

**Massachusetts Institute of Technology**  
**Department of Electrical Engineering and Computer Science**

**6.685 Electric Machines**

**Class Notes 3: Eddy Currents, Surface Impedances and Loss Mechanisms**

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## 1 Introduction

Losses in electric machines arise from conduction and magnetic hysteresis. Conduction losses are attributed to straightforward transport conduction and to eddy currents. Transport losses are relatively easy to calculate so we will not pay them much attention. Eddy currents are more interesting and result in frequency dependent conduction losses in machines.

Eddy currents in linear materials can often be handled rigorously, but eddy currents in saturating material are more difficult and are often handled in a heuristic fashion. We present here both analytical and semi-empirical ways of dealing with such losses.

We start with surface impedance: the ratio of electric field to surface current. This is important not just in calculating machine losses, but also in describing how some machines operate.

## 2 Surface Impedance of Uniform Conductors

The objective of this section is to describe the calculation of the surface impedance presented by a layer of conductive material. Two problems are considered here. The first considers a layer of *linear* material backed up by an infinitely permeable surface. This is approximately the situation presented by, for example, surface mounted permanent magnets and is probably a decent approximation to the conduction mechanism that would be responsible for loss due to asynchronous harmonics in these machines. It is also appropriate for use in estimating losses in solid rotor induction machines and in the poles of turbogenerators. The second problem, which we do not work here but simply present the previously worked solution, concerns saturating ferromagnetic material.

### 2.1 Linear Case

The situation and coordinate system are shown in Figure 1. The conductive layer is of thickness  $T$  and has conductivity  $\sigma$  and permeability  $\mu_0$ . To keep the mathematical expressions within bounds, we assume rectilinear geometry. This assumption will present errors which are small to the extent that curvature of the problem is small compared with the wavenumbers encountered. We presume that the situation is excited, as it would be in an electric machine, by a current sheet of the form  $K_z = \text{Re} \left\{ \underline{K} e^{j(\omega t - kx)} \right\}$

In the conducting material, we must satisfy the diffusion equation:

$$\nabla^2 \bar{H} = \mu_0 \sigma \frac{\partial \bar{H}}{\partial t}$$

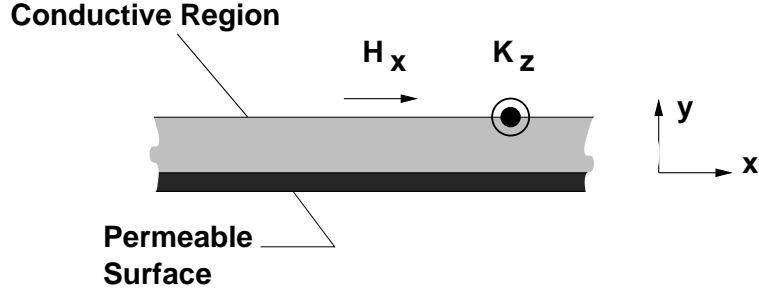


Figure 1: Axial View of Magnetic Field Problem

In view of the boundary condition at the back surface of the material, taking that point to be  $y = 0$ , a general solution for the magnetic field in the material is:

$$\begin{aligned} H_x &= \operatorname{Re} \left\{ A \sinh \alpha y e^{j(\omega t - kx)} \right\} \\ H_y &= \operatorname{Re} \left\{ j \frac{k}{\alpha} A \cosh \alpha y e^{j(\omega t - kx)} \right\} \end{aligned}$$

where the coefficient  $\alpha$  satisfies:

$$\alpha^2 = j\omega\mu_0\sigma + k^2$$

and note that the coefficients above are chosen so that  $\overline{\mathbf{H}}$  has no divergence.

Note that if  $k$  is small (that is, if the wavelength of the excitation is large), this spatial coefficient  $\alpha$  becomes

$$\alpha = \frac{1 + j}{\delta}$$

where the skin depth is:

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

Faraday's law:

$$\nabla \times \overline{\mathbf{E}} = -\frac{\partial \overline{\mathbf{B}}}{\partial t}$$

gives:

$$\underline{E}_z = -\mu_0 \frac{\omega}{k} \underline{H}_y$$

Now: the "surface current" is just

$$\underline{K}_s = -\underline{H}_x$$

so that the equivalent surface impedance is:

$$\underline{Z} = \frac{\underline{E}_z}{-\underline{H}_x} = j\mu_0 \frac{\omega}{\alpha} \coth \alpha T$$

A pair of limits are interesting here. Assuming that the wavelength is long so that  $k$  is negligible, then if  $\alpha T$  is *small* (i.e. thin material),

$$\underline{Z} \rightarrow j\mu_0 \frac{\omega}{\alpha^2 T} = \frac{1}{\sigma T}$$

On the other hand as  $\alpha T \rightarrow \infty$ ,

$$\underline{Z} \rightarrow \frac{1+j}{\sigma\delta}$$

Next it is necessary to transfer this surface impedance across the air-gap of a machine. So, assume a new coordinate system in which the surface of impedance  $\underline{Z}_s$  is located at  $y = 0$ , and we wish to determine the impedance  $\underline{Z} = -\underline{E}_z/\underline{H}_x$  at  $y = g$ .

In the gap there is no current, so magnetic field can be expressed as the gradient of a scalar potential which obeys Laplace's equation:

$$\overline{H} = -\nabla\psi$$

and

$$\nabla^2\psi = 0$$

Ignoring a common factor of  $e^{j(\omega t - kx)}$ , we can express  $\overline{H}$  in the gap as:

$$\begin{aligned}\underline{H}_x &= jk(\underline{\psi}_+ e^{ky} + \underline{\psi}_- e^{-ky}) \\ \underline{H}_y &= -k(\underline{\psi}_+ e^{ky} - \underline{\psi}_- e^{-ky})\end{aligned}$$

At the surface of the rotor,

$$\underline{E}_z = -\underline{H}_x \underline{Z}_s$$

or

$$-\omega\mu_0(\underline{\psi}_+ - \underline{\psi}_-) = jk\underline{Z}_s(\underline{\psi}_+ + \underline{\psi}_-)$$

and then, at the surface of the stator,

$$\underline{Z} = -\frac{\underline{E}_z}{\underline{H}_x} = j\mu_0 \frac{\omega}{k} \left[ \frac{\underline{\psi}_+ e^{kg} - \underline{\psi}_- e^{-kg}}{\underline{\psi}_+ e^{kg} + \underline{\psi}_- e^{-kg}} \right]$$

A bit of manipulation is required to obtain:

$$\underline{Z} = j\mu_0 \frac{\omega}{k} \left\{ \frac{e^{kg}(\omega\mu_0 - jk\underline{Z}_s) - e^{-kg}(\omega\mu_0 + jk\underline{Z}_s)}{e^{kg}(\omega\mu_0 - jk\underline{Z}_s) + e^{-kg}(\omega\mu_0 + jk\underline{Z}_s)} \right\}$$

It is useful to note that, in the limit of  $\underline{Z}_s \rightarrow \infty$ , this expression approaches the *gap impedance*

$$\underline{Z}_g = j \frac{\omega\mu_0}{k^2 g}$$

and, if the gap is small enough that  $kg \rightarrow 0$ ,

$$\underline{Z} \rightarrow \underline{Z}_g || \underline{Z}_s$$

### 3 Iron

Electric machines employ ferromagnetic materials to carry magnetic flux from and to appropriate places within the machine. Such materials have properties which are interesting, useful and problematical, and the designers of electric machines must deal with this stuff. The purpose of this note is to introduce the most salient properties of the kinds of magnetic materials used in electric machines.

We will be concerned here with materials which exhibit *magnetization*: flux density is something other than  $\vec{B} = \mu_0 \vec{H}$ . Generally, we will speak of *hard* and *soft* magnetic materials. Hard materials are those in which the magnetization tends to be permanent, while soft materials are used in magnetic circuits of electric machines and transformers. Since they are related we will find ourselves talking about them either at the same time or in close proximity, even though their uses are widely disparate.

#### 3.1 Magnetization:

It is possible to relate, in all materials, magnetic flux density to magnetic field intensity with a constitutive relationship of the form:

$$\vec{B} = \mu_0 (\vec{H} + \vec{M})$$

where magnetic field intensity  $H$  and magnetization  $M$  are the two important properties. Now, in linear magnetic material magnetization is a simple linear function of magnetic field:

$$\vec{M} = \chi_m \vec{H}$$

so that the flux density is also a linear function:

$$\vec{B} = \mu_0 (1 + \chi_m) \vec{H}$$

Note that in the most general case the magnetic susceptibility  $\chi_m$  might be a tensor, leading to flux density being non-colinear with magnetic field intensity. But such a relationship would still be linear. Generally this sort of complexity does not have a major effect on electric machines.

#### 3.2 Saturation and Hysteresis

In useful magnetic materials this nice relationship is not correct and we need to take a more general view. We will not deal with the microscopic picture here, except to note that the magnetization is due to the alignment of groups of magnetic dipoles, the groups often called *domains*. There are only so many magnetic dipoles available in any given material, so that once the flux density is high enough the material is said to saturate, and the relationship between magnetic flux density and magnetic field intensity is nonlinear.

Shown in Figure 2, for example, is a “saturation curve” for a magnetic sheet steel that is sometimes used in electric machinery. Note the magnetic field intensity is on a logarithmic scale. If this were plotted on linear coordinates the saturation would appear to be quite abrupt.

At this point it is appropriate to note that the units used in magnetic field analysis are not always the same nor even consistent. In almost all systems the unit of flux is the weber (Wb),

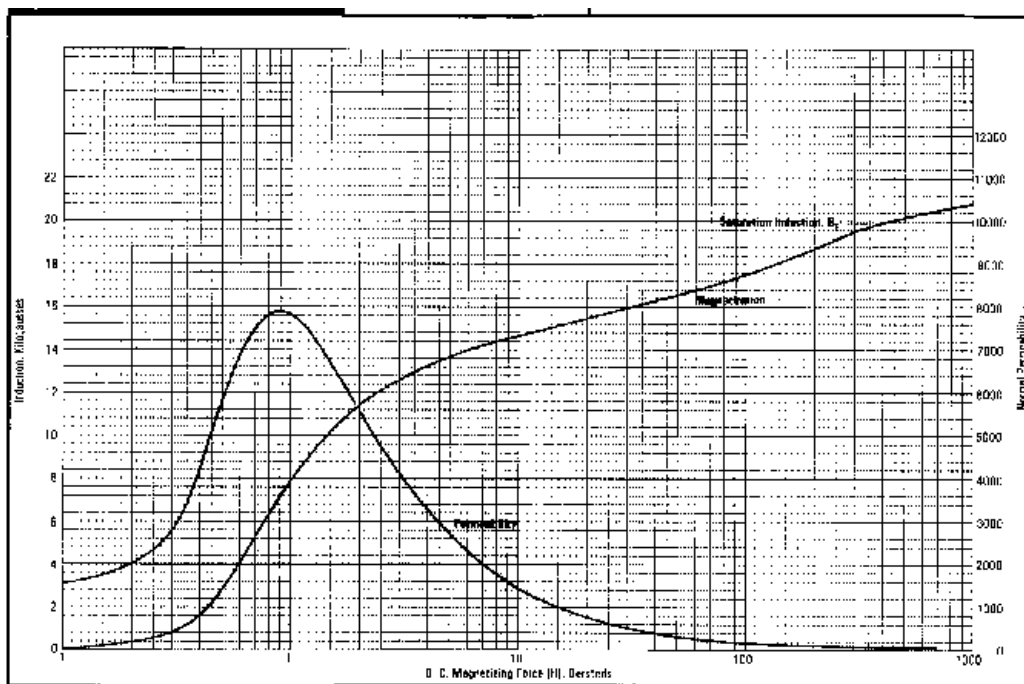


Figure 2: Saturation Curve: Commercial M-19 Silicon Iron

Courtesy of United States Steel Corporation. (U.S. Steel).

U.S. Steel accepts no liability for reliance on any information contained in the graphs shown above.

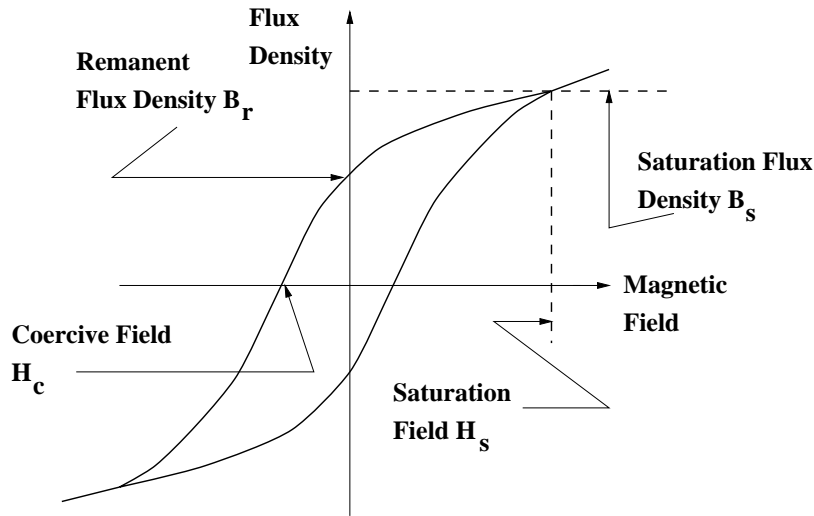


Figure 3: Hysteresis Curve Nomenclature

which is the same as a volt-second. In SI the unit of flux density is the tesla (T), but many people refer to the gauss (G), which has its origin in CGS.  $10,000 \text{ G} = 1 \text{ T}$ . Now it gets worse, because there is an English system measure of flux density generally called kilo-lines per square inch. This is because in the English system the unit of flux is the line.  $10^8$  lines is equal to a weber. Thus a Tesla is 64.5 kilolines per square inch.

The SI and CGS units of flux density are easy to reconcile, but the units of magnetic field are a bit harder. In SI we generally measure  $H$  in amperes/meter (or ampere-turns per meter). Often, however, you will see magnetic field represented as Oersteds (Oe). One Oe is the same as the magnetic field required to produce one gauss in free space. So  $79.577 \text{ A/m}$  is one Oe.

In most useful magnetic materials the magnetic domains tend to be somewhat “sticky”, and a more-than-incremental magnetic field is required to get them to move. This leads to the property called “hysteresis”, both useful and problematical in many magnetic systems.

Hysteresis loops take many forms; a generalized picture of one is shown in Figure 3. Salient features of the hysteresis curve are the remanent magnetization  $B_r$  and the coercive field  $H_c$ . Note that the actual loop that will be traced out is a function of field amplitude and history. Thus there are many other “minor loops” that might be traced out by the B-H characteristic of a piece of material, depending on just what the fields and fluxes have done and are doing.

Hysteresis is important for two reasons. First, it represents the mechanism for “trapping” magnetic flux in a piece of material to form a permanent magnet. We will have more to say about that anon. Second, hysteresis is a loss mechanism. To show this, consider some arbitrary chunk of material for which we can characterize an MMF and a flux:

$$F = NI = \int \vec{H} \cdot d\vec{\ell}$$

$$\Phi = \int \frac{V}{N} dt = \iint_{\text{Area}} \vec{B} \cdot d\vec{A}$$

Energy input to the chunk of material over some period of time is

$$w = \int V I dt = \int F d\Phi = \int_t \int \vec{H} \cdot d\vec{\ell} \iint d\vec{B} \cdot d\vec{A} dt$$

Now, imagine carrying out the second (double) integral over a continuous set of surfaces which are perpendicular to the magnetic field  $H$ . (This IS possible!). The energy becomes:

$$w = \int_t \iiint \vec{H} \cdot d\vec{B} d\text{vol} dt$$

and, done over a complete cycle of some input waveform, that is:

$$w = \iiint_{\text{vol}} W_m d\text{vol}$$

$$W_m = \oint_t \vec{H} \cdot d\vec{B}$$

That last expression simply expresses the area of the hysteresis loop for the particular cycle.

Generally, for most electric machine applications we will use magnetic material characterized as “soft”, having as narrow a hysteresis loop (and therefore as low a hysteretic loss) as possible. At the other end of the spectrum are “hard” magnetic materials which are used to make permanent magnets. The terminology comes from steel, in which soft, annealed steel material tends to have narrow loops and hardened steel tends to have wider loops. However permanent magnet technology has advanced to the point where the coercive forces possible in even cheap ceramic magnets far exceed those of the hardest steels.

### 3.3 Conduction, Eddy Currents and Laminations:

Steel, being a metal, is an electrical conductor. Thus when time varying magnetic fields pass through it they cause eddy currents to flow, and of course those produce dissipation. In fact, for almost all applications involving “soft” iron, eddy currents are the dominant source of loss. To reduce the eddy current loss, magnetic circuits of transformers and electric machines are almost invariably laminated, or made up of relatively thin sheets of steel. To further reduce losses the steel is alloyed with elements (often silicon) which poison the electrical conductivity.

There are several approaches to estimating the loss due to eddy currents in steel sheets and in the surface of solid iron, and it is worthwhile to look at a few of them. It should be noted that this is a “hard” problem, since the behavior of the material itself is difficult to characterize.

### 3.4 Complete Penetration Case

Consider the problem of a stack of laminations. In particular, consider one sheet in the stack represented in Figure 4. It has thickness  $t$  and conductivity  $\sigma$ . Assume that the “skin depth” is much greater than the sheet thickness so that magnetic field penetrates the sheet completely. Further, assume that the applied magnetic flux density is parallel to the surface of the sheets:

$$\vec{B} = \vec{i}_z \text{Re} \left\{ \sqrt{2} B_0 e^{j\omega t} \right\}$$

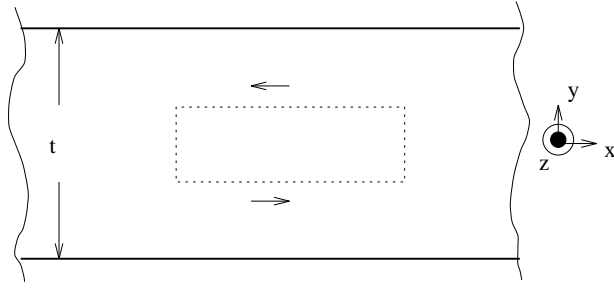


Figure 4: Lamination Section for Loss Calculation

Now we can use Faraday's law to determine the electric field and therefore current density in the sheet. If the problem is uniform in the x- and z- directions,

$$\frac{\partial \underline{E}_x}{\partial y} = -j\omega_0 B_0$$

Note also that, unless there is some net transport current in the x- direction,  $E$  must be anti-symmetric about the center of the sheet. Thus if we take the origin of  $y$  to be in the center, electric field and current are:

$$\begin{aligned} \underline{E}_x &= -j\omega B_0 y \\ \underline{J}_x &= -j\omega B_0 \sigma y \end{aligned}$$

Local power dissipated is

$$P(y) = \omega^2 B_0^2 \sigma y^2 = \frac{|J|^2}{\sigma}$$

To find average power dissipated we integrate over the thickness of the lamination:

$$\langle P \rangle = \frac{2}{t} \int_0^{\frac{t}{2}} P(y) dy = \frac{2}{t} \omega^2 B_0^2 \sigma \int_0^{\frac{t}{2}} y^2 dy = \frac{1}{12} \omega^2 B_0^2 t^2 \sigma$$

Pay attention to the orders of the various terms here: power is proportional to the square of flux density and to the square of frequency. It is also proportional to the square of the lamination thickness (this is average volume power dissipation).

As an aside, consider a simple magnetic circuit made of this material, with some length  $\ell$  and area  $A$ , so that volume of material is  $\ell A$ . Flux lined by a coil of  $N$  turns would be:

$$\Lambda = N\Phi = NAB_0$$

and voltage is of course just  $V = j\omega L$ . Total power dissipated in this core would be:

$$P_c = A\ell \frac{1}{12} \omega^2 B_0^2 t^2 \sigma = \frac{V^2}{R_c}$$

where the equivalent core resistance is now

$$R_c = \frac{A}{\ell} \frac{12N^2}{\sigma t^2}$$



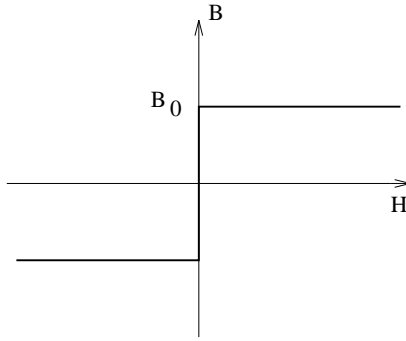


Figure 5: Idealized Saturating Characteristic

### 3.5 Eddy Currents in Saturating Iron

The same geometry holds for this pattern, although we consider only the one-dimensional problem ( $k \rightarrow 0$ ). The problem was worked by McLean and his graduate student Agarwal [2] [1]. They assumed that the magnetic field at the surface of the flat slab of material was sinusoidal in time and of high enough amplitude to saturate the material. This is true if the material has high permeability and the magnetic field is strong. What happens is that the impressed magnetic field saturates a region of material near the surface, leading to a magnetic flux density parallel to the surface. The depth of the region affected changes with time, and there is a separating surface (in the flat problem this is a plane) that moves away from the top surface in response to the change in the magnetic field. An electric field is developed to move the surface, and that magnetic field drives eddy currents in the material.

Assume that the material has a perfectly rectangular magnetization curve as shown in Figure 5, so that flux density in the x- direction is:

$$B_x = B_0 \text{sign}(H_x)$$

The flux per unit width (in the z- direction) is:

$$\Phi = \int_0^{-\infty} B_x dy$$

and Faraday's law becomes:

$$E_z = \frac{\partial \Phi}{\partial t}$$

while Ampere's law in conjunction with Ohm's law is:

$$\frac{\partial H_x}{\partial y} = \sigma E_z$$

Now, McLean suggested a solution to this set in which there is a "separating surface" at depth  $\zeta$  below the surface, as shown in Figure 6 . At any given time:

$$\begin{aligned} H_x &= H_s(t) \left( 1 + \frac{y}{\zeta} \right) \\ J_z &= \sigma E_z = \frac{H_s}{\zeta} \end{aligned}$$

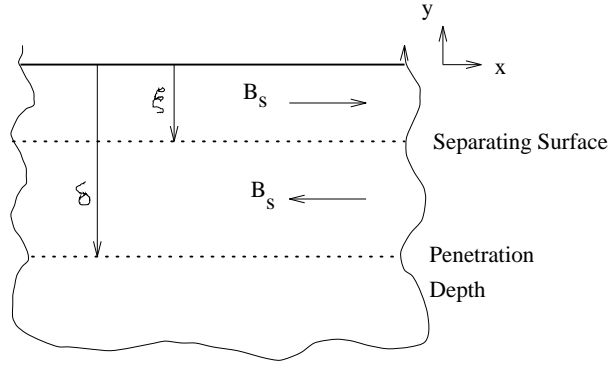


Figure 6: Separating Surface and Penetration Depth

That is, in the region between the separating surface and the top of the material, electric field  $E_z$  is uniform and magnetic field  $H_x$  is a linear function of depth, falling from its impressed value at the surface to zero at the separating surface. Now: electric field is produced by the rate of change of flux which is:

$$E_z = \frac{\partial \Phi}{\partial t} = 2B_x \frac{\partial \zeta}{\partial t}$$

Eliminating E, we have:

$$2\zeta \frac{\partial \zeta}{\partial t} = \frac{H_s}{\sigma B_x}$$

and then, if the impressed magnetic field is sinusoidal, this becomes:

$$\frac{d\zeta^2}{dt} = \frac{H_0}{\sigma B_0} |\sin \omega t|$$

This is easy to solve, assuming that  $\zeta = 0$  at  $t = 0$ ,

$$\zeta = \sqrt{\frac{2H_0}{\omega\sigma B_0}} \sin \frac{\omega t}{2}$$

Now: the surface always moves in the downward direction (as we have drawn it), so at each half cycle a new surface is created: the old one just stops moving at a maximum position, or penetration depth:

$$\delta = \sqrt{\frac{2H_0}{\omega\sigma B_0}}$$

This penetration depth is analogous to the “skin depth” of the linear theory. However, it is an absolute penetration depth.

The resulting electric field is:

$$E_z = \frac{2H_0}{\sigma\delta} \cos \frac{\omega t}{2} \quad 0 < \omega t < \pi$$

This may be Fourier analyzed: noting that if the impressed magnetic field is sinusoidal, only the time fundamental component of electric field is important, leading to:

$$E_z = \frac{8}{3\pi} \frac{H_0}{\sigma\delta} (\cos \omega t + 2 \sin \omega t + \dots)$$

Complex surface impedance is the ratio between the complex amplitude of electric and magnetic field, which becomes:

$$\underline{Z}_s = \frac{\underline{E}_z}{\underline{H}_x} = \frac{8}{3\pi} \frac{1}{\sigma \delta} (2 + j)$$

Thus, in practical applications, we can handle this surface much as we handle linear conductive surfaces, by establishing a skin depth and assuming that current flows within that skin depth of the surface. The resistance is modified by the factor of  $\frac{16}{3\pi}$  and the “power factor” of this surface is about 89 % (as opposed to a linear surface where the “power factor” is about 71 %).

Agarwal suggests using a value for  $B_0$  of about 75 % of the saturation flux density of the steel.

## 4 Semi-Empirical Method of Handling Iron Loss

Neither of the models described so far are fully satisfactory in describing the behavior of laminated iron, because losses are a combination of eddy current and hysteresis losses. The rather simple model employed for eddy currents is precise because of its assumption of abrupt saturation. The hysteresis model, while precise, would require an empirical determination of the size of the hysteresis loops anyway. So we must often resort to empirical loss data. Manufacturers of lamination steel sheets will publish data, usually in the form of curves, for many of their products. Here are a few ways of looking at the data.

A low frequency flux density vs. magnetic field (“saturation”) curve was shown in Figure 2. Included with that was a measure of the incremental permeability

$$\mu' = \frac{dB}{dH}$$

In *some* machine applications either the “total” inductance (ratio of flux to MMF) or “incremental” inductance (slope of the flux to MMF curve) is required. In the limit of low frequency these numbers may be useful.

For designing electric machines, however, a second way of looking at steel may be more useful. This is to measure the real and reactive power as a function of magnetic flux density and (sometimes) frequency. In principal, this data is immediately useful. In any well-designed electric machine the flux density in the core is distributed fairly uniformly and is not strongly affected by eddy currents, etc. in the core. Under such circumstances one can determine the flux density in each part of the core. With that information one can go to the published empirical data for real and reactive power and determine core loss and reactive power requirements.

Figure 7 shows core loss and “apparent” power per unit mass as a function of (RMS) induction for 29 gage, fully processed M-19 steel. The two left-hand curves are the ones we will find most useful. “ $P$ ” denotes real power while “ $P_a$ ” denotes “apparent power”. The use of this data is quite straightforward. If the flux density in a machine is estimated for each part of the machine and the mass of steel calculated, then with the help of this chart a total core loss and apparent power can be estimated. Then the effect of the core may be approximated with a pair of elements in parallel with the terminals, with:

$$R_c = \frac{q|V|^2}{P}$$

$$X_c = \frac{q|V|^2}{Q}$$

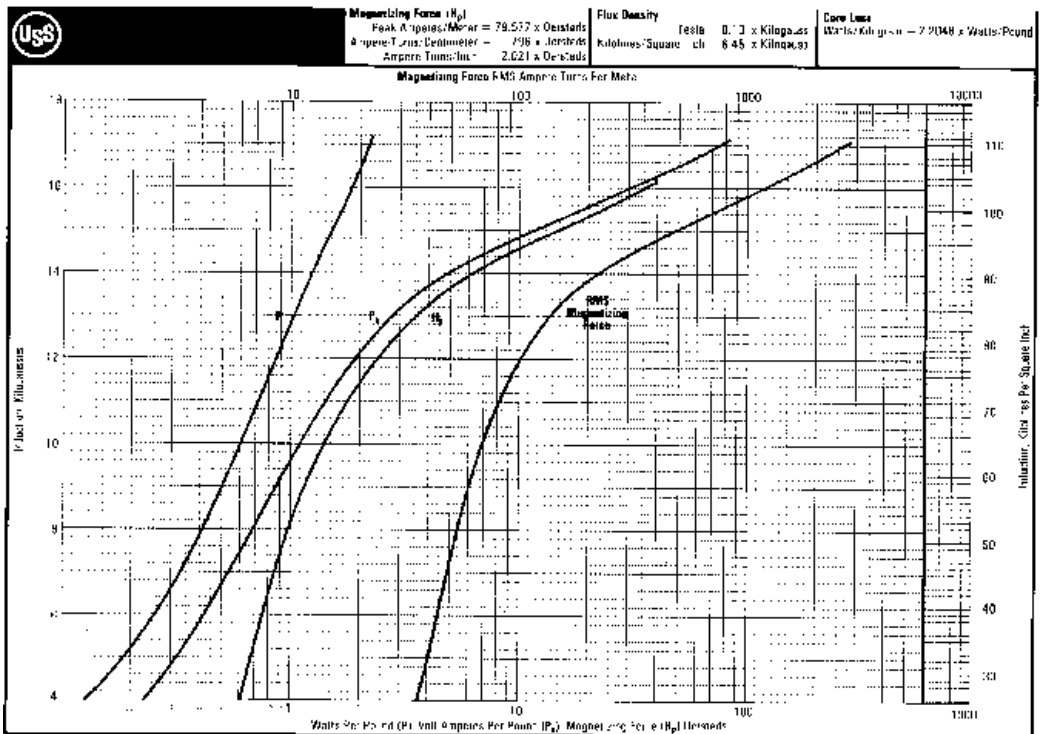


Figure 7: Real and Apparent Loss: M19, Fully Processed, 29 Ga

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### M-19, 29 Ga, Fully Processed

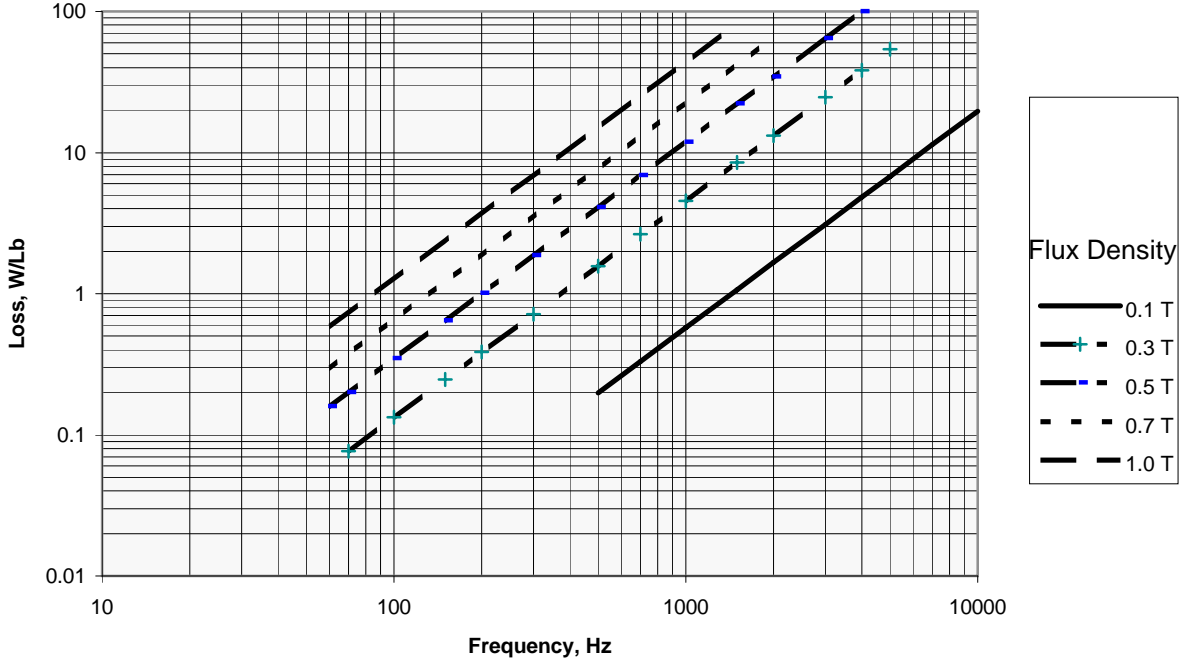


Figure 8: Steel Sheet Core Loss Fit vs. Flux Density and Frequency

$$Q = \sqrt{P_a^2 - P^2}$$

Where  $q$  is the number of machine phases and  $V$  is *phase* voltage. Note that this picture is, strictly speaking, only valid for the voltage and frequency for which the flux density was calculated. But it will be approximately true for small excursions in either voltage or frequency and therefore useful for estimating voltage drop due to exciting current and such matters. In design program applications these parameters can be re-calculated repeatedly if necessary.

“Looking up” this data is a bit awkward for design studies, so it is often convenient to do a “curve fit” to the published data. There are a large number of possible ways of doing this. One method that has been found to work reasonably well for silicon iron is an “exponential fit”:

$$P \approx P_0 \left( \frac{B}{B_0} \right)^{\epsilon_B} \left( \frac{f}{f_0} \right)^{\epsilon_F}$$

This fit is appropriate if the data appears on a log-log plot to lie in approximately straight lines. Figure 8 shows such a fit for the same steel sheet as the other figures.

For “apparent power” the same sort of method can be used. It appears, however, that the simple exponential fit which works well for real power is inadequate, at least if relatively high inductions are to be used. This is because, as the steel saturates, the reactive component of exciting current rises rapidly. I have had some success with a “double exponential” fit:

$$VA \approx VA_0 \left( \frac{B}{B_0} \right)^{\epsilon_0} + VA_1 \left( \frac{B}{B_0} \right)^{\epsilon_1}$$

Table 1: Exponential Fit Parameters for Two Steel Sheets

		29 Ga, Fully Processed	
		M-19	M-36
Base Flux Density	$B_0$	1 T	1 T
Base Frequency	$f_0$	60 Hz	60 Hz
Base Power (w/lb)	$P_0$	0.59	0.67
Flux Exponent	$\epsilon_B$	1.88	1.86
Frequency Exponent	$\epsilon_F$	1.53	1.48
Base Apparent Power 1	$VA_0$	1.08	1.33
Base Apparent Power 2	$VA_1$	.0144	.0119
Flux Exponent	$\epsilon_0$	1.70	2.01
Flux Exponent	$\epsilon_1$	16.1	17.2

To first order the reactive component of exciting current will be linear in frequency.

## References

- [1] W. MacLean, "Theory of Strong Electromagnetic Waves in Massive Iron", Journal of Applied Physics, V.25, No 10, October, 1954
- [2] P.D. Agarwal, "Eddy-Current Losses in Solid and Laminated Iron", Trans. AIEE, V. 78, pp 169-171, 1959