Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.685 Electric Machines

Problem Set 1 Solutions September 10, 2005

Problem 1: If we assume, as suggested in the problem statement, that fields outside the coil can be ignored, magnetic field inside the coil is simply

$$\vec{H} = \vec{i}_z \frac{Ni}{L}$$

and outside the coil magnetic field is zero. On the inner surface of the coil the normal vector is $-\vec{i}_r$ and

$$T_{rr} = -\frac{\mu_0}{2} H_z^2$$

so that 'traction' on the surface is pressure pushing out:

$$P_r = \frac{\mu_0}{2} \left(\frac{Ni}{L} \right)^2$$

Hoop force per unit length is just pressure times radius:

$$\frac{F_h}{L} = RP_r = \frac{\mu_0}{2} \left(\frac{Ni}{L} \right)^2 R$$

To do this same problem using the principal of virtual work, see that co-energy is just co-energy per unit volume times volume, or

$$W'_m = \frac{u_0}{2} \left(\frac{Ni}{L} \right)^2 \pi R^2 L$$

Hoop force is the first derivative of co-energy with respect to hoop circumference, which is $C = 2\pi R$:

$$F_h = \frac{\partial W'_m}{\partial C} = \frac{1}{2\pi} \frac{\partial W'_m}{\partial R} = \frac{1}{2\pi} \frac{\partial}{\partial R} \left(\frac{\mu_0}{2} \left(\frac{Ni}{L} \right)^2 \pi R^2 L \right) = \frac{\mu_0}{2} \left(\frac{Ni}{L} \right)^2 RL$$

to get hoop force per unit length, divide by L and we get the same answer.

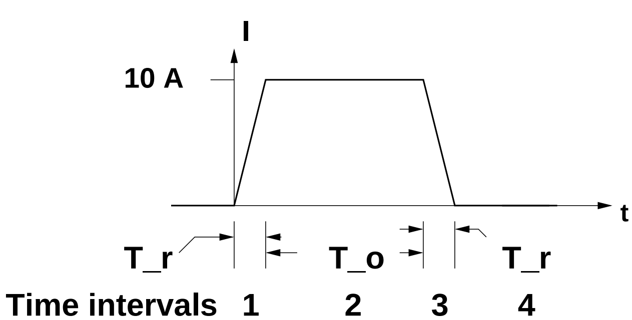

Figure 1: Current in coil for Problem 2

Problem 2: I have re-drawn the current waveform to show my notation for time intervals (Figure 1). If we note H_z as magnetic field within the conductive cylinder, current in that cylinder must be:

$$K_\theta = -\frac{\sigma t_s}{2\pi R_i}\frac{\partial}{\partial t}\left(\mu_0 \pi R_i^2 H_z\right) = T_s \frac{\partial H_z}{\partial t}$$

where t_s and R_i are shell thickness and radius, respectively. The shell time constant is

$$T_s = \frac{\mu_0 \sigma t_s R}{2}$$

If K_θ is azimuthal current in the shell, field inside the shell is:

$$H_z = K_\theta + \frac{Ni}{L}$$

so that

$$T_s \frac{\partial H_z}{\partial t} + H_z = \frac{Ni}{L}$$

Now we have to patch together a solution. Note that if we note $H_{z0} = \frac{Ni}{L}$, the step response of our differential equation would be:

$$H_z^s = H_{z0}\left(1 - e^{-\frac{t}{T_s}}\right)$$

Since the ramp which is the current waveform during interval 1 is simply the integral of a step, and noting that the response of an ODE to the integral of a waveform is the integral of the response to the waveform itself, the response during the first interval is

$$H_{z1} = \int_0^t \frac{H_{z0}}{T_r}\left(1 - e^{-\frac{t'}{T_s}}\right) dt = H_{z0}\left(\frac{t}{T_r} - \frac{T_s}{T_r}\left(1 - e^{-\frac{t}{T_s}}\right)\right)$$

At the end of this interval the field has risen to:

$$H_z^{(1)} = H_{z0}\left(1 - \frac{T_s}{T_r}\left(1 - e^{-\frac{T_r}{T_s}}\right)\right)$$

During the second time interval the excitation is constant and we have the equivalent of a step response with an initial condition:

$$H_{z2} = H_{z0} - \left(H_{z0} - H_z^{(1)}\right) e^{-\frac{t_2}{T_s}}$$

where t_2 is time from the start of interval 2. With a little bit of manipulation this is found to be:

$$H_{z2} = H_{z0}\left(1 - \frac{T_s}{T_r}\left(1 - e^{-\frac{T_r}{T_s}}\right) e^{-\frac{t_2}{T_s}}\right)$$

In the third time interval we have an excitation which is the same as the steady state *minus* the same ramp as started the problem:

$$H_{z3} = H_{z2} - H_{z0}\left(\frac{t_3}{T_r} - \frac{T_s}{T_r}\left(1 - e^{-\frac{t_3}{T_s}}\right)\right)$$

Writing that out and evaluating at the end of the third time interval, when excitation current reaches zero,

$$H_z^{(3)} = H_{z0}\left(\frac{T_s}{T_r}\left(1 - e^{-\frac{T_r}{T_s}}\right)\left(1 - e^{-\frac{T_o + T_r}{T_s}}\right)\right)$$

and in the fourth time interval the system is homogeneous:

$$H_{z4} = H_z^{(3)} e^{-\frac{t_4}{T_s}}$$

It is relatively easy in MATLAB to build up the waveforms for H_z by simply concatenating the time periods. Shown in Figure 2 are the field inside of the cylinder and outside (which is just $\frac{Ni}{L}$).

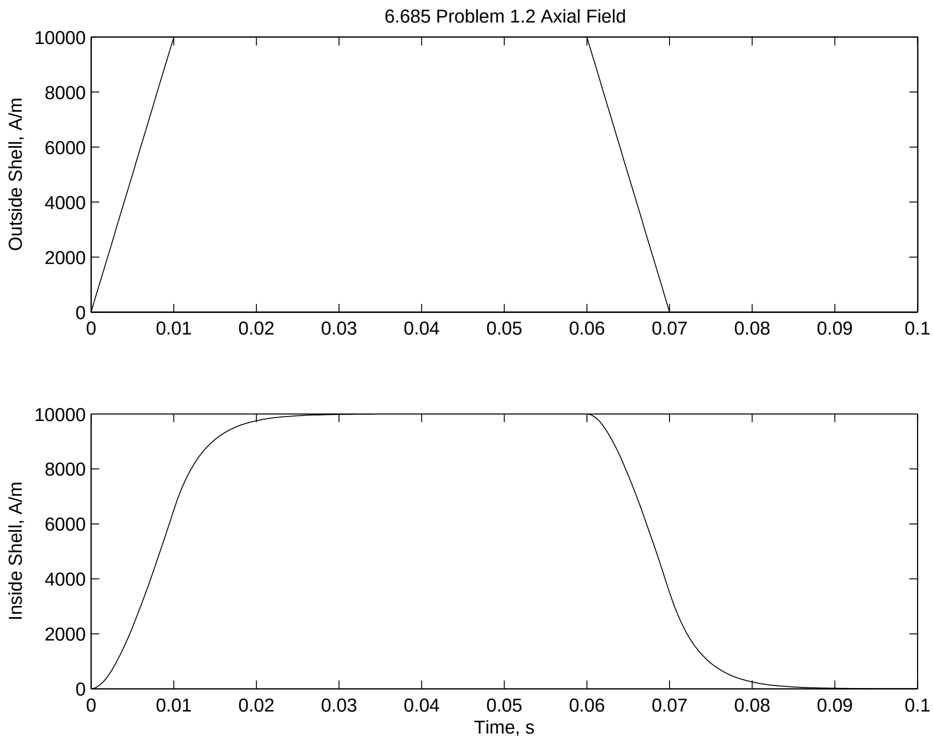

Figure 2: Magnetic Fields

Net pressure on the cylinder is, noting that the normal vector inside is just $-\vec{i}_r$ and outside is $\vec{i}_r$,

$$P_r = \frac{\mu_0}{2}\left(H_z^2 - \left(\frac{Ni}{L}\right)^2\right)$$

so that hoop force is

$$\frac{F_H}{L} = RP_r$$

This is shown in Figure 3. Note that it starts out negative as the field outside is greater than the field inside.

To compute coil voltage we need:

$$v = Ri + N\frac{d\phi}{dt}$$

Now, the resistive term Ri is straightforward. Flux consists of two parts:

$$\Phi = \mu_0 A_1 H_z + \mu_0 A_2 \frac{Ni}{L}$$

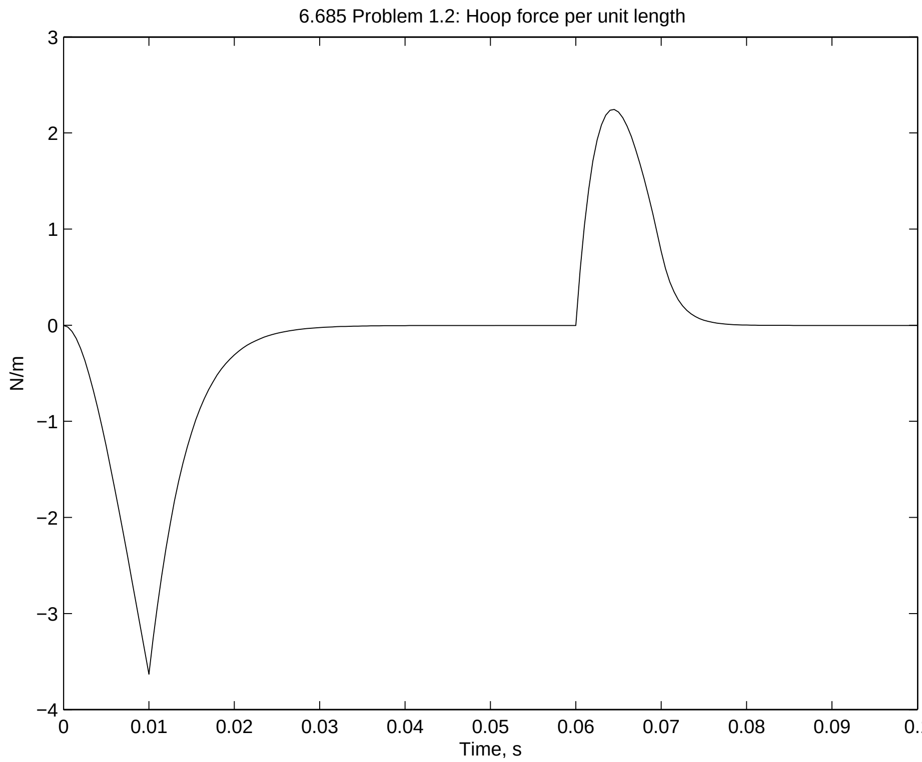

Figure 3: Net Pressure

where $A_1 = \pi R_i^2$ (R_i is the shell radius) and $A_2 = \pi(R_o^2 - R_i^2)$ (R_o is the coil radius).

The rate of change of current is $\pm\frac{I_0}{T_r}$ or zero, depending on time interval and that can be pieced together in MATLAB. Rate of change of field inside the cylinder is:

$$\frac{dH_z}{dt} = \frac{1}{T_s}\left(\frac{Ni}{L} - H_z\right)$$

The result is shown in Figure 4

To get dissipation in the cylinder we can simply find the current in the cylinder:

$$K_\theta = \frac{Ni}{L} - H_z$$

and then

$$P_d = 2\pi R_i L \frac{K_\theta^2}{\sigma t_s}$$

Cylinder current is shown in Figure 5 and resulting dissipation in Figure 6

Problem 3 We assume here that speed is very close to synchronous, so that:

$$\Omega = \frac{2\pi \times 60}{p}$$

Required torque is simply:

$$T = \frac{P}{\Omega}$$

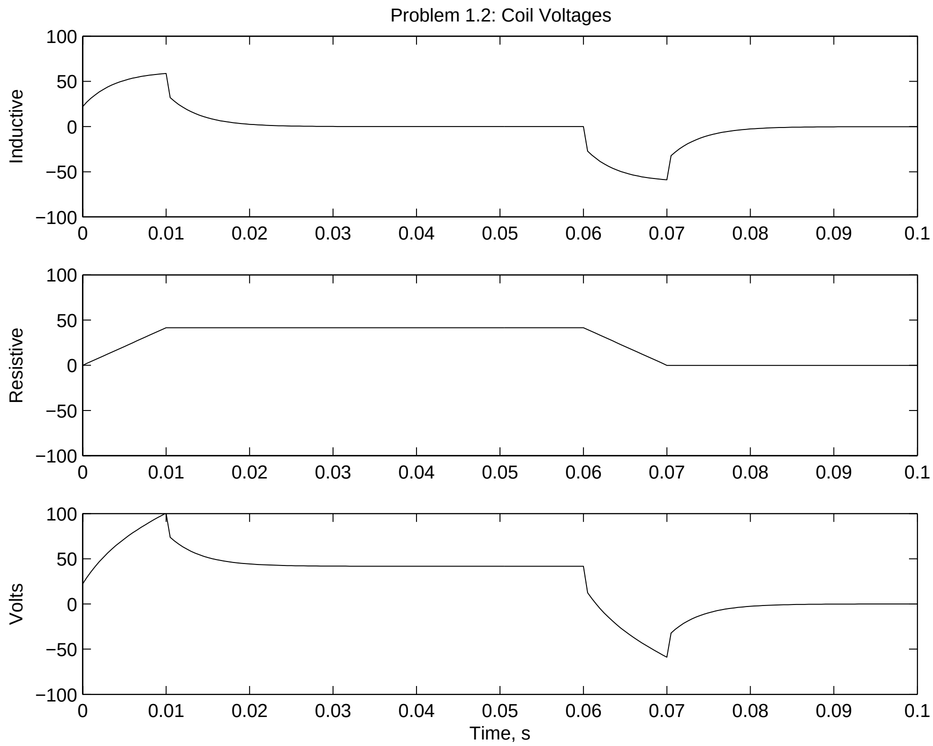

Figure 4: Voltage Induced in the Coil

We established in class that:

$$T = 2\pi R^2 L = 2 \times V_{\text{rotor}} \times \tau$$

where R and L are rotor radius and length, and τ is peripheral shear stress. If length is twice diameter,

$$V_{\text{rotor}} = 4\pi R^3$$

I have written a simple script that carries out these calculations (appended) and here is the result (I have removed a bunch of blank lines)

```
Om  =   376.9911   188.4956   125.6637
T  =    1.0e+03  *     2.6526     5.3052     7.9577
Vol  =     0.0133     0.0265     0.0398
D  =     0.2036     0.2566     0.2937
L  =     0.4073     0.5131     0.5874
```

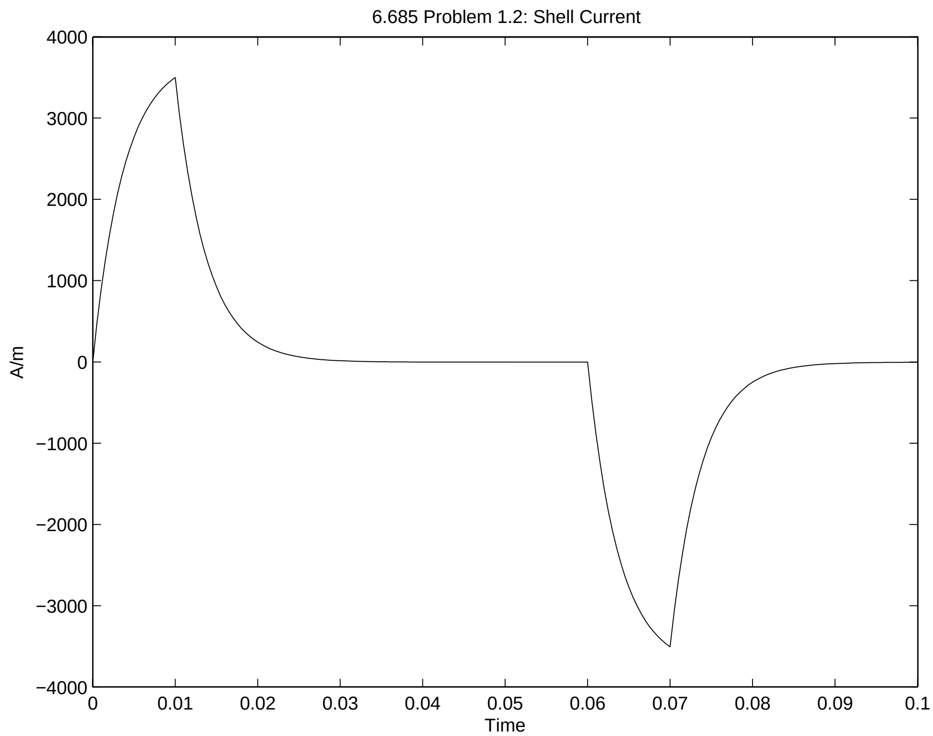

Figure 5: Azumuthal Current in Conductive Cylinder

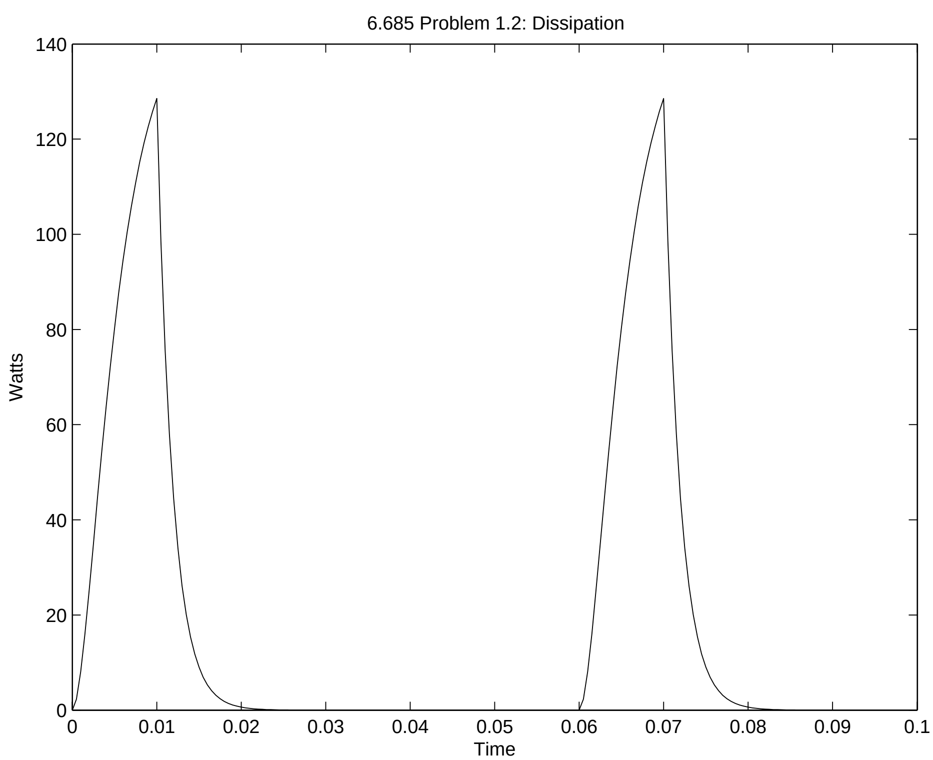

Figure 6: Dissipation in Cylinder

Matlab Code for Problem 2

```
% Massachusetts Institute of Technology
% 6.685, Fall Term, 2005
% Problem set 1, Problem 2 Solution
% Parameters, etc.
muzero = pi*4e-7;
sig = 6e7;              % conductivity of the cylinder
ts = .001;              % thickness of the cylinder
Ri = .1;                % radius of the cylinder
Ro = .125;              % radius of the coil
L = 1;                  % axial extent of the system
N = 1000;               % turns
I0 = 10;                % peak current
T_r = .01;              % rise time
T_o = .05;              % steady current time
T_f = .03;              % end time (kinda arbitrary)
dw = .002;              % coil wire diameter

% minor side calculations
R_c = N*2*pi*Ro/(sig*(pi/4)*dw^2);   % coil resistance
T_s = muzero*sig*ts*Ri/2;         % shell time constant

% now build the times
dt = T_r/20;                      % basic time step
t1 = 0:dt:T_r;                    % first interval
t2i = dt:dt:T_o;                  % constant current time
t2 = t2i + T_r;                   % so we can put things together
t3i = dt:dt:T_r;                  % downgoing ramp
t2e = t3i + T_o;                  % extension of interval 2
t3 = t3i + T_r + T_o;             % for assembly purposes
t4i = dt:dt:T_f;                  % not sure how far to carry this...
t4 = t4i + 2*T_r + T_o;
t = [t1 t2 t3 t4];                % complete time line
T_e = 2*T_r+T_o+T_f;

% construct current waveform
i1 = (I0/T_r) .* t1;              % first interval
i2 = I0 .* ones(size(t2i));       % constant current interval
i3 = I0 .* (1 - (1/T_r) .* t3i);      % downward ramp
i4 = zeros(size(t4));
I = [i1 i2 i3 i4];
H_o = (N/L) .* I;

% now construct magnetic field
H1 = (N*I0/L) .* (t1 ./ T_r - (T_s/T_r) .* (1 - exp(-t1 ./ T_s)));
H2 = (N*I0/L) .* (1 - (T_s/T_r)*(1-exp(-T_r/T_s)) .* exp(-t2i ./ T_s));
H3 = (N*I0/L) .* (1 - (T_s/T_r)*(1-exp(-T_r/T_s)) .* exp(-t2e ./ T_s))...
```

```
    - (N*I0/L) .* (t3i ./ T_r - (T_s/T_r) .* (1 - exp(-t3i ./ T_s)));
H4 = (N*I0/L) .* (T_s/T_r) * (1 - exp(-T_r/T_s)) * (1 - exp(-(T_o+T_r)/T_s)) .* ex
H = [H1 H2 H3 H4];
figure(1)
subplot 211
plot(t, H_o)
title('6.685 Problem 1.2 Axial Field')
ylabel('Outside Shell, A/m')
subplot 212
plot(t, H)
ylabel('Inside Shell, A/m')
xlabel('Time, s')

% hoop force is easily computed from the fields:
% this is force per unit length
F_h = (Ri*muzero/2) .* (H .^2 - H_o .^2);
figure(2)
plot(t,F_h)
title('6.685 Problem 1.2: Hoop force per unit length')
ylabel('N/m')
xlabel('Time, s')

% Now to compute voltage
dhz0dt = (N*I0/(L*T_r)) .* [ones(size(t1)) zeros(size(t2)) -ones(size(t3)) zeros(size(t
dhzdt = (1/T_s) .* (H_o - H);
A1 = pi*Ri^2;
A2 = pi*(Ro^2 - Ri^2);
v_i = muzero*N .* (A1 .* dhzdt + A2 .* dhz0dt);
v_r = R_c .* I;
v = v_r + v_i;

figure(3)
subplot 311
plot(t, v_i)
axis([0 T_e -100 100])
title('Problem 1.2: Coil Voltages')
ylabel('Inductive')
subplot 312
plot(t, v_r)
axis([0 T_e -100 100])
ylabel('Resistive')
subplot 313
plot(t, v)
axis([0 T_e -100 100])
ylabel('Volts')
xlabel('Time, s')

% and, finally current in the shell and then loss
```

```
Kth = H_o - H;                     % this is current in the shell
P_d = (2*pi*Ri/(sig*ts)) .* Kth .^ 2;  % dissipation

figure(4)
plot(t, Kth)
title('6.685 Problem 1.2: Shell Current')
ylabel('A/m')
xlabel('Time')

figure(5)
plot(t, P_d)
title('6.685 Problem 1.2: Dissipation')
ylabel('Watts')
xlabel('Time')
```

Matlab Code for Problem 3

```
% 6.685 Problem Set 1, Problem 3
p = [1 2 3];
Om = 2*pi*60 ./ p
T = 1e6 ./ Om
Vol = T ./ 2e5
R = (Vol ./ (4*pi)) .^ (1/3);
D = 2 .* R
L = 2 .* D
```