6.231 Dynamic Programming and Stochastic Control
Fall 2008

6.231 DYNAMIC PROGRAMMING

LECTURE 14

LECTURE OUTLINE

- Review of stochastic shortest path problems
- Computational methods
  - Value iteration
  - Policy iteration
  - Linear programming
- Discounted problems as special case of SSP

STOCHASTIC SHORTEST PATH PROBLEMS

- Assume finite-state system: States $1, \ldots, n$ and special cost-free termination state t
  - Transition probabilities $p_{ij}(u)$
  - Control constraints $u \in U(i)$
  - Cost of policy $\pi = \{\mu_0, \mu_1, \ldots\}$ is

$$J_\pi(i) = \lim_{N \to \infty} E\left\{ \sum_{k=0}^{N-1} g\big(x_k, \mu_k(x_k)\big) \,\Big|\, x_0 = i \right\}$$

  - Optimal policy if $J_\pi(i) = J^*(i)$ for all i.
  - Special notation: For stationary policies $\pi = \{\mu, \mu, \ldots\}$, we use $J_\mu(i)$ in place of $J_\pi(i)$.

- Assumption (Termination inevitable): There exists integer m such that for every policy and initial state, there is positive probability that the termination state will be reached after no more that m stages; for all π, we have

$$\rho_\pi = \max_{i=1,\ldots,n} P\{x_m \neq t \mid x_0 = i, \pi\} < 1$$

MAIN RESULT

- Given any initial conditions $J_0(1), \ldots, J_0(n)$, the sequence $J_k(i)$ generated by value iteration

$$J_{k+1}(i) = \min_{u \in U(i)} \left[g(i,u) + \sum_{j=1}^{n} p_{ij}(u) J_k(j) \right], \ \forall\ i$$

converges to the optimal cost $J^*(i)$ for each i.

- Bellman's equation has $J^*(i)$ as unique solution:

$$J^*(i) = \min_{u \in U(i)} \left[g(i,u) + \sum_{j=1}^{n} p_{ij}(u) J^*(j) \right], \ \forall\ i$$

- A stationary policy μ is optimal if and only if for every state i, $\mu(i)$ attains the minimum in Bellman's equation.

- Key proof idea: The "tail" of the cost series,

$$\sum_{k=mK}^{\infty} E\left\{ g\big(x_k, \mu_k(x_k)\big) \right\}$$

vanishes as K increases to ∞.

BELLMAN'S EQUATION FOR A SINGLE POLICY

- Consider a stationary policy μ

- $J_\mu(i)$, $i = 1, \ldots, n$, are the unique solution of the linear system of n equations

$$J_\mu(i) = g\big(i, \mu(i)\big) + \sum_{j=1}^{n} p_{ij}\big(\mu(i)\big) J_\mu(j), \quad \forall\, i = 1, \ldots, n$$

- Proof: This is just Bellman's equation for a modified/restricted problem where there is only one policy, the stationary policy μ, i.e., the control constraint set at state i is $\tilde{U}(i) = \{\mu(i)\}$

- The equation provides a way to compute $J_\mu(i)$, $i = 1, \ldots, n$, but the computation is substantial for large n [$O(n^3)$]

- For large n, value iteration may be preferable. (Typical case of a large linear system of equations, where an iterative method may be better than a direct solution method.)

POLICY ITERATION

- It generates a sequence $\mu^1, \mu^2, \ldots$ of stationary policies, starting with any stationary policy μ^0.

- At the typical iteration, given μ^k, we perform a *policy evaluation step*, that computes the $J_{\mu^k}(i)$ as the solution of the (linear) system of equations

$$J(i) = g\big(i, \mu^k(i)\big) + \sum_{j=1}^{n} p_{ij}\big(\mu^k(i)\big) J(j), \quad i = 1, \ldots, n,$$

in the n unknowns $J(1), \ldots, J(n)$. We then perform a *policy improvement step*, which computes a new policy μ^{k+1} as

$$\mu^{k+1}(i) = \arg\min_{u \in U(i)} \left[g(i, u) + \sum_{j=1}^{n} p_{ij}(u) J_{\mu^k}(j) \right], \ \forall\, i$$

- The algorithm stops when $J_{\mu^k}(i) = J_{\mu^{k+1}}(i)$ for all i

- Note the connection with the rollout algorithm, which is just a single policy iteration

JUSTIFICATION OF POLICY ITERATION

- We can show that$J_{\mu^{k+1}}(i) \leq J_{\mu^k}(i)$ for all i, k
- Fix k and consider the sequence generated by

$$J_{N+1}(i) = g\big(i, \mu^{k+1}(i)\big) + \sum_{j=1}^{n} p_{ij}\big(\mu^{k+1}(i)\big) J_N(j)$$

where $J_0(i) = J_{\mu^k}(i)$. We have

$$J_0(i) = g\big(i, \mu^k(i)\big) + \sum_{j=1}^{n} p_{ij}\big(\mu^k(i)\big) J_0(j)$$

$$\geq g\big(i, \mu^{k+1}(i)\big) + \sum_{j=1}^{n} p_{ij}\big(\mu^{k+1}(i)\big) J_0(j) = J_1(i)$$

Using the monotonicity property of DP,

$$J_0(i) \geq J_1(i) \geq \cdots \geq J_N(i) \geq J_{N+1}(i) \geq \cdots, \qquad \forall\, i$$

Since $J_N(i) \to J_{\mu^{k+1}}(i)$ as $N \to \infty$, we obtain $J_{\mu^k}(i) = J_0(i) \geq J_{\mu^{k+1}}(i)$ for all i. Also if $J_{\mu^k}(i) = J_{\mu^{k+1}}(i)$ for all i, J_{μ^k} solves Bellman's equation and is therefore equal to J^*

- A policy cannot be repeated, there are finitely many stationary policies, so the algorithm terminates with an optimal policy

LINEAR PROGRAMMING

- We claim that J^* is the "largest" J that satisfies the constraint

$$J(i) \le g(i,u) + \sum_{j=1}^{n} p_{ij}(u) J(j), \qquad (1)$$

for all $i = 1, \ldots, n$ and $u \in U(i)$.

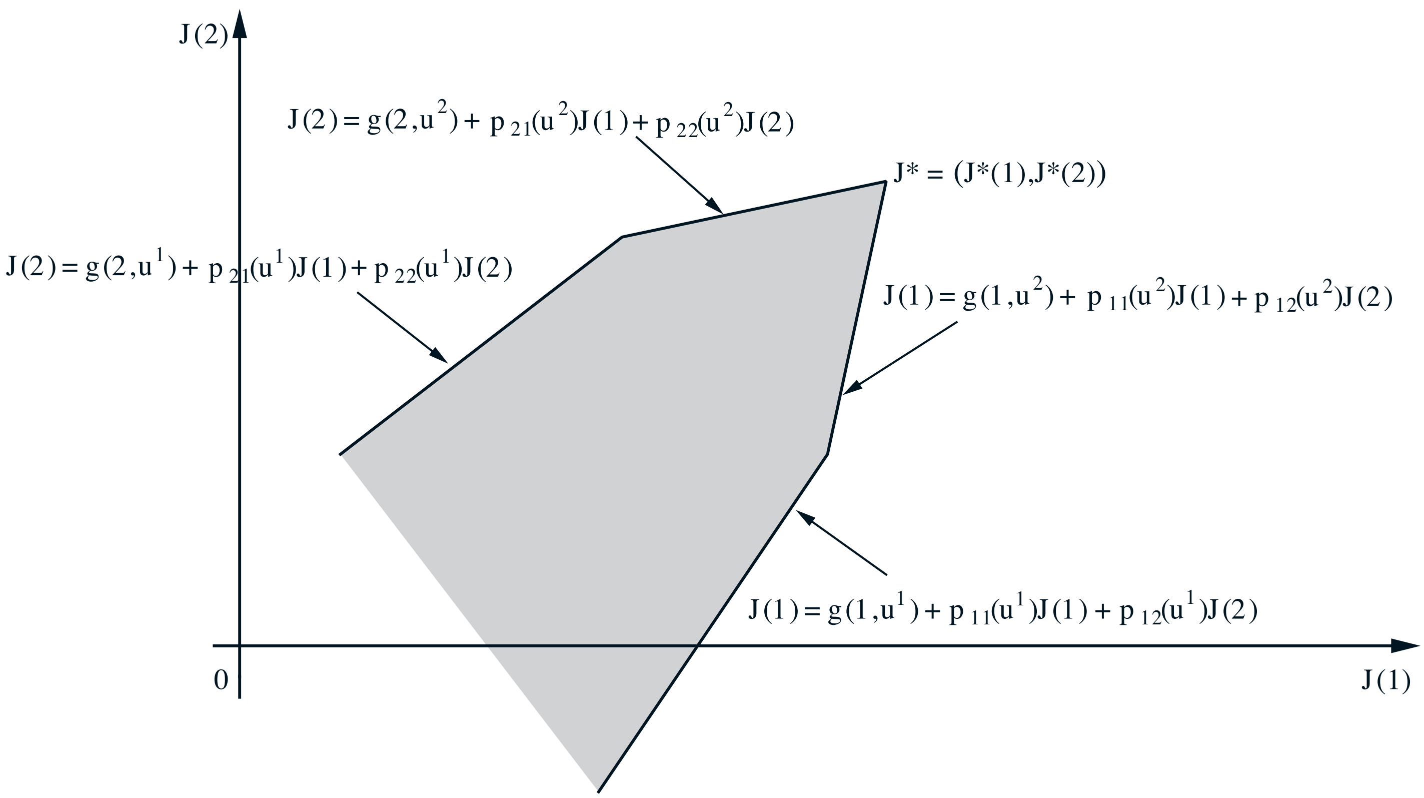
- Proof: If we use value iteration to generate a sequence of vectors $J_k = \big(J_k(1), \ldots, J_k(n)\big)$ starting with a J_0 such that

$$J_0(i) \le \min_{u \in U(i)} \left[g(i,u) + \sum_{j=1}^{n} p_{ij}(u) J_0(j) \right], \quad \forall\ i$$

Then, $J_k(i) \le J_{k+1}(i)$ for all k and i (monotonicity property of DP) and $J_k \to J^*$, so that $J_0(i) \le J^*(i)$ for all i.

- So $J^* = (J^*(1), \ldots, J^*(n))$ is the solution of the linear program of maximizing $\sum_{i=1}^{n} J(i)$ subject to the constraint (1).

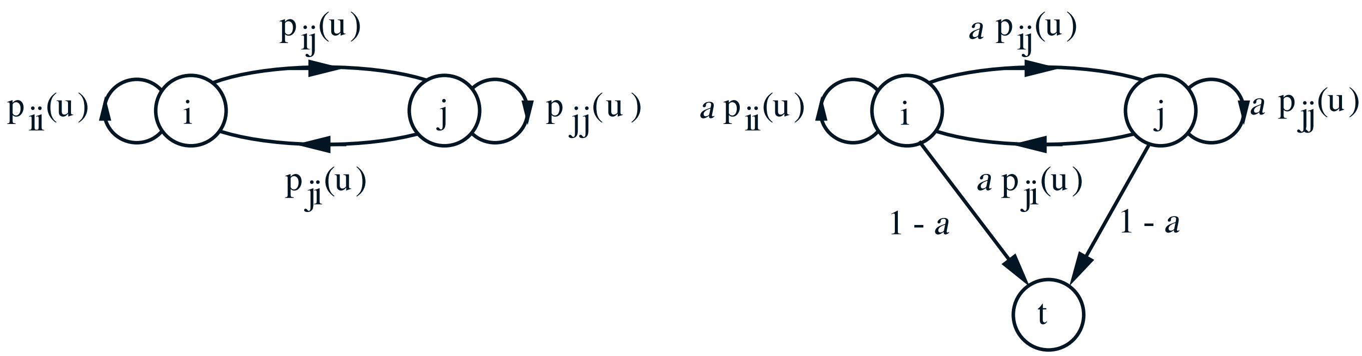
LINEAR PROGRAMMING (CONTINUED)



- Drawback: For large n the dimension of this program is very large. Furthermore, the number of constraints is equal to the number of state-control pairs.

DISCOUNTED PROBLEMS

- Assume a discount factor $\alpha < 1$.
- Conversion to an SSP problem.



- Value iteration converges to J^* for all initial J_0:

$$J_{k+1}(i) = \min_{u \in U(i)} \left[g(i,u) + \alpha \sum_{j=1}^{n} p_{ij}(u) J_k(j) \right], \ \forall\, i$$

- J^* is the unique solution of Bellman's equation:

$$J^*(i) = \min_{u \in U(i)} \left[g(i,u) + \alpha \sum_{j=1}^{n} p_{ij}(u) J^*(j) \right], \ \forall\, i$$

DISCOUNTED PROBLEMS (CONTINUED)

- Policy iteration converges finitely to an optimal policy, and linear programming works.

- Example: Asset selling over an infinite horizon. If accepted, the offer x_k of period k, is invested at a rate of interest r.

- By depreciating the sale amount to period 0 dollars, we view $(1+r)^{-k}x_k$ as the reward for selling the asset in period k at a price x_k, where $r > 0$ is the rate of interest. So the discount factor is $\alpha = 1/(1+r)$.

- J^* is the unique solution of Bellman's equation

$$J^*(x) = \max\left[x, \frac{E\{J^*(w)\}}{1+r}\right].$$

- An optimal policy is to sell if and only if the current offer x_k is greater than or equal to $\bar{\alpha}$, where

$$\bar{\alpha} = \frac{E\{J^*(w)\}}{1+r}.$$