Lectures on Dynamic Systems and Control

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Chapter 14

Internal Stability for LTI Systems

14.1 Introduction

Constructing a Lyapunov function for an arbitrary nonlinear system is not a trivial exercise. The complication arises from the fact that we cannot restrict the class of functions to search from in order to prove stability. The situation is different for LTI systems. In this chapter, we address the question of constructing Lyapunov functions for linear systems and then we present and verify Lyapunov indirect method for proving stability of a nonlinear system.

14.2 Quadratic Lyapunov Functions for LTI Systems

Consider the continuous-time system

$$\dot{x}(t) = Ax(t) . \tag{14.1}$$

We have already established that the system (14.1) is asymptotically stable if and only if all the eigenvalues of A are in the open left half plane. In this section we will show that this result can be inferred from Lyapunov theory. Moreover, it will be shown that *quadratic* Lyapunov functions suffice. A consequence of this is that stability can be assessed by methods that may be computationally simpler than eigenanalysis. More importantly, quadratic Lyapunov functions and the associated mathematics turn up in a variety of other problems, so they are worth mastering in the context of stability evaluation.

Quadratic Positive-Definite Functions

Consider the function
$$V(x) = x^T P x, \quad x \in \mathbb{R}^n$$

where P is a symmetric matrix. This is the general form of a quadratic function in $\mathbb{R}^n$. It is sufficient to consider symmetric matrices; if P is not symmetric, we can define $P_1 = \frac{1}{2}(P + P^T)$. It follows immediately that $x^T P x = x^T P_1 x$ (verify, using the fact that $x^T P x$ is a scalar).

Proposition 14.1 $V(x)$ is a positive definite function if and only if all the eigenvalues of P are positive.

Proof: Since P is symmetric, it can be diagonalized by an orthogonal matrix, *i.e.*,

$$P = U^T D U \ \text{ with } U^T U = I \text{ and } D \text{ diagonal.}$$

Then, if $y = Ux$

$$V(x) = x^T U^T D U x = y^T D y = \sum_i \lambda_i |y_i|^2.$$

Thus,

$$V(x) > 0 \ \ \forall x \neq 0 \Leftrightarrow \lambda_i > 0, \ \forall i.$$

Definition 14.1 A matrix P that satisfies

$$x^T P x > 0 \quad \forall x \neq 0 \tag{14.2}$$

is called *positive definite*. When P is symmetric (which is usually the case of interest, for the reason mentioned above), we will denote its positive definiteness by $P > 0$. If $x^T P x \geq 0 \ \ \forall x \neq 0$, then P is positive semi-definite, which we denote in the symmetric case by $P \geq 0$.

For a symmetric positive definite matrix, it follows that

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2.$$

This inequality follows directly from the proof of Proposition 14.1.

It is also evident from the above discussion that the singular values and eigenvalues of any positive definite matrix coincide.

Exercise: Show that $P > 0$ if and only if $P = G^T G$ where G is nonsingular. The matrix G is called a square root of P and is denoted by $P^{\frac{1}{2}}$. Show that H is another square root of P if and only if $G = WH$ for some orthogonal matrix W. Can you see how to construct a *symmetric* square root? (You may find it helpful to begin with the eigen-decomposition $P = U^T D U$, where U is orthogonal and D is diagonal.)

Quadratic Lyapunov Functions for CT LTI Systems

Consider defining a Lyapunov function candidate of the form

$$V(x) = x^T P x, \quad P > 0, \tag{14.3}$$

for the system (14.1). Then

$$
\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T A^T P x + x^T P A x \\
&= x^T (A^T P + P A) x \\
&= -x^T Q x \, ,
\end{aligned}
$$

where we have introduced the notation $Q = -(A^T P + P A)$; note that Q is symmetric. Now invoking the Lyapunov stability results from Lecture 5, we see that V is a Lyapunov function if $Q \geq 0$, in which case the equilibrium point at the origin of the system (14.1) is stable i.s.L. If $Q > 0$, then the equilibrium point at the origin is *globally asymptotically stable*. In this latter case, the origin must be the only equilibrium point of the system, so we typically say the *system* (rather than just the equilibrium point) is asymptotically stable.

The preceding relationships show that in order to find a quadratic Lyapunov function for the system (14.1), we can pick $Q > 0$ and then try to solve the equation

$$A^T P + P A = -Q \tag{14.4}$$

for P. This equation is referred to as a *Lyapunov equation*, and is a *linear* system of equations in the entries of P. If it has a solution, then it has a symmetric solution (show this!), so we only consider symmetric solutions. If it has a positive definite solution $P > 0$, then we evidently have a Lyapunov function $x^T P x$ that will allow us to prove the asymptotic stability of the system (14.1). The interesting thing about LTI systems is that the converse also holds: If the system is asymptotically stable, then the Lyapunov equation (14.4) has positive definite solution $P > 0$ (which, as we shall show, is unique). This result is stated and proved in the following theorem.

Theorem 14.1 Given the dynamic system (14.1) and any $Q > 0$, there exists a positive definite solution P of the Lyapunov equation

$$A^T P + P A = -Q$$

if and only if all the eigenvalues of A are in the open left half plane (OLHP). The solution P in this case is unique.

Proof: If $P > 0$ is a solution of (14.4), then $V(x) = x^T P x$ is a Lyapunov function of system (14.1) with $\dot{V}(x) < 0$ for any $x \neq 0$. Hence, system (14.1) is (globally) asymptotically stable and thus the eigenvalues of A are in the OLHP.

To prove the converse, suppose A has all eigenvalues in the OLHP, and $Q > 0$ is given. Define the symmetric matrix P by

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt \, . \tag{14.5}$$

This integral is well defined because the integrand decays exponentially to the origin, since the eigenvalues of A are in the OLHP. Now

$$
\begin{aligned}
A^T P + PA &= \int_0^\infty A^T e^{tA^T} Q e^{tA} dt + \int_0^\infty e^{tA^T} Q e^{tA} A dt \\
&= \int_0^\infty \frac{d}{dt} \left[e^{tA^T} Q e^{tA} \right] dt \\
&= -Q
\end{aligned}
$$

so P satisfies the Lyapunov equation.

To prove that P is positive definite, note that

$$
\begin{aligned}
x^T P x &= \int_0^\infty x^T e^{tA^T} Q e^{tA} x \, dt \\
&= \int_0^\infty \| Q^{\frac{1}{2}} e^{tA} x \|^2 dt \geq 0
\end{aligned}
$$

and

$$x^T P x = 0 \Rightarrow Q^{\frac{1}{2}} e^{tA} x = 0 \Rightarrow x = 0 \, ,$$

where $Q^{\frac{1}{2}}$ denotes a square root of Q. Hence P is positive definite.

To prove that the P defined in (14.5) is the unique solution to (14.4) when A has all eigenvalues in the OLHP, suppose that P_2 is another solution. Then

$$
\begin{aligned}
P_2 &= -\int_0^\infty \frac{d}{dt} \left[e^{tA^T} P_2 e^{tA} \right] dt \qquad \text{(verify this identity)} \\
&= -\int_0^\infty e^{tA^T} \left(A^T P_2 + P_2 A \right) e^{tA} dt \\
&= \int_0^\infty e^{tA^T} Q e^{tA} dt = P
\end{aligned}
$$

This completes the proof of the theorem.

A variety of generalizations of this theorem are known.

Quadratic Lyapunov Functions for DT LTI Systems

Consider the system

$$x(t+1) = Ax(t) = f\big(x(t)\big) \tag{14.6}$$

If

$$V(x) = x^T P x,$$

then
$$\dot{V}(x) \stackrel{\triangle}{=} V(f(x)) - V(x) = x^T A^T P A x - x^T P x.$$

Thus the resulting Lyapunov equation to study is

$$A^T P A - P = -Q . \tag{14.7}$$

The following theorem is analogous to what we proved in the CT case, and we leave its proof as an exercise.

Theorem 14.2 Given the dynamic system (14.6) and any $Q > 0$, there exists a positive definite solution P of the Lyapunov equation

$$A^T P A + P = -Q$$

if and only if all the eigenvalues of A have magnitude less than 1 (i.e. are in the open unit disc). The solution P in this case is unique.

Example 14.1 Differential Inclusion

In many situations, the evolution of a dynamic system can be uncertain. One way of modeling this uncertainty is by differential (difference) inclusion which can be described as follows:
$$\dot{x}(t) \subset \{Ax(t) \mid A \subset \mathcal{A}\}$$

where $\mathcal{A}$ is a set of matrices. Consider the case where $\mathcal{A}$ is a finite set of matrices and their convex combinations:

$$\mathcal{A} = \{A = \sum_{i=1}^{m} \alpha_i A_i \mid \sum_{i=1}^{m} \alpha_i = 1\}$$

One way to guarantee the stability of this system is to find *one* Lyapunov function for all systems defined by $\mathcal{A}$. If we look for a quadratic Lyapunov function, then it suffices to find a P that satisfies:

$$A_i^T P + P A_i < -Q, \quad i = 1, 2, \ldots m$$

for some positive definite Q. Then $V(x) = x^T P x$ satisfies $\dot{V}(x) < -x^T Q x$ (verify) showing that the system is asymptotically stable.

Example 14.2 Set of Bounded Norm

In this problem, we are interested in studying the stability of linear time-invariant systems of the form $\dot{x}(t) = (A + \Delta)x(t)$ where Δ is a real matrix perturbation with bounded norm. In particular, we are interested in calculating a good bound on the size of the smallest perturbation that will destabilize a stable matrix A.

This problem can be cast as a differntial inclusion problem as in the previous example with
$$\mathcal{A} = \{A + \Delta \,\|\, \|\Delta\| \leq \gamma, \; \Delta \text{ is a real matrix}\}$$

Since A is stable, we can calculate a quadratic Lyapunov function with a matrix P satisfying $A^T P + PA < -Q$ and Q is positive definite. Applying the same Lyapunov function to the perturbed system we get:

$$\dot{V}(x) = x^T \left(A^T P + PA + \Delta^T P + P\Delta \right) x$$

It is evident that all perturbations satisfying

$$\Delta^T P + P\Delta < Q$$

will result in a stable system. This can be guaranteed if

$$2\sigma_{max}(P)\sigma_{max}(\Delta) < \sigma_{min}(Q)$$

This provides a bound on the perturbation although it is potentially conservative.

Example 14.3 Bounded Perturbation

Casting the perturbation in the previous example in terms of differential inclusion introduces a degree of conservatism in that the value Δ takes can change as a function of time. Consider the system:

$$\dot{x}(t) = (A + \Delta)x(t)$$

where A is a known fixed stable matrix and Δ is an unknown fixed real perturbation matrix. The *stability margin* of this system is defined as

$$\gamma(A) = \min_{\Delta \in \mathbb{R}^{n \times n}} \{ \|\Delta\| \,|\, A + \Delta \text{ is unstable} \}.$$

We desire to compute a good lower bound on $\gamma(A)$. The previous example gave one such bound.

First, it is easy to argue that the minimizing solution Δ_o of the above problem results in $A + \Delta_0$ having eigenvalues at the imaginary axis (either at the origin, or in two complex conjugate locations). This is a consequence of the fact that the eigenvalues of $A + p\Delta_0$ will move continuously in the complex plane as the parameter p varies from 0 to 1. The intersection with the imaginary axis will happen at $p = 1$; if not, a perturbation of smaller size can be found.

We can get a lower bound on γ by dropping the condition that Δ is a real matrix, and allowing complex matrices (is it clear why this gives a lower bound?). We can show:

$$\min_{\Delta \in \mathbf{C}^{n \times n}} \{ \|\Delta\| \,|\, A + \Delta \text{ is unstable} \} = \min_{\omega \in \mathbb{R}} \sigma_{min}(A - j\omega I).$$

To verify this, notice that if the minimizing solution has an eigenvalue at the imaginary axis, then $j\omega_0 I - A - \Delta_0$ should be singular while we know that $j\omega_0 - A$ is not. The smallest possible perturbation that achieves this has size $\sigma_{min}(A - j\omega_0 I)$. We can then choose ω_0 that gives the smallest possible size. In the exercises, we further improve this bound.

14.3 Lyapunov's Indirect Method: Analyzing the Linearization

Suppose the system

$$\dot{x} = f(x) \tag{14.8}$$

has an equilibrium point at $\overline{x} = 0$ (an equilibrium at any other location can be dealt with by a preliminary change of variables to move that equilibrium to the origin). Assume we can write

$$f(x) = Ax + h(x)$$

where

$$\lim_{\|x\| \to 0} \frac{\|h(x)\|}{\|x\|} = 0$$

i.e. $h(x)$ denotes terms that are higher order than linear, and A is the Jacobian matrix associated with the linearization of (14.8) about the equilibrium point. The linearized system is thus given by

$$\dot{x} = Ax . \tag{14.9}$$

We might expect that if (14.9) is asymptotically stable, then in a small neighborhood around the equilibrium point, the system (14.8) behaves like (14.9) and will be stable. This is made precise in the following theorem.

Theorem 14.3 If the system (14.9) is asymptotically stable, then the equilibrium point of system (14.8) at the origin is (locally) asymptotically stable.

Proof: If system (14.9) is asymptotically stable, then for any $Q > 0$, there exists $P > 0$ such that

$$A^T P + PA = -Q$$

and $V(x) = x^T Px$ is a Lyapunov function for system (14.9). Consider $V(x)$ as a Lyapunov function candidate for system (14.8). Then

$$
\begin{aligned}
\dot{V}(x) &= x^T(A^T P + PA)x + 2x^T Ph(x) \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\|x\| \cdot \|h(x)\| \cdot \lambda_{\max}(P) \\
&\leq -\left[\lambda_{\min}(Q) - 2\lambda_{\max}(P)\frac{\|h(x)\|}{\|x\|}\right] \cdot \|x\|^2
\end{aligned}
$$

From the assumption on h, for every $\epsilon > 0$, there exists $r > 0$ such that

$$\|h(x)\| < \epsilon \|x\|, \quad \forall \, \|x\| < r.$$

This implies that $\dot{V}$ is strictly negative for all $\|x\| < r$, where r is chosen for

$$\epsilon < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}.$$

This concludes the proof.

Notice that asymptotic stability of the equilibrium point of the system (14.8) can be concluded from the asymptotic stability of the linearized system (14.9) only when the eigenvalues of A have negative real parts. It can also be shown that if there is any eigenvalue of A in the right half plane, i.e. if the linearization is exponentially unstable, then the equilibrium point of the nonlinear system is unstable. The above theorem is inconclusive if there are eigenvalues on the imaginary axis, but none in the right half plane. The higher-order terms of the nonlinear model can in this case play a decisive role in determining stability; for instance, if the linearization is polynomially (rather than exponentially) unstable, due to the presence of one or more Jordan blocks of size greater than 1 for eigenvalues on the imaginary axis (and the absence of eigenvalues in the right half plane), then the higher-order terms can still cause the equilibrium point to be stable.

It turns out that stronger versions of the preceding theorem hold if A has no eigenvalues on the imaginary axis: not only the stability properties of the equilibrium point, but also the local behavior of (14.8) can be related to the behavior of (14.9). We will not discuss these results further here.

Similar results hold for discrete-time systems.

Example 14.4

The equations of motion for a pendulum with friction are

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 - \sin x_1
\end{aligned}$$

The two equilibrium points of the system are at $(0, 0)$ and $(\pi, 0)$. The linearized system at the origin is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2
\end{aligned}$$

or

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x = Ax \ .$$

This A has all its eigenvalues in the OLHP. Hence the equilibrium point at the origin is asymptotically stable. Note, however, that if there were no damping, then the linearized system would be

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

and the resulting matrix A has eigenvalues on the imaginary axis. No conclusions can be drawn from this situation using Lyapunov linearization methods. Lyapunov's direct method, by contrast, allowed us to conclude stability even in the case of zero damping, and also permitted some detailed global conclusions in the case with damping.

The linearization around the equilibrium point at $(\pi, 0)$ is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= +z_1 - z_2 \end{aligned}$$

where $z_1 = x_1 - \pi$ and $z_2 = x_2$, so these variables denote the (small) deviations of x_1 and x_2 from their respective equilibrium values. Hence

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x = Ax \,,$$

which has one eigenvalues in the RHP, indicating that this equilibrium point is unstable.

Exercises

Exercise 14.1 Bounded Perturbation Recall Example 14.3. In this problem we want to improve the lower bound on $\gamma(A)$.

(a) To improve the lower bound, we use the information that if Δ is real, then poles appear in complex conjugate pair. Define
$$A_w = \begin{pmatrix} A & wI \\ -wI & A \end{pmatrix}.$$
Show that
$$\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{min}[A_w].$$

(b) If you think harder about your proof above, you will be able to further improve the lower bound. In fact, it follows that
$$\gamma(A) \geq \min_{w \in \mathbb{R}} \sigma_{2n-1}[A_w]$$
where σ_{2n-1} is the next to last singular value. Show this result.

Exercise 14.2 Consider the LTI unforced system given below:
$$\dot{x} = Ax = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{N-1} & -a_{N-2} & \ldots & \ldots & \ldots & -a_0 \end{pmatrix} x$$

(a) Under what conditions is this system asymptotically stable?

Assume the system above is asymptotically stable. Now, consider the perturbed system
$$\dot{x} = Ax + \Delta x,$$
where Δ is given by
$$\Delta = \begin{pmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\delta_{N-1} & -\delta_{N-2} & \ldots & \ldots & \ldots & -\delta_0 \end{pmatrix}, \qquad \delta_i \in \mathbb{R}.$$

(b) Argue that the perturbation with the smallest Frobenius norm that destabilizes the system (makes the system not asymptotically stable) will result in $A + \Delta$ having an eigenvalue at the imaginary axis.

(c) Derive an exact expression for the smallest Frobenius norm of Δ necessary to destabilize the above system (i.e., $\dot{x} = (A+\Delta)x$ is not asymptotically stable). Give an expression for the perturbation Δ that attains the minimum.

(d) Evaluate your answer in part 3 for the case $N = 2$, and $a_0 = a_1$.

Exercise 14.3 Periodic Controllers

(a) Show that the periodically varying system in Exercise 7.4 is asymptotically stable if and only if all the eigenvalues of the matrix $[A_{N-1}...A_0]$ have magnitude less than 1.

(b) (i) Given the system

$$x(k+1) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) , \qquad y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(k)$$

write down a linear state-space representation of the closed-loop system obtained by implementing the linear output feedback control $u(k) = g(k)y(k)$.

(ii) It turns out that there is *no* constant gain $g(k) = g$ for which the above system is asymptotically stable. (**Optional:** Show this.) However, consider the periodically varying system obtained by making the gain take the value -1 for even k and the value 3 for odd k. Show that any nonzero initial condition in the resulting system will be brought to the origin in at most 4 steps. (The moral of this is that periodically varying output feedback can do more than constant output feedback.)

Exercise 14.4 Delay Systems

The material we covered in class has focused on finite-dimensional systems, i.e., systems that have state-space descriptions with a finite number of state variables. One class of systems that does not belong to the class of finite-dimensional systems is continuous-time systems with delays.

Consider the following forced continuous-time system:

$$y(t) + a_1 y(t-1) + a_2 y(t-2) + \ldots + a_N y(t-N) = u(t) \qquad t \geq N, \ t \in \mathbb{R}.$$

This is known as a delay system with commensurate delays (multiple of the same delay unit). We assume that $u(t) = 0$ for all $t < N$.

(a) Show that we can compute the solution $y(t)$, $t \geq N$, if $y(t)$ is completely known in the interval [0,N). Explain why this system cannot have a finite-dimensional state space description.

(b) To compute the solution $y(t)$ given the initial values (denote those by the function $f(t)$, $t \in [0, N)$, which we will call the initial function) and the input u, it is useful to think of every non-negative real number as $t = \tau + k$ with $\tau \in [0, 1)$ and k being a non-negative integer. Show that for every fixed τ, the solution evaluated at $\tau + k$ ($y(\tau + k)$) can be computed using discrete-time methods and can be expressed in terms of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_N & -a_{N-1} & \ldots & \ldots & \ldots & -a_1 \end{pmatrix}$$

and the initial vector

$$\begin{pmatrix} f(\tau) & f(\tau+1) & \ldots & f(\tau+N-1) \end{pmatrix}^T .$$

Write down the general solution for $y(t)$.

(c) Compute the solution for $N = 2$, $f(t) = 1$ for $t \in [0, 2)$, and $u(t) = e^{-(t-2)}$ for $t \geq 2$.

(d) This system is asymptotically stable if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all initial functions with $|f(t)| < \delta$, $t \in [0, N)$, and $u = 0$, it follows that $|y(t)| < \epsilon$, and $\lim_{t \to \infty} y(t) = 0$. Give a necessary and sufficient condition for the asymptotic stability of this system. Explain your answer.

(e) Give a necessary and sufficient condition for the above system to be BIBO stable (∞-stable). Verify your answer.

Exercise 14.5 Local Stabilization

(a) One method for stabilizing a nonlinear system is to linearize it around an equilibrium point and then stabilize the resulting linear system. More formally, consider a nonlinear time-invariant system
$$\dot{x} = f(x, u)$$
and its linearization around an equilibrium point $(\tilde{x}, \tilde{u})$
$$\dot{\delta x} = A\delta x + B\delta u.$$

As usual, $\delta x = x - \tilde{x}$ and $\delta u = u - \tilde{u}$. Suppose that the feedback $\delta u = K\delta x$ asymptotically stabilizes the linearized system.

  1. What can you say about the eigenvalues of the matrix $A + BK$.
  2. Show that $\dot{x} = f(x, Kx)$ is (locally) asymptotically stable around $\tilde{x}$.

(b) Consider the dynamic system S_1 governed by the following differential equation:
$$\ddot{y} + \dot{y}^4 + \dot{y}^2 u + y^3 = 0$$

where u is the input.

  1. Write down a state space representation for the system S_1 and find its unique equilibrium point x^*.
  2. Now try to apply the above method to the system S_1 at the equilibrium point x^* and $u^* = 0$. Does the linearized system provide information about the stability of S_1. Explain why the method fails.

(c) To find a stabilizing controller for S_1, we need to follow approaches that are not based on local linearization. One approach is to pick a positive definite function of the states and then construct the control such that this function becomes a Lyapunov function. This can be a very frustrating exercise. A trick that is commonly used is to find an input as a function of the states so that the resulting system belongs to a class of systems that are known to be stable (e.g. a nonlinear circuit or a mechanical system that are known to be stable). Use this idea to find an input u as function of the states such that S_1 is stable.

Exercise 14.6 For the system

$$\dot{x}(t) = sin[x(t) + y(t)]$$
$$\dot{y}(t) = e^{x(t)} - 1$$

determine *all* equilibrium points, and using Lyapunov's indirect method (i.e. linearization), classify each equilibrium point as asymptotically stable or unstable.

Exercise 14.7 For each of the following parts, all of them **optional**, use Lyapunov's indirect method to determine, if possible, whether the origin is an asymptotically stable or unstable equilibrium point.

(a)

$$\dot{x}_1 = -x_1 + x_2^2$$
$$\dot{x}_2 = -x_2(x_1 + 1)$$

(b)

$$\dot{x}_1 = x_1^3 + x_2$$
$$\dot{x}_2 = x_1 - x_2$$

(c)

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = -x_2 + x_1^2$$

(d)

$$x_1(k + 1) = 2x_1(k) + x_2(k)^2$$
$$x_2(k + 1) = x_1(k) + x_2(k)$$

(e)

$$x_1(k + 1) = 1 - e^{x_1(k)x_2(k)}$$
$$x_2(k + 1) = x_1(k) + 2x_2(k)$$

Exercise 14.8 For each of the nonlinear systems below, construct a linearization for the equilibrium point at the origin, assess the stability of the linearization, and decide (using the results of Lyapunov's *indirect* method) whether you can infer something about the stability of the equilibrium of the nonlinear system at the origin. Then use Lyapunov's *direct* method prove that the origin is actually stable in each case; if you can make further arguments to actually deduce *asymptotic* stability or even global asymptotic stability, do so. [Hints: In part (a), find a suitable Lyapunov (energy) function by interpreting the model as the dynamic equation for a mass attached to a nonlinear (cubic) spring. In parts (b) and (c), try a simple quadratic Lyapunov function of the form $px^2 + qy^2$, then choose p and q appropriately. In part (d), use the indicated Lyapunov function.]

(a)

$$\dot{x} = y$$
$$\dot{y} = -x^3$$

(b)

$$\dot{x} = -x^3 - y^2$$
$$\dot{y} = xy - y^3$$

(c)

$$x_1(k+1) = \frac{x_2(k)}{1 + x_2^2(k)}$$
$$x_2(k+1) = \frac{x_1(k)}{1 + x_2^2(k)}$$

(d)

$$\dot{x} = y(1 - x)$$
$$\dot{y} = -x(1 - y)$$
$$V(x, y) = -x - \ln(1 - x) - y - \ln(1 - y)$$

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