Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.685 Electric Machinery

Class Notes 9: Synchronous Machine Simulation Models   September 5, 2005
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1 Introduction

In this document we develop models useful for calculating the dynamic behavior of synchronous machines. We start with a commonly accepted picture of the synchronous machine, assuming that the rotor can be fairly represented by three equivalent windings: one being the field and the other two, the d- and q- axis "damper" windings, representing the effects of rotor body, wedge chain, amortisseur and other current carrying paths.

While a synchronous machine is assumed here, the results are fairly directly applicable to induction machines. Also, extension to situations in which the rotor representation must have more than one extra equivalent winding per axis should be straightforward.

2 Phase Variable Model

To begin, assume that the synchronous machine can be properly represented by six equivalent windings. Four of these, the three armature phase windings and the field winding, really are windings. The other two, representing the effects of distributed currents on the rotor, are referred to as the "damper" windings. Fluxes are, in terms of currents:

$$\left[\begin{array}{c} \underline{\lambda}_{ph} \\ \underline{\lambda}_R \end{array}\right] = \left[\begin{array}{cc} \underline{\underline{L}}_{ph} & \underline{\underline{M}} \\ \underline{\underline{M}}^T & \underline{\underline{L}}_R \end{array}\right] \left[\begin{array}{c} I_{ph} \\ I_R \end{array}\right] \tag{1}$$

where *phase* and *rotor* fluxes (and, similarly, currents) are:

$$\underline{\lambda}_{ph} = \left[\begin{array}{c} \lambda_a \\ \lambda_b \\ \lambda_c \end{array}\right] \tag{2}$$

$$\underline{\lambda}_R = \left[\begin{array}{c} \lambda_f \\ \lambda_{kd} \\ \lambda_{kq} \end{array}\right] \tag{3}$$

There are three inductance sub- matrices. The first of these describes armature winding inductances:

$$\underline{\underline{L}}_{ph} = \left[\begin{array}{ccc} L_a & L_{ab} & L_{ac} \\ L_{ab} & L_b & L_{bc} \\ L_{ac} & L_{bc} & L_c \end{array}\right] \tag{4}$$

where, for a machine that may have some saliency:

$$L_a = L_{a0} + L_2 \cos 2\theta \tag{5}$$

$$L_b = L_{a0} + L_2 \cos 2(\theta - \frac{2\pi}{3}) \tag{6}$$

$$L_c = L_{a0} + L_2 \cos 2(\theta + \frac{2\pi}{3}) \tag{7}$$

$$L_{ab} = L_{ab0} + L_2 \cos 2(\theta - \frac{\pi}{3}) \tag{8}$$

$$L_{bc} = L_{ab0} + L_2 \cos 2\theta \tag{9}$$

$$L_{ac} = L_{ab0} + L_2 \cos 2(\theta + \frac{\pi}{3}) \tag{10}$$

Note that, in this last set of expressions, we have assumed a particular form for the mutual inductances. This is seemingly restrictive, because it constrains the form of phase- to- phase mutual inductance variations with rotor position. The coefficient L_2 is actually the *same* in all six of these last expressions. As it turns out, this assumption does not really restrict the accuracy of the model very much. We will have more to say about this a bit later.

The rotor inductances are relatively simply stated:

$$\underline{\underline{L}}_R = \begin{bmatrix} L_f & L_{fkd} & 0 \\ L_{fkd} & L_{kd} & 0 \\ 0 & 0 & L_{kq} \end{bmatrix} \tag{11}$$

And the stator- to- rotor mutual inductances are:

$$\underline{\underline{M}} = \begin{bmatrix} M\cos\theta & L_{akd}\cos\theta & -L_{akq}\sin\theta \\ M\cos(\theta - \frac{2\pi}{3}) & L_{akd}\cos(\theta - \frac{2\pi}{3}) & -L_{akq}\sin(\theta - \frac{2\pi}{3}) \\ M\cos(\theta + \frac{2\pi}{3}) & L_{akd}\cos(\theta + \frac{2\pi}{3}) & -L_{akq}\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \tag{12}$$

3   Park's Equations

The first step in the development of a suitable model is to *transform* the armature winding variables to a coordinate system in which the rotor is stationary. We identify equivalent armature windings in the *direct* and *quadrature* axes. The *direct axis*armature winding is the equivalent of one of the phase windings, but aligned directly with the field. The *quadrature* winding is situated so that its axis *leads* the field winding by 90 *electrical* degrees. The transformation used to map the armature currents, fluxes and so forth onto the *direct* and *quadrature* axes is the celebrated *Park's Transformation* named after Robert H. Park, an early investigator into transient behavior in synchronous machines. The mapping takes the form:

$$\begin{bmatrix} u_d \\ u_q \\ u_0 \end{bmatrix} = \underline{u}_{dq} = \underline{\underline{T}}\underline{u}_{ph} = \underline{\underline{T}} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} \tag{13}$$

Where the transformation and its inverse are:

$$\underline{\underline{T}} = \frac{2}{3} \begin{bmatrix} \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin\theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{14}$$

$$\underline{\underline{T}}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix} \tag{15}$$

This transformation maps *balanced* sets of phase currents into *constant* currents in the d-q frame. That is, if rotor angle is $\theta = \omega t + \theta_0$, and phase currents are:

$$\begin{aligned} I_a &= I\cos\omega t \\ I_b &= I\cos(\omega t - \frac{2\pi}{3}) \\ I_c &= I\cos(\omega t + \frac{2\pi}{3}) \end{aligned}$$

Then the transformed set of currents is:

$$\begin{aligned} I_d &= I\cos\theta_0 \\ I_q &= -I\sin\theta_0 \end{aligned}$$

Now, we apply this transformation to (1) to express fluxes and currents in the armature in the d-q reference frame. To do this, extract the top line in (1):

$$\underline{\lambda}_{ph} = \underline{\underline{L}}_{ph}\underline{I}_{ph} + \underline{\underline{M}}\underline{I}_R \tag{16}$$

The transformed flux is obtained by premultiplying this whole expression by the transformation matrix. Phase current may be obtained from d-q current by multiplying by the inverse of the transformation matrix. Thus:

$$\underline{\lambda}_{dq} = \underline{\underline{T}}\underline{\underline{L}}_{ph}\underline{\underline{T}}^{-1}\underline{I}_{dq} + \underline{\underline{T}}\underline{\underline{M}}\underline{I}_R \tag{17}$$

The same process carried out for the lower line of (1) yields:

$$\underline{\lambda}_R = \underline{\underline{M}}^T\underline{\underline{T}}^{-1}\underline{I}_{dq} + \underline{\underline{L}}_R\underline{I}_R \tag{18}$$

Thus the fully transformed version of (1) is:

$$\begin{bmatrix} \underline{\lambda}_{dq} \\ \underline{\lambda}_R \end{bmatrix} = \begin{bmatrix} \underline{\underline{L}}_{dq} & \underline{\underline{L}}_C \\ \frac{3}{2}\underline{\underline{L}}_C^T & \underline{\underline{L}}_R \end{bmatrix} \begin{bmatrix} \underline{I}_{dq} \\ \underline{I}_R \end{bmatrix} \tag{19}$$

If the conditions of (5) through (10) are satisfied, the inductance submatrices of (19) wind up being of particularly simple form. (Please note that a substantial amount of algebra has been left out here!)

$$\underline{\underline{L}}_{dq} = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & L_0 \end{bmatrix} \tag{20}$$

$$\underline{\underline{L}}_C = \begin{bmatrix} M & L_{akd} & 0 \\ 0 & 0 & L_{akq} \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

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Note that (19) through (21) express three *separate* sets of apparently independent flux/current relationships. These may be re-cast into the following form:

$$\begin{bmatrix} \lambda_d \\ \lambda_{kd} \\ \lambda_f \end{bmatrix} = \begin{bmatrix} L_d & L_{akd} & M \\ \frac{3}{2}L_{akd} & L_{kd} & L_{fkd} \\ \frac{3}{2}M & L_{fkd} & L_f \end{bmatrix} \begin{bmatrix} I_d \\ I_{kd} \\ I_f \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} \lambda_q \\ \lambda_{kq} \end{bmatrix} = \begin{bmatrix} L_q & L_{akq} \\ \frac{3}{2}L_{akq} & L_{kq} \end{bmatrix} \begin{bmatrix} I_q \\ I_{kq} \end{bmatrix} \tag{23}$$

$$\lambda_0 = L_0 I_0 \tag{24}$$

Where the component inductances are:

$$L_d = L_{a0} - L_{ab0} + \frac{3}{2}L_2 \tag{25}$$

$$L_q = L_{a0} - L_{ab0} - \frac{3}{2}L_2 \tag{26}$$

$$L_0 = L_{a0} + 2L_{ab0} \tag{27}$$

Note that the apparently restrictive assumptions embedded in (5) through (10) have resulted in the very simple form of (21) through (24). In particular, we have three mutually independent sets of fluxes and currents. While we may be concerned about the restrictiveness of these expressions, note that the orthogonality between the d- and q- axes is not unreasonable. In fact, because these axes are orthogonal in *space*, it seems reasonable that they should not have mutual flux linkages. The principal consequence of these assumptions is the de-coupling of the *zero-sequence* component of flux from the d- and q- axis components. We are not in a position at this time to determine the reasonableness of this. However, it should be noted that departures from this form (that is, coupling between the "direct" and "zero" axes) must be through higher harmonic fields that will not couple well to the armature, so that any such coupling will be weak.

Next, armature voltage is, ignoring resistance, given by:

$$\underline{V}_{ph} = \frac{d}{dt}\underline{\lambda}_{ph} = \frac{d}{dt}\underline{\underline{T}}^{-1}\underline{\lambda}_{dq} \tag{28}$$

and that the *transformed* armature voltage must be:

$$\begin{aligned} \underline{V}_{dq} &= \underline{\underline{T}}\,\underline{V}_{ph} \\ &= \underline{\underline{T}}\frac{d}{dt}(\underline{\underline{T}}^{-1}\underline{\lambda}_{dq}) \\ &= \frac{d}{dt}\underline{\lambda}_{dq} + (\underline{\underline{T}}\frac{d}{dt}\underline{\underline{T}}^{-1})\underline{\lambda}_{dq} \end{aligned} \tag{29}$$

A good deal of manupulation goes into reducing the second term of this, resulting in:

$$\underline{\underline{T}}\frac{d}{dt}\underline{\underline{T}}^{-1} = \begin{bmatrix} 0 & -\frac{d\theta}{dt} & 0 \\ \frac{d\theta}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{30}$$

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This expresses the *speed voltage* that arises from a coordinate transformation. The two voltage/flux relationships that are affected are:

$$V_d = \frac{d\lambda_d}{dt} - \omega\lambda_q \tag{31}$$

$$V_q = \frac{d\lambda_q}{dt} + \omega\lambda_d \tag{32}$$

where we have used

$$\omega = \frac{d\theta}{dt} \tag{33}$$

4   Power and Torque

Instantaneous *power* is given by:

$$P = V_a I_a + V_b I_b + V_c I_c \tag{34}$$

Using the transformations given above, this can be shown to be:

$$P = \frac{3}{2}V_d I_d + \frac{3}{2}V_q I_q + 3V_0 I_0 \tag{35}$$

which, in turn, is:

$$P = \omega\frac{3}{2}(\lambda_d I_q - \lambda_q I_d) + \frac{3}{2}(\frac{d\lambda_d}{dt}I_d + \frac{d\lambda_q}{dt}I_q) + 3\frac{d\lambda_0}{dt}I_0 \tag{36}$$

Then, noting that *electrical* speed ω and shaft speed Ω are related by $\omega = p\Omega$ and that (36) describes electrical terminal power as the sum of shaft power and rate of change of stored energy, we may deduce that torque is given by:

$$T = \frac{3}{2}p(\lambda_d I_q - \lambda_q I_d) \tag{37}$$

5   Per-Unit Normalization

The next thing for us to do is to investigate the way in which electric machine system are *normalized*, or put into what is called a *per-unit* system. The reason for this step is that, when the voltage, current, power and impedance are referred to normal operating parameters, the behavior characteristics of all types of machines become quite similar, giving us a better way of relating how a particular machine works to some reasonable standard. There are also numerical reasons for normalizing performance parameters to some standard.

The first step in normalization is to establish a set of *base* quantities. We will be normalizing voltage, current, flux, power, impedance and torque, so we will need base quantities for each of these. Note, however, that the base quantities are *not* independent. In fact, for the armature, we need only specify three quantities: voltage (V_B), current (I_B) and frequency (ω_0). Note that we do *not* normalize time nor frequency. Having done this for the armature circuits, we can derive each of the other base quantities:

- Base Power

$$P_B = \frac{3}{2} V_B I_B$$

- Base Impedance

$$Z_B = \frac{V_B}{I_B}$$

- Base Flux

$$\lambda_B = \frac{V_B}{\omega_0}$$

- Base Torque

$$T_B = \frac{p}{\omega_0} P_B$$

Note that, for our purposes, base *voltage* and *current* are expressed as *peak* quantities. Base voltage is taken on a phase basis (line to neutral for a "wye" connected machine), and base current is similarly taken on a phase basis, (line current for a "wye" connected machine).

Normalized, or *per-unit* quantities are derived by dividing the *ordinary* variable (with units) by the corresponding *base*. For example, per-unit flux is:

$$\psi = \frac{\lambda}{\lambda_B} = \frac{\omega_0 \lambda}{V_B} \tag{38}$$

In this derivation, per- unit quantities will usually be designated by lower case letters. Two notable exceptions are flux, where we use the letter ψ, and torque, where we will still use the upper case T and risk confusion.

Now, we note that there will be *base* quantities for voltage, current and frequency for each of the different coils represented in our model. While it is reasonable to expect that the *frequency* base will be the same for all coils in a problem, the *voltage* and *current* bases may be different. We might write (22) as:

$$\begin{bmatrix} \psi_d \\ \psi_{kd} \\ \psi_f \end{bmatrix} = \begin{bmatrix} \frac{\omega_0 I_{dB}}{V_{db}} L_d & \frac{\omega_0 I_{kB}}{V_{db}} L_{akd} & \frac{\omega_0 I_{fB}}{V_{db}} M \\ \frac{\omega_0 I_{dB}}{V_{kb}} \frac{3}{2} L_{akd} & \frac{\omega_0 I_{kB}}{V_{kb}} L_{kd} & \frac{\omega_0 I_{fB}}{V_{kdb}} L_{fkd} \\ \frac{\omega_0 I_{dB}}{V_{fb}} \frac{3}{2} M & \frac{\omega_0 I_{kB}}{V_{fb}} L_{fkd} & \frac{\omega_0 I_{fB}}{V_{fb}} L_f \end{bmatrix} \begin{bmatrix} i_d \\ i_{kd} \\ i_f \end{bmatrix} \tag{39}$$

where $i = I/I_B$ denotes *per-unit*, or normalized current.

Note that (39) may be written in simple form:

$$\begin{bmatrix} \psi_d \\ \psi_{kd} \\ \psi_f \end{bmatrix} = \begin{bmatrix} x_d & x_{akd} & x_{ad} \\ x_{akd} & x_{kd} & x_{fkd} \\ x_{ad} & x_{fkd} & x_f \end{bmatrix} \begin{bmatrix} i_d \\ i_{kd} \\ i_f \end{bmatrix} \tag{40}$$

It is important to note that (40) *assumes* reciprocity in the normalized system. To wit, the following expressions are implied:

$$x_d = \omega_0 \frac{I_{dB}}{V_{dB}} L_d \tag{41}$$

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$$x_{kd} = \omega_0 \frac{I_{kB}}{V_{kB}} L_{kd} \tag{42}$$

$$x_f = \omega_0 \frac{I_{fB}}{V_{fB}} L_f \tag{43}$$

$$x_{akd} = \omega_0 \frac{I_{kB}}{V_{dB}} L_{akd}$$
$$= \frac{3}{2} \omega_0 \frac{I_{dB}}{V_{kB}} L_{akd} \tag{44}$$

$$x_{ad} = \omega_0 \frac{I_{fB}}{V_{dB}} M$$
$$= \frac{3}{2} \omega_0 \frac{I_{dB}}{V_{fB}} M \tag{45}$$

$$x_{fkd} = \omega_0 \frac{I_{kB}}{V_{fb}} L_{fkd}$$
$$= \omega_0 \frac{I_{fB}}{V_{kb}} L_{fkd} \tag{46}$$

These in turn imply:

$$\frac{3}{2} V_{dB} I_{dB} = V_{fB} I_{fB} \tag{47}$$

$$\frac{3}{2} V_{dB} I_{dB} = V_{kB} I_{kB} \tag{48}$$

$$V_{fB} I_{fB} = V_{kB} I_{kB} \tag{49}$$

These expressions imply the same *power* base on all of the windings of the machine. This is so because the *armature* base quantities V_{db} and I_{db} are stated as *peak* values, while the *rotor* base quantities are stated as *DC* values. Thus power base for the *three- phase* armature is $\frac{3}{2}$ times the product of *peak* quantities, while the power base for the rotor is simply the product of those quantities.

The quadrature axis, which may have fewer equivalent elements than the direct axis and which may have different numerical values, still yields a similar structure. Without going through the details, we can see that the per-unit flux/current relationship for the q- axis is:

$$\begin{bmatrix} \psi_q \\ \psi_{kq} \end{bmatrix} = \begin{bmatrix} x_q & x_{akq} \\ x_{akq} & x_{kq} \end{bmatrix} \begin{bmatrix} i_q \\ i_{kq} \end{bmatrix} \tag{50}$$

The voltage equations, including speed voltage terms, (31) and (32), may be augmented to reflect armature resistance:

$$V_d = \frac{d\lambda_d}{dt} - \omega\lambda_q + R_a I_d \tag{51}$$

$$V_q = \omega\lambda_d + \frac{d\lambda_q}{dt} + R_a I_q \tag{52}$$

The *per-unit* equivalents of these are:

$$v_d = \frac{1}{\omega_0} \frac{d\psi_d}{dt} - \frac{\omega}{\omega_0} \psi_q + r_a i_d \tag{53}$$

$$v_q = \frac{\omega}{\omega_0}\psi_d + \frac{1}{\omega_0}\frac{d\psi_q}{dt} + r_a i_q \tag{54}$$

Where the per-unit armature resistance is just $r_a = \frac{R_a}{Z_B}$

Note that none of the other circuits in this model have *speed voltage* terms, so their voltage expressions are exactly what we might expect:

$$v_f = \frac{1}{\omega_0}\frac{d\psi_f}{dt} + r_f i_f \tag{55}$$

$$v_{kd} = \frac{1}{\omega_0}\frac{d\psi_{kd}}{dt} + r_{kd} i_{kd} \tag{56}$$

$$v_{kq} = \frac{1}{\omega_0}\frac{d\psi_{kq}}{dt} + r_{kq} i_{kq} \tag{57}$$

$$v_0 = \frac{1}{\omega_0}\frac{d\psi_0}{dt} + r_a i_0 \tag{58}$$

It should be noted that the *damper* winding circuits represent closed conducting paths on the rotor, so the two voltages v_{kd} and v_{kq} are always zero.

Per-unit torque is simply:

$$T_e = \psi_d i_q - \psi_q i_d \tag{59}$$

Often, we need to represent the dynamic behavior of the machine, including electromechanical dynamics involving rotor inertia. If we note J as the rotational inertia constant of the machine system, the rotor dynamics are described by the two ordinary differential equations:

$$\frac{1}{p}J\frac{d\omega}{dt} = T^e + T^m \tag{60}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{61}$$

where T^e and T^m represent *electrical* and *mechanical* torques in "ordinary" variables. The angle δ represents rotor phase angle with respect to some synchronous reference.

It is customary to define an "inertia constant" which is not dimensionless but which nevertheless fits into the per-unit system of analysis. This is:

$$H \equiv \frac{\text{Rotational kinetic energy at rated speed}}{\text{Base Power}} \tag{62}$$

Or:

$$H = \frac{\frac{1}{2}J\left(\frac{\omega_0}{p}\right)^2}{P_B} = \frac{J\omega_0}{2pT_B} \tag{63}$$

Then the per-unit equivalent to (60) is:

$$\frac{2H}{\omega_0}\frac{d\omega}{dt} = T_e + T_m \tag{64}$$

where now we use T_e and T_m to represent *per-unit* torques.

6   Equal Mutual's Base

In normalizing the differential equations that make up our model, we have used a number of *base quantities*. For example, in deriving (40), the *per-unit* flux- current relationship for the *direct* axis, we used six base quantities: V_B, I_B, V_{fB}, I_{fB}, V_{kB} and I_{kB}. Imposing reciprocity on (40) results in two constraints on these six variables, expressed in (47) through (49). Presumably the two armature base quantities will be fixed by machine rating. That leaves two more "degrees of freedom" in selection of base quantities. Note that the selection of base quantities will affect the reactance matrix in (40).

While there are different schools of thought on just how to handle these degrees of freedom, a commonly used convention is to employ what is called the *equal mutuals* base system. The two degrees of freedom are used to set the field and damper base impedances so that all three mutual inductances of (40) are equal:

$$x_{akd} = x_{fkd} = x_{ad} \tag{65}$$

The direct- axis flux- current relationship becomes:

$$\begin{bmatrix} \psi_d \\ \psi_{kd} \\ \psi_f \end{bmatrix} = \begin{bmatrix} x_d & x_{ad} & x_{ad} \\ x_{ad} & x_{kd} & x_{ad} \\ x_{ad} & x_{ad} & x_f \end{bmatrix} \begin{bmatrix} i_d \\ i_{kd} \\ i_f \end{bmatrix} \tag{66}$$
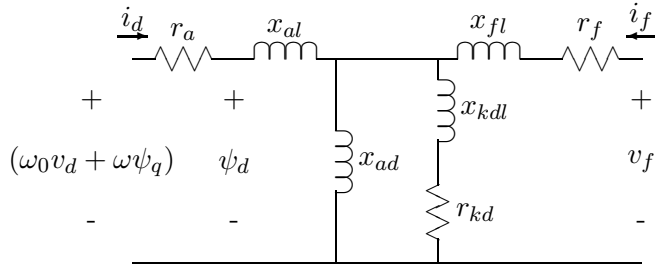
7   Equivalent Circuit

Figure 1: D- Axis Equivalent Circuit

The flux- current relationship of (66) is represented by the equivalent circuit of Figure 1, if the "leakage" inductances are defined to be:

$$x_{al} = x_d - x_{ad} \tag{67}$$

$$x_{kdl} = x_{kd} - x_{ad} \tag{68}$$

$$x_{fl} = x_f - x_{ad} \tag{69}$$

Many of the interesting features of the electrical dynamics of the synchronous machine may be discerned from this circuit. While a complete explication of this thing is beyond the scope of this note, it is possible to make a few observations.

The apparent inductance measured from the terminals of this equivalent circuit (ignoring resistance r_a) will, in the frequency domain, be of the form:

$$x(s) = \frac{\psi_d(s)}{i_d(s)} = x_d \frac{P_n(s)}{P_d(s)} \tag{70}$$

Both the numerator and denominator polynomials in s will be second order. (You may convince yourself of this by writing an expression for terminal impedance). Since this is a "diffusion" type circuit, having only resistances and inductances, all poles and zeros must be on the negative real axis of the "s-plane". The per-unit inductance is, then:

$$x(s) = x_d \frac{(1 + T'_d s)(1 + T''_d s)}{(1 + T'_{do} s)(1 + T''_{do} s)} \tag{71}$$

The two time constants T'_d and T''_d are the reciprocals of the *zeros* of the impedance, which are the *poles* of the admittance. These are called the *short circuit* time constants.

The other two time constants T'_{do} and T''_{do} are the reciprocals of the *poles* of the impedance, and so are called the *open circuit* time constants.

We have cast this thing as if there are two sets of well- defined time constants. These are the *transient* time constants T'_d and T'_{do}, and the *subtransient* time constants T''_d and T''_{do}. In many cases, these are indeed well separated, meaning that:

$$T'_d \gg T''_d \tag{72}$$
$$T'_{do} \gg T''_{do} \tag{73}$$

If this is true, then the reactance is described by the pole-zero diagram shown in Figure 2. Under this circumstance, the apparent terminal inductance has three distinct values, depending on frequency. These are the *synchronous* inductance, the *transient* inductance, and the *subtransient* inductance, given by:

$$x'_d = x_d \frac{T'_d}{T'_{do}} \tag{74}$$
$$x''_d = x'_d \frac{T''_d}{T''_{do}}$$
$$= x_d \frac{T'_d}{T'_{do}} \frac{T''_d}{T''_{do}} \tag{75}$$

A *Bode Plot* of the terminal reactance is shown in Figure 3.

If the time constants are spread widely apart, they are given, approximately, by:

$$T'_{do} = \frac{x_f}{\omega_0 r_f} \tag{76}$$
$$T''_{do} = \frac{x_{kdl} + x_{fl} || x_{ad}}{\omega_0 r_{kd}} \tag{77}$$
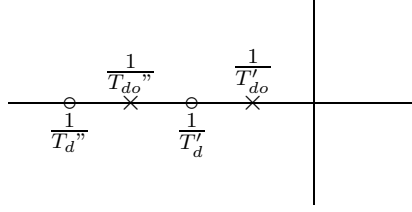
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Figure 2: Pole-Zero Diagram For Terminal Inductance

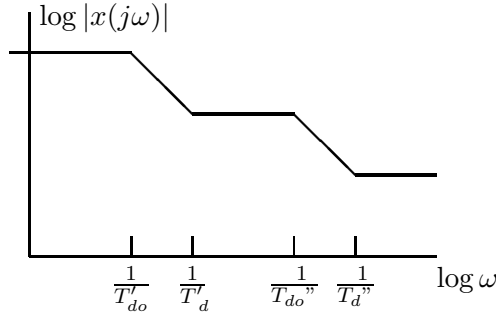

Figure 3: Frequency Response of Terminal Inductance

Finally, note that the three reactances are found simply from the model:

$$
\begin{align}
x_d &= x_{al} + x_{ad} \tag{78}\\
x_d' &= x_{al} + x_{ad}||x_{fl} \tag{79}\\
x_d'' &= x_{al} + x_{ad}||x_{fl}||x_{kdl} \tag{80}
\end{align}
$$

8 Statement of Simulation Model

Now we can write down the simulation model. Actually, we will derive more than one of these, since the machine can be driven by either voltages or currents. Further, the expressions for permanent magnet machines are a bit different. So the first model is one in which the terminals are all constrained by *voltage*.

The state variables are the two stator fluxes ψ_d, ψ_q, two "damper" fluxes ψ_{kd}, ψ_{kq}, field flux ψ_f, and rotor speed ω and torque angle δ. The most straightforward way of stating the model employs currents as auxiliary variables, and these are:

$$
\begin{bmatrix} i_d \\ i_{kd} \\ i_f \end{bmatrix} = \begin{bmatrix} x_d & x_{ad} & x_{ad} \\ x_{ad} & x_{kd} & x_{ad} \\ x_{ad} & x_{ad} & x_f \end{bmatrix}^{-1} \begin{bmatrix} \psi_d \\ \psi_{kd} \\ \psi_f \end{bmatrix} \tag{81}
$$

$$
\begin{bmatrix} i_q \\ i_{kq} \end{bmatrix} = \begin{bmatrix} x_q & x_{aq} \\ x_{aq} & x_{kq} \end{bmatrix}^{-1} \begin{bmatrix} \psi_q \\ \psi_{kq} \end{bmatrix} \tag{82}
$$

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Then the state equations are:

$$\frac{d\psi_d}{dt} = \omega_0 v_d + \omega\psi_q - \omega_0 r_a i_d \tag{83}$$

$$\frac{d\psi_q}{dt} = \omega_0 v_q - \omega\psi_d - \omega_0 r_a i_q \tag{84}$$

$$\frac{d\psi_{kd}}{dt} = -\omega_0 r_{kd} i_{kd} \tag{85}$$

$$\frac{d\psi_{kq}}{dt} = -\omega_0 r_{kq} i_{kq} \tag{86}$$

$$\frac{d\psi_f}{dt} = \omega_0 v_f - \omega_0 r_f i_f \tag{87}$$

$$\frac{d\omega}{dt} = \frac{\omega_0}{2H}(T_e + T_m) \tag{88}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{89}$$

and, of course,

$$T_e = \psi_d i_q - \psi_q i_d$$

8.1 Statement of Parameters:

Note that often data for a machine may be given in terms of the reactances x_d, x'_d, x''_d, T'_{do} and T''_{do}, rather than the elements of the equivalent circuit model. Note that there are four inductances in the equivalent circuit so we have to assume one. There is no loss in generality in doing so. Usually one assumes a value for the stator leakage inductance, and if this is done the translation is straightforward:

$$x_{ad} = x_d - x_{al}$$

$$x_{fl} = \frac{x_{ad}(x'_d - x_{al})}{x_{ad} - x'_d + x_{al}}$$

$$x_{kdl} = \frac{1}{\frac{1}{x''_d - x_{al}} - \frac{1}{x_{ad}} - \frac{1}{x_{fl}}}$$

$$r_f = \frac{x_{fl} + x_{ad}}{\omega_0 T'_{do}}$$

$$r_{kd} = \frac{x_{kdl} + x_{ad}||x_{fl}}{\omega_0 T''_{do}}$$

8.2 Linearized Model

Often it becomes desirable to carry out a linearized analysis of machine operation to, for example, examine the damping of the swing mode at a particular operating point. What is done, then, is to assume a steady state operating point and examine the dynamics for deviations from that operating point that are "small". The definition of "small" is really "small enough" that everything important appears in the first-order term of a Taylor series about the steady operating point.

Note that the expressions in the machine model are, for the most part, linear. There are, however, a few cases in which products of state variables cause us to do the expansion of the

Taylor series. Assuming a steady state operating point $[\psi_{d0} \ \psi_{kd0} \ \psi_{f0} \ \psi_{q0} \ \psi_{kq0} \ \omega_0 \ \delta_0]$, the first-order (small-signal) variations are described by the following set of equations. First, since the flux-current relationship is linear:

$$\begin{bmatrix} i_{d1} \\ i_{kd1} \\ i_{f1} \end{bmatrix} = \begin{bmatrix} x_d & x_{ad} & x_{ad} \\ x_{ad} & x_{kd} & x_{ad} \\ x_{ad} & x_{ad} & x_f \end{bmatrix}^{-1} \begin{bmatrix} \psi_{d1} \\ \psi_{kd1} \\ \psi_{f1} \end{bmatrix} \tag{90}$$

$$\begin{bmatrix} i_{q1} \\ i_{kq1} \end{bmatrix} = \begin{bmatrix} x_q & x_{aq} \\ x_{aq} & x_{kq} \end{bmatrix}^{-1} \begin{bmatrix} \psi_{q1} \\ \psi_{kq1} \end{bmatrix} \tag{91}$$

Terminal voltage will be, for operation against a voltage source:

$$V_d = V \sin \delta \quad V_q = V \cos \delta$$

Then the differential equations governing the first-order variations are:

$$\frac{d\psi_{d1}}{dt} = \omega_0 V \cos \delta_0 \delta_1 + \omega_0 \psi_{q1} + \omega_1 \psi_{q0} - \omega_0 r_a i_{d1} \tag{92}$$

$$\frac{d\psi_{q1}}{dt} = -\omega_0 V \sin \delta_0 \delta_1 - \omega_0 \psi_{d1} - \omega_1 \psi_{d0} - \omega_0 r_a i_{q1} \tag{93}$$

$$\frac{d\psi_{kd1}}{dt} = -\omega_0 r_{kd} i_{kd1} \tag{94}$$

$$\frac{d\psi_{kq1}}{dt} = -\omega_0 r_{kq} i_{kq1} \tag{95}$$

$$\frac{d\psi_{f1}}{dt} = -\omega_0 r_f i_{f1} \tag{96}$$

$$\frac{d\omega_1}{dt} = \frac{\omega_0}{2H} (T_{e1} + T_{m1}) \tag{97}$$

$$\frac{d\delta_1}{dt} = \omega_1 \tag{98}$$

$$T_e = \psi_{d0} i_{q1} + \psi_{d1} i_{q0} - \psi_{q0} i_{d1} - \psi_{q1} i_{d0}$$

8.3 Reduced Order Model for Electromechanical Transients

In many situations the two armature variables contribute little to the dynamic response of the machine. Typically the armature resistance is small enough that there is very little voltage drop across it and transients in the difference between armature flux and the flux that would exist in the "steady state" decay rapidly (or are not even excited). Further, the relatively short armature time constant makes for very short time steps. For this reason it is often convenient, particularly when studying the relatively slow electromechanical transients, to omit the first two differential equations and set:

$$\psi_d = v_q = V \cos \delta \tag{99}$$

$$\psi_q = -v_d = -V \sin \delta \tag{100}$$

The set of differential equations changes only a little when this approximation is made. Note, however, that it can be simulated with far fewer "cycles" if the armature time constant is short.

9 Current Driven Model: Connection to a System

The simulation expressions developed so far are useful in a variety of circumstances. They are, however, difficult to tie to network simulation programs because they use terminal voltage as an input. Generally, it is more convenient to use *current* as the input to the machine simulation and accept *voltage* as the output. Further, it is difficult to handle unbalanced situations with this set of equations.

An alternative to this set would be to employ the *phase* currents as state variables. Effectively, this replaces ψ_d, ψ_q and ψ_0 with i_a, i_b, and i_c. The resulting model will, as we will show, interface nicely with network simulations.

To start, note that we could write an expression for terminal flux, on the d- axis:

$$\psi_d = x_d'' i_d + \psi_f \frac{x_{ad}||x_{kdl}}{x_{ad}||x_{kdl} + x_{fl}} + \psi_{kd} \frac{x_{ad}||x_{fl}}{x_{ad}||x_{fl} + x_{kdl}} \tag{101}$$

and here, of course,

$$x_d'' = x_{al} + x_{ad}||x_{kdl}||x_{fl}$$

This leads us to define a "flux behind subtransient reactance":

$$\psi_d'' = \frac{x_{ad}x_{kdl}\psi_f + x_{ad}x_{fl}\psi_{kd}}{x_{ad}x_{kdl} + x_{ad}x_{fl} + x_{kdl}x_{fl}} \tag{102}$$

So that

$$\psi_d = \psi_d'' + x_d'' i_d$$

On the quadrature axis the situation is essentially the same, but one step easier if there is only one quadrature axis rotor winding:

$$\psi_q = x_q'' i_q + \psi_{kq} \frac{x_{aq}}{x_{aq} + x_{kql}} \tag{103}$$

where

$$x_q'' = x_{al} + x_{aq}||x_{kql}$$

Very often these fluxes are referred to as "voltage behind subtransient reactance, with $\psi_d'' = e_q''$ and $\psi_q'' = -e_d''$. Then:

$$\psi_d = x_d'' i_d + e_q'' \tag{104}$$
$$\psi_q = x_q'' i_q - e_d'' \tag{105}$$

Now, if i_d and i_q are determined, it is a bit easier to find the other currents required in the simulation. Note we can write:

$$\begin{bmatrix} \psi_{kd} \\ \psi_f \end{bmatrix} = \begin{bmatrix} x_{kd} & x_{ad} \\ x_{ad} & x_f \end{bmatrix} \begin{bmatrix} i_{kd} \\ i_f \end{bmatrix} + \begin{bmatrix} x_{ad} \\ x_{ad} \end{bmatrix} i_d \tag{106}$$

and this inverts easily:

$$\begin{bmatrix} i_{kd} \\ i_f \end{bmatrix} = \begin{bmatrix} x_{kd} & x_{ad} \\ x_{ad} & x_f \end{bmatrix}^{-1} \left(\begin{bmatrix} \psi_{kd} \\ \psi_f \end{bmatrix} - \begin{bmatrix} x_{ad} \\ x_{ad} \end{bmatrix} i_d \right) \tag{107}$$

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The quadrature axis rotor current is simply:

$$i_{kq} = \frac{1}{x_{kq}}\psi_{kq} - \frac{x_{aq}}{x_{kq}}i_q \tag{108}$$

The torque equation is the same, but since it is usually convenient to assemble the fluxes behind subtransient reactance, it is possible to use:

$$T_e = e''_q i_q + e''_d i_d + (x''_d - x''_q)i_d i_q \tag{109}$$

Now it is necessary to consider terminal voltage. This is most conveniently cast in matrix notation. The vector of phase voltages is:

$$\underline{v}_{ph} = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \tag{110}$$

Then, with similar notation for phase flux, terminal voltage is, ignoring armature resistance:

$$\begin{aligned}
\underline{v}_{ph} &= \frac{1}{\omega_0}\frac{d\underline{\psi}_{ph}}{dt} \\
&= \frac{1}{\omega_0}\frac{d}{dt}\left\{\underline{\underline{T}}^{-1}\underline{\psi}_{dq}\right\}
\end{aligned} \tag{111}$$

Note that we may define the transformed vector of fluxes to be:

$$\underline{\psi}_{dq} = \underline{\underline{x}}''\underline{i}_{dq} + \underline{e}'' \tag{112}$$

where the matrix of reactances shows orthogonality:

$$\underline{\underline{x}}'' = \begin{bmatrix} x''_d & 0 & 0 \\ 0 & x''_q & 0 \\ 0 & 0 & x_0 \end{bmatrix} \tag{113}$$

and the vector of internal fluxes is:

$$\underline{e}'' = \begin{bmatrix} e''_q \\ -e''_d \\ 0 \end{bmatrix} \tag{114}$$

Now, of course, $\underline{i}_{dq} = \underline{\underline{T}}\underline{i}_{ph}$, so that we may re-cast (111) as:

$$\underline{v}_{ph} = \frac{1}{\omega_0}\frac{d}{dt}\left\{\underline{\underline{T}}^{-1}\underline{\underline{x}}''\underline{\underline{T}}\underline{i}_{ph} + \underline{\underline{T}}^{-1}\underline{e}''\right\} \tag{115}$$

Now it is necessary to make one assumption and one definition. The assumption, which is only moderately restrictive, is that subtransient saliency may be ignored. That is, we assume that $x''_d = x''_q$. The definition separates the "zero sequence" impedance into phase and neutral components:

$$x_0 = x_d'' + 3x_g \tag{116}$$

Note that according to this definition the reactance x_g accounts for any impedance in the neutral of the synchronous machine as well as mutual coupling between phases.

Then, the impedance matrix becomes:

$$\underline{\underline{x}}'' = \begin{bmatrix} x_d'' & 0 & 0 \\ 0 & x_d'' & 0 \\ 0 & 0 & x_d'' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3x_g \end{bmatrix} \tag{117}$$

In compact notation, this is:

$$\underline{\underline{x}}'' = x_d''\underline{\underline{I}} + \underline{\underline{x}}_g \tag{118}$$

where $\underline{\underline{I}}$ is the identity matrix.

Now the vector of phase voltages is:

$$\underline{v}_{ph} = \frac{1}{\omega_0}\frac{d}{dt}\left\{x_d''\underline{i}_{ph} + \underline{\underline{T}}^{-1}\underline{\underline{x}}_g\underline{\underline{T}}\underline{i}_{ph} + \underline{\underline{T}}^{-1}\underline{e}''\right\} \tag{119}$$

Note that in (119), we have already factored out the multiplication by the identity matrix. The next step is to carry out the matrix multiplication in the third term of (119). This operation turns out to produce a remarkably simple result:

$$\underline{\underline{T}}^{-1}\underline{\underline{x}}_g\underline{\underline{T}} = x_g \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{120}$$

The impact of this is that each of the three phase voltages has the same term, and that is related to the time derivative of the sum of the three currents, multiplied by x_g.

The third and final term in (119) describes voltages induced by rotor fluxes. It can be written as:

$$\frac{1}{\omega_0}\frac{d}{dt}\left\{\underline{\underline{T}}^{-1}\underline{e}''\right\} = \frac{1}{\omega_0}\frac{d}{dt}\left\{\underline{\underline{T}}^{-1}\right\}\underline{e}'' + \frac{1}{\omega_0}\underline{\underline{T}}^{-1}\frac{d\underline{e}''}{dt} \tag{121}$$

Now, the time derivative of the inverse transform is:

$$\frac{1}{\omega_0}\frac{d}{dt}\underline{\underline{T}}^{-1} = \frac{\omega}{\omega_0}\begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ -\sin(\theta - \frac{2\pi}{3}) & -\cos(\theta - \frac{2\pi}{3}) & 0 \\ -\sin(\theta + \frac{2\pi}{3}) & -\cos(\theta + \frac{2\pi}{3}) & 0 \end{bmatrix} \tag{122}$$

Now the three phase voltages can be extracted from all of this matrix algebra:

$$v_a = \frac{x_d''}{\omega_0}\frac{di_a}{dt} + \frac{x_g}{\omega_0}\frac{d}{dt}(i_a + i_b + i_c) + e_a'' \tag{123}$$

$$v_b = \frac{x_d''}{\omega_0}\frac{di_b}{dt} + \frac{x_g}{\omega_0}\frac{d}{dt}(i_a + i_b + i_c) + e_b'' \tag{124}$$

$$v_c = \frac{x_d''}{\omega_0}\frac{di_c}{dt} + \frac{x_g}{\omega_0}\frac{d}{dt}(i_a + i_b + i_c) + e_c'' \tag{125}$$

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Where the internal voltages are:

$$e''_a = -\frac{\omega}{\omega_0}(e''_q \sin(\theta) - e''_d \cos(\theta))$$

$$+\frac{1}{\omega_0}\cos(\theta)\frac{de''_q}{dt} + \frac{1}{\omega_0}\sin(\theta)\frac{de''_d}{dt} \quad (126)$$

$$e''_b = -\frac{\omega}{\omega_0}(e''_q \sin(\theta - \frac{2\pi}{3}) - e''_d \cos(\theta - \frac{2\pi}{3}))$$

$$+\frac{1}{\omega_0}\cos(\theta - \frac{2\pi}{3})\frac{de''_q}{dt} + \frac{1}{\omega_0}\sin(\theta - \frac{2\pi}{3})\frac{de''_d}{dt} \quad (127)$$

$$e''_c = -\frac{\omega}{\omega_0}(e''_q \sin(\theta + \frac{2\pi}{3}) - e''_d \cos(\theta + \frac{2\pi}{3}))$$

$$+\frac{1}{\omega_0}\cos(\theta + \frac{2\pi}{3})\frac{de''_q}{dt} + \frac{1}{\omega_0}\sin(\theta + \frac{2\pi}{3})\frac{de''_d}{dt} \quad (128)$$

This set of expressions describes the equivalent circuit shown in Figure 4.
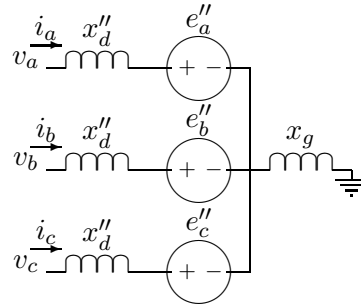

Figure 4: Equivalent Network Model

10   Restatement Of The Model

The synchronous machine model which uses the three phase currents as state variables may now be stated in the form of a set of differential and algebraic equations:

$$\frac{d\psi_{kd}}{dt} = -\omega_0 r_{kd} i_{kd} \quad (129)$$

$$\frac{d\psi_{kq}}{dt} = -\omega_0 r_{kq} i_{kq} \quad (130)$$

$$\frac{d\psi_f}{dt} = -\omega_0 r_f i_f \quad (131)$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \quad (132)$$

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$$\frac{d\omega}{dt} \quad = \quad \frac{\omega_0}{2H}\left(T_m + e_q'' i_q + e_d'' i_d\right) \tag{133}$$

where:

$$\begin{bmatrix} i_{kd} \\ i_f \end{bmatrix} = \begin{bmatrix} x_{kd} & x_{ad} \\ x_{ad} & x_f \end{bmatrix}^{-1} \left(\begin{bmatrix} \psi_{kd} \\ \psi_f \end{bmatrix} - \begin{bmatrix} x_{ad} \\ x_{ad} \end{bmatrix} i_d \right)$$

and

$$i_{kq} = \frac{1}{x_{kq}}\psi_{kq} - \frac{x_{aq}}{x_{kq}} i_q$$

(It is assumed here that the difference between subtransient reactances is small enough to be neglected.)

The network interface equations are, from the network to the machine:

$$i_d = i_a \cos(\theta) + i_b \cos(\theta - \frac{2\pi}{3}) + i_c \cos(\theta + \frac{2\pi}{3}) \tag{134}$$

$$i_q = -i_a \sin(\theta) - i_b \sin(\theta - \frac{2\pi}{3}) - i_c \sin(\theta + \frac{2\pi}{3}) \tag{135}$$

and, in the reverse direction, from the machine to the network:

$$e_a'' \quad = \quad -\frac{\omega}{\omega_0}(e_q'' \sin(\theta) - e_d'' \cos(\theta))$$

$$+ \frac{1}{\omega_0} \cos(\theta)\frac{de_q''}{dt} + \frac{1}{\omega_0} \sin(\theta)\frac{de_d''}{dt} \tag{136}$$

$$e_b'' \quad = \quad -\frac{\omega}{\omega_0}(e_q'' \sin(\theta - \frac{2\pi}{3}) - e_d'' \cos(\theta - \frac{2\pi}{3}))$$

$$+ \frac{1}{\omega_0} \cos(\theta - \frac{2\pi}{3})\frac{de_q''}{dt} + \frac{1}{\omega_0} \sin(\theta - \frac{2\pi}{3})\frac{de_d''}{dt} \tag{137}$$

$$e_c'' \quad = \quad -\frac{\omega}{\omega_0}(e_q'' \sin(\theta + \frac{2\pi}{3}) - e_d'' \cos(\theta + \frac{2\pi}{3}))$$

$$+ \frac{1}{\omega_0} \cos(\theta + \frac{2\pi}{3})\frac{de_q''}{dt} + \frac{1}{\omega_0} \sin(\theta + \frac{2\pi}{3})\frac{de_d''}{dt} \tag{138}$$

And, of course,

$$\theta \quad = \quad \omega_0 t + \delta \tag{139}$$

$$e_q'' \quad = \quad \psi_d'' \tag{140}$$

$$e_d'' \quad = \quad -\psi_q'' \tag{141}$$

$$\psi_d'' \quad = \quad \frac{x_{ad}x_{kdl}\psi_f + x_{ad}x_{fl}\psi_{kd}}{x_{ad}x_{kdl} + x_{ad}x_{fl} + x_{kdl}x_{fl}} \tag{142}$$

$$\psi_q'' \quad = \quad \frac{x_{aq}}{x_{aq} + x_{kql}}\psi_{kq} \tag{143}$$

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11   Network   Constraints

This model may be embedded in a number of networks. Different configurations will result in different constraints on currents. Consider, for example, the situation in which all of the terminal voltages are constrained, but perhaps by unbalanced (not entirely positive sequence) sources. In that case, the differential equations for the three phase currents would be:

$$\frac{x_d''}{\omega_0}\frac{di_a}{dt} = (v_a - e_a'')\frac{x_d'' + 2x_g}{x_d'' + 3x_g} - [(v_b - e_b'') + (v_c - e_c'')]\frac{x_g}{x_d'' + 3x_g} \tag{144}$$

$$\frac{x_d''}{\omega_0}\frac{di_b}{dt} = (v_b - e_b'')\frac{x_d'' + 2x_g}{x_d'' + 3x_g} - [(v_a - e_a'') + (v_c - e_c'')]\frac{x_g}{x_d'' + 3x_g} \tag{145}$$

$$\frac{x_d''}{\omega_0}\frac{di_c}{dt} = (v_c - e_c'')\frac{x_d'' + 2x_g}{x_d'' + 3x_g} - [(v_b - e_b'') + (v_a - e_a'')]\frac{x_g}{x_d'' + 3x_g} \tag{146}$$

12   Example: Line-Line   Fault

We are not, however, constrained to situations defined in this way. This model is suitable for embedding into network analysis routines. It is also possible to handle many different situations directly. Consider, for example, the unbalanced fault represented by the network shown in Figure 5. This shows a line-line fault situation, with one phase still connected to the network.
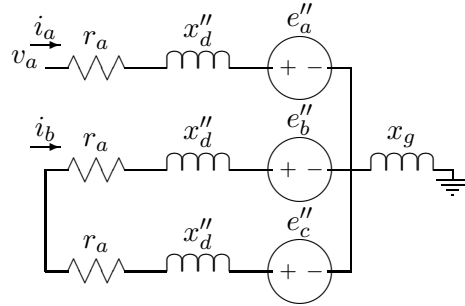

Figure 5: Line-Line Fault Network Model

In this situation, we have only two currents to worry about, and their differential equations would be:

$$\frac{di_b}{dt} = \frac{\omega_0}{2x_d''}(e_c'' - e_b'' - 2r_a i_b) \tag{147}$$

$$\frac{di_a}{dt} = \frac{\omega_0}{x_d'' + x_g}(v_a - e_a'' - r_a i_a) \tag{148}$$

and, of course, $i_c = -i_b$.

Note that here we have included the effects of armature resistance, ignored in the previous section but obviously important if the results are to be believed.

13 Permanent Magnet Machines

Permanent Magnet machines are one state variable simpler than their wound-field counterparts. They may be accurately viewed as having *constant* field current. Assuming that we can define the internal (field) flux as:

$$\psi_0 = x_{ad} i_{f0} \tag{149}$$

13.1 Model: Voltage Driven Machine

We have a reasonably simple expression for the rotor currents, in the case of a voltage driven machine:

$$\begin{bmatrix} i_d \\ i_{kd} \end{bmatrix} = \begin{bmatrix} x_d & x_{ad} \\ x_{ad} & x_{kd} \end{bmatrix}^{-1} \begin{bmatrix} \psi_d - \psi_0 \\ \psi_{kd} - \psi_0 \end{bmatrix} \tag{150}$$

$$\begin{bmatrix} i_q \\ i_{kq} \end{bmatrix} = \begin{bmatrix} x_q & x_{aq} \\ x_{aq} & x_{kq} \end{bmatrix}^{-1} \begin{bmatrix} \psi_q \\ \psi_{kq} \end{bmatrix} \tag{151}$$

The simulation model then has six states:

$$\frac{d\psi_d}{dt} = \omega_0 v_d + \omega \psi_q - \omega_0 r_a i_d \tag{152}$$

$$\frac{d\psi_q}{dt} = \omega_0 v_q - \omega \psi_d - \omega_0 r_a i_q \tag{153}$$

$$\frac{d\psi_{kd}}{dt} = -\omega_0 r_{kd} i_{kd} \tag{154}$$

$$\frac{d\psi_{kq}}{dt} = -\omega_0 r_{kq} i_{kq} \tag{155}$$

$$\frac{d\omega}{dt} = \frac{\omega_0}{2H} \left(\psi_d i_q - \psi_q i_d + T_m \right) \tag{156}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{157}$$

13.2 Curent-Driven Machine Model

In the case of a current-driven machine, rotor currents required in the simulation are:

$$i_{kd} = \frac{1}{x_{kd}} \left(\psi_{kd} - x_{ad} i_d - \psi_0 \right) \tag{158}$$

$$i_{kq} = \frac{1}{x_{kq}} \left(\psi_{kq} - x_{aq} i_q \right) \tag{159}$$

Here, the "flux behind subtransient reactance" is, on the direct axis:

$$\psi_d'' = \frac{x_{kdl} \psi_0 + x_{ad} \psi_{kd}}{x_{ad} + x_{kdl}} \tag{160}$$

and the subtransient reactance is:

$$x_d'' = x_{al} + x_{ad} || x_{kdl} \tag{161}$$

On the quadrature axis,

$$\psi_q'' = \frac{x_{ad}\psi_{kq}}{x_{ad} + x_{kql}} \tag{162}$$

and

$$x_q'' = x_{al} + x_{aq}||x_{kql} \tag{163}$$

In this case there are only four state equations:

$$\frac{d\psi_{kd}}{dt} = -\omega_0 r_{kd} i_{kd} \tag{164}$$

$$\frac{d\psi_{kq}}{dt} = -\omega_0 r_{kq} i_{kq} \tag{165}$$

$$\frac{d\omega}{dt} = \frac{\omega_0}{2H}\left(e_q'' i_q + e_d'' i_d + T_m\right) \tag{166}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{167}$$

The interconnections to and from the network are the same as in the case of a wound-field machine: in the "forward" direction, from network to machine:

$$i_d = i_a \cos(\theta) + i_b \cos(\theta - \frac{2\pi}{3}) + i_c \cos(\theta + \frac{2\pi}{3}) \tag{168}$$

$$i_q = -i_a \sin(\theta) - i_b \sin(\theta - \frac{2\pi}{3}) - i_c \sin(\theta + \frac{2\pi}{3}) \tag{169}$$

and, in the reverse direction, from the machine to the network:

$$\begin{aligned}
e_a'' &= -\frac{\omega}{\omega_0}(e_q'' \sin(\theta) - e_d'' \cos(\theta)) \\
&\quad + \frac{1}{\omega_0}\cos(\theta)\frac{de_q''}{dt} + \frac{1}{\omega_0}\sin(\theta)\frac{de_d''}{dt}
\end{aligned} \tag{170}$$

$$\begin{aligned}
e_b'' &= -\frac{\omega}{\omega_0}(e_q'' \sin(\theta - \frac{2\pi}{3}) - e_d'' \cos(\theta - \frac{2\pi}{3})) \\
&\quad + \frac{1}{\omega_0}\cos(\theta - \frac{2\pi}{3})\frac{de_q''}{dt} + \frac{1}{\omega_0}\sin(\theta - \frac{2\pi}{3})\frac{de_d''}{dt}
\end{aligned} \tag{171}$$

$$\begin{aligned}
e_c'' &= -\frac{\omega}{\omega_0}(e_q'' \sin(\theta + \frac{2\pi}{3}) - e_d'' \cos(\theta + \frac{2\pi}{3})) \\
&\quad + \frac{1}{\omega_0}\cos(\theta + \frac{2\pi}{3})\frac{de_q''}{dt} + \frac{1}{\omega_0}\sin(\theta + \frac{2\pi}{3})\frac{de_d''}{dt}
\end{aligned} \tag{172}$$

13.3 PM Machines with no damper

PM machines without much rotor conductivity may often behave as if they have no damper winding at all. In this case the model simplifies even further. Armature currents are:

$$i_d = \frac{1}{x_d}(\psi_d - \psi_0) \tag{173}$$

$$i_q = \frac{1}{x_q}\psi_q \tag{174}$$

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The state equations are:

$$\frac{d\psi_d}{dt} = \omega_0 v_d + \omega\psi_q - \omega_0 r_a i_d \tag{175}$$

$$\frac{d\psi_q}{dt} = \omega_0 v_q - \omega\psi_d - \omega_0 r_a i_q \tag{176}$$

$$\frac{d\omega}{dt} = \frac{\omega_0}{2H}\left(\psi_d i_q - \psi_q i_d + T_m\right) \tag{177}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{178}$$

13.4   Current Driven PM Machines with no damper

In the case of no damper the machine becomes quite simple. There is no "internal flux" on the quadrature axis. Further, there are no time derivatives of the internal flux on the d- axis. The only machine state equations are mechanical:

$$\frac{d\omega}{dt} = \frac{\omega_0}{2H}\left(\psi_0 i_q + T_m\right) \tag{179}$$

$$\frac{d\delta}{dt} = \omega - \omega_0 \tag{180}$$

The "forward" network interface is as before:

$$i_d = i_a \cos(\theta) + i_b \cos(\theta - \frac{2\pi}{3}) + i_c \cos(\theta + \frac{2\pi}{3}) \tag{181}$$

$$i_q = -i_a \sin(\theta) - i_b \sin(\theta - \frac{2\pi}{3}) - i_c \sin(\theta + \frac{2\pi}{3}) \tag{182}$$

and, in the reverse direction, from the machine to the network, things are a bit simpler than before:

$$e_a'' = -\frac{\omega}{\omega_0}\psi_0 \sin(\theta) \tag{183}$$

$$e_b'' = -\frac{\omega}{\omega_0}\psi_0 \sin(\theta - \frac{2\pi}{3}) \tag{184}$$

$$e_c'' = -\frac{\omega}{\omega_0}\psi_0 \sin(\theta + \frac{2\pi}{3}) \tag{185}$$

$$\tag{186}$$