdot \overline{dl} = I_{\text{enclosed}} \Rightarrow H_\phi 2\pi r = I_1 \Rightarrow H_\phi = \frac{I}{2\pi r}$. Lorentz Force on the

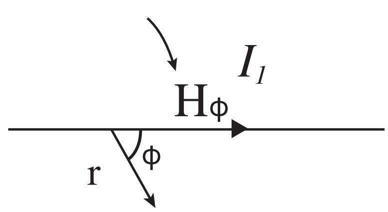

Figure 9: The magnetic field from a line current is determined by using Ampere's circuital law. (Image by MIT OpenCourseWare.)

conducting strings

$$\frac{\overline{F}}{l} = (I_2\bar{i}_x) \times (\mu_0\overline{H}\Big|_{r=a-\xi}) = I_2\frac{\mu_0 I_1}{2\pi(a-\xi)}\bar{i}_r$$

$$\Rightarrow \frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi}\frac{1}{(a-\xi)} = \frac{\mu_0 I_1 I_2}{2\pi a}\frac{1}{(1-\frac{\xi}{a})} \approx \frac{\mu_0 I_1 I_2}{2\pi a}\left(1+\frac{\xi}{a}\right)$$

B

Question: What is the governing linearized differential equation of motion of the membrane?

Solution: The equation of motion for the strings is given by

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} - mg + \frac{\mu_0 I_1 I_2}{2\pi a}\left(1+\frac{\xi}{a}\right)$$

C

Question: What must I_1 be in terms of I_2, m and other relevant parameters so that the membrane is in static equilibrium with $\xi(x,t) = 0$?

Solution: In static equilibrium at $\xi = 0$, $\frac{d}{dt} \to 0$, therefore

$$0 = -mg + \frac{\mu_0 I_1 I_2}{2\pi a} \Rightarrow I_1 = \frac{2\pi a m g}{\mu_0 I_2}$$

D

Question: For small membrane deflections of the form $\xi(x,t) = \mathrm{Re}\left[\hat{\xi}e^{j(\omega t - kx)}\right]$ find the $\omega - k$ dispersion relation. Plot the $\omega - k$ relationship showing significant intercepts on the axes and slope asymptotes. Assume that k is real and that ω can be pure real or pure imaginary.

Solution: For small deflections from equilibrium point at $\xi = 0$, the perturbation equation is

$$m\frac{\partial^2 \xi'}{\partial t^2} = T\frac{d^2 \xi'}{dx^2} + \frac{\mu_0 I_1 I_2}{2\pi a^2}\xi'$$

$$\Rightarrow \frac{\partial^2 \xi'}{\partial t^2} = \underbrace{\frac{T}{m}}_{v_p^2}\frac{\partial^2 \xi'}{\partial x^2} + \underbrace{\frac{\mu_0 I_1 I_2}{2\pi a^2 m}}_{\omega_c^2}\xi' \qquad v_p^2 = \frac{T}{m}, w_c^2 = \frac{\mu_0 I_2 I_1}{2\pi m a^2} = \frac{g}{a}$$

$$\frac{\partial^2 \xi'}{\partial t^2} = v_p^2\frac{\partial^2 \xi'}{\partial x^2} + \omega_c^2 \xi'$$

For deflections of the form $\xi'(x,t) = \mathrm{Re}\left[\hat{\xi}e^{j(\omega t - kx)}\right]$

$$-\omega^2\hat{\xi} = -v_p^2 k^2\hat{\xi} + \omega_c^2\hat{\xi} \Rightarrow \omega^2 = v_p^2 k^2 - \omega_c^2$$

dispersion relationship: for $I_1 I_2 > 0, \omega^2 = v_p^2 k^2 - \omega_c^2$, with $\omega_c^2 > 0$.

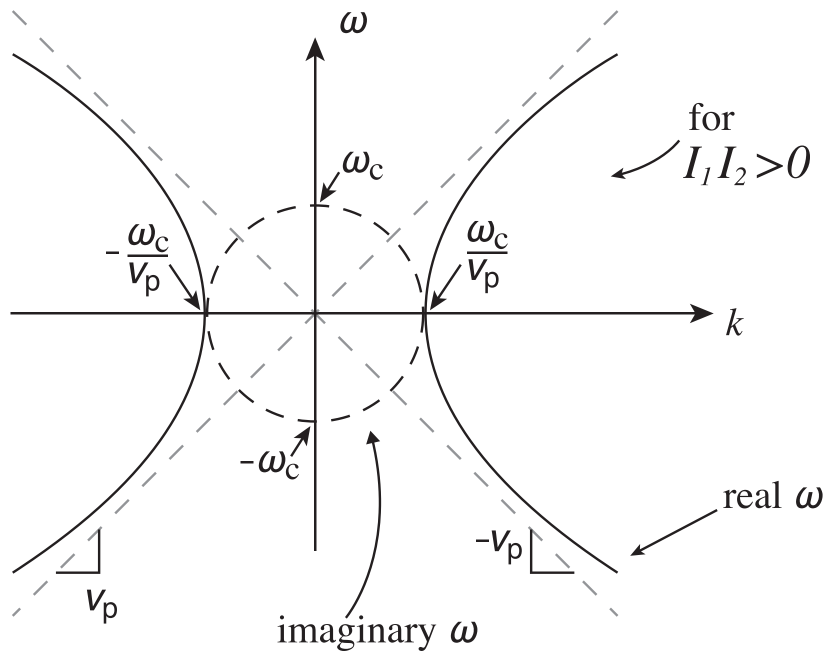

Figure 10: $\omega - k$ dispersion relation with $I_1 I_2 > 0$. (Image by MIT OpenCourseWare.)

E

Question: What are the allowed values of k that satisfy the zero deflection boundary conditions at $x = 0$ and $x = L$?

Solution: From dispersion relationships

$$\omega^2 = v_p^2 k^2 - \omega_c^2 \Rightarrow k^2 = \frac{\omega^2 + \omega_c^2}{v_p^2} \Rightarrow k = \pm\sqrt{\frac{\omega^2 + \omega_c^2}{v_p^2}}$$

For $k_0 = \sqrt{\frac{\omega^2+\omega_0^2}{v_p^2}}$ where k_0 is the positive root of solution for k the displacement will have solution of the form

$$\xi(x,t) = \text{Re}\left\{\left(\hat{\xi}_1 e^{-jk_0 x} + \hat{\xi}_2 e^{jk_0 x}\right) e^{j\omega t}\right\}$$

with boundary conditions B.C. I $\xi(x = 0, t) = 0, \Rightarrow \hat{\xi}_1 + \hat{\xi}_2 = 0 \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2$ and
B.C. II $\xi(x = L, t) = 0 \Rightarrow \hat{\xi}_1 e^{-jk_0 L} + \hat{\xi}_2 e^{jk_0 L} = 0$.

$$\Rightarrow \hat{\xi}_1 e^{-jk_0 L} - \hat{\xi}_1 e^{jk_0 L} = 0$$

$$\hat{\xi}_1(-2j \sin k_0 L) = 0 \Rightarrow k_0 = \frac{n\pi}{L}, n = 1, 2, 3$$

F

Question: Under what conditions will the membrane equilibrium with $\xi(x,t) = 0$ first become unstable?

Solution: First unstable condition happens when $k = \frac{\pi}{L} = \frac{\omega_c}{v_p}$

$$\frac{\omega_c^2}{v_p^2} = \left(\frac{\pi}{L}\right)^2 \Rightarrow \frac{g}{a}\frac{m}{T} = \left(\frac{\pi}{L}\right)^2$$