6.852: Distributed Algorithms
Fall, 2009

Class 8

Today's plan

- Basic asynchronous system model, continued
  - Hierarchical proofs
  - Safety and liveness properties
- Asynchronous networks
- Asynchronous network algorithms:
  - Leader election in a ring
  - Leader election in a general network
- Reading: Sections 8.5.3 and 8.5.5, Chapter 14, Sections 15.1-15.2.
- Next:
  - Constructing a spanning tree
  - Breadth-first search
  - Shortest paths
  - Minimum spanning trees
  - Reading: Section 15.3-15.5, [Gallager, Humblet, Spira]

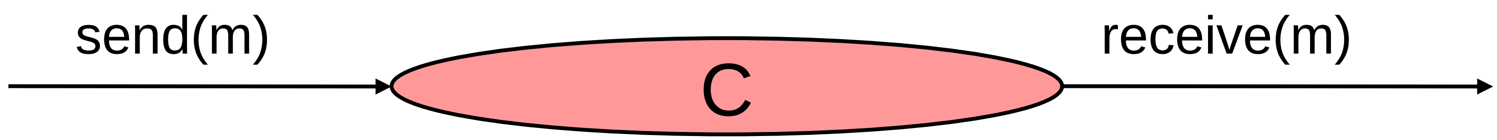
Last time

- Defined basic math framework for modeling asynchronous systems.
- I/O automata
- Executions, traces
- Operations: Composition, hiding
- Proof methods and concepts
  - Compositional methods
  - Invariants
  - Trace properties, including safety and liveness properties.
  - Hierarchical proofs

Input/output automaton

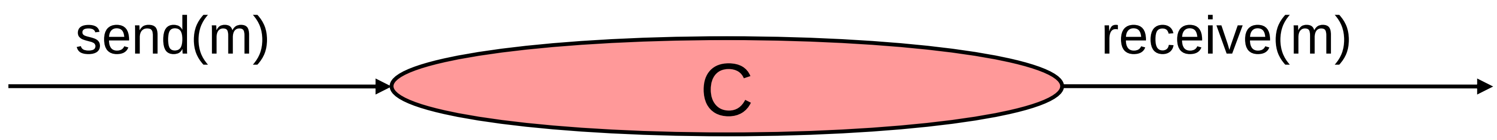
- sig = (in, out, int)
  - input, output, internal actions (disjoint)
  - acts = in $\cup$ out $\cup$ int
  - ext = in $\cup$ out
  - local = out $\cup$ int
- states: Not necessarily finite
- start $\subseteq$ states
- trans $\subseteq$ states $\times$ acts $\times$ states
  - Input-enabled: Any input "enabled" in any state.
- tasks, partition of locally controlled actions
  - Used for liveness.

Channel automaton



- Reliable unidirectional FIFO channel between two processes.
  - Fix message alphabet M.
- signature
  - input actions: send(m), $m \in M$
  - output actions: receive(m), $m \in M$
  - no internal actions
- states
  - queue: FIFO queue of M, initially empty

Channel automaton



- trans
  - send(m)
    - effect: add m to (end of) queue
  - receive(m)
    - precondition: m is at head of queue
    - effect: remove head of queue
- tasks
  - All receive actions in one task.

Executions

- An I/O automaton executes as follows:
  - Start at some start state.
  - Repeatedly take step from current state to new state.
- Formally, an execution is a finite or infinite sequence:
  - $s_0\ \pi_1\ s_1\ \pi_2\ s_2\ \pi_3\ s_3\ \pi_4\ s_4\ \pi_5\ s_5$ ... (if finite, ends in state)
  - s_0 is a start state
  - $(s_i, \pi_{i+1}, s_{i+1})$ is a step (i.e., in trans)

λ, send(a), a, send(b), ab, receive(a), b, receive(b), λ

Execution fragments

- An I/O automaton executes as follows:
  - Start at some start state.
  - Repeatedly take step from current state to new state.
- Formally, an ~~execution~~ **execution fragment** is a sequence:
  - $s_0\ \pi_1\ s_1\ \pi_2\ s_2\ \pi_3\ s_3\ \pi_4\ s_4\ \pi_5\ s_5\ \ldots$
  - ~~s_0 is a start state~~
  - $(s_i, \pi_{i+1}, s_{i+1})$ is a step.

Traces

- Models external behavior, useful for defining correctness.
- A trace of an execution is the subsequence of external actions in the execution.
  - Denoted trace(α), where α is an execution.
  - No states, no internal actions.

λ, send(a), a, send(b), ab, receive(a), b, receive(b), λ

send(a), send(b), receive(a), receive(b)

Composition of compatible automata

- Compose two automata A and B (see book for general case).
- $out(A \times B) = out(A) \cup out(B)$
- $int(A \times B) = int(A) \cup int(B)$
- $in(A \times B) = in(A) \cup in(B) – (out(A) \cup out(B))$
- $states(A \times B) = states(A) \times states(B)$
- $start(A \times B) = start(A) \times start(B)$
- $trans(A \times B)$: includes (s, π, s') iff
  - $(s_A, \pi, s'_A) \in trans(A)$ if $\pi \in acts(A)$; $s_A = s'_A$ otherwise.
  - $(s_B, \pi, s'_B) \in trans(B)$ if $\pi \in acts(B)$; $s_B = s'_B$ otherwise.
- $tasks(A \times B) = tasks(A) \cup tasks(B)$

- Notation: $\Pi_{i \in I} A_i$, for composition of $A_i : i \in I$ (I countable)

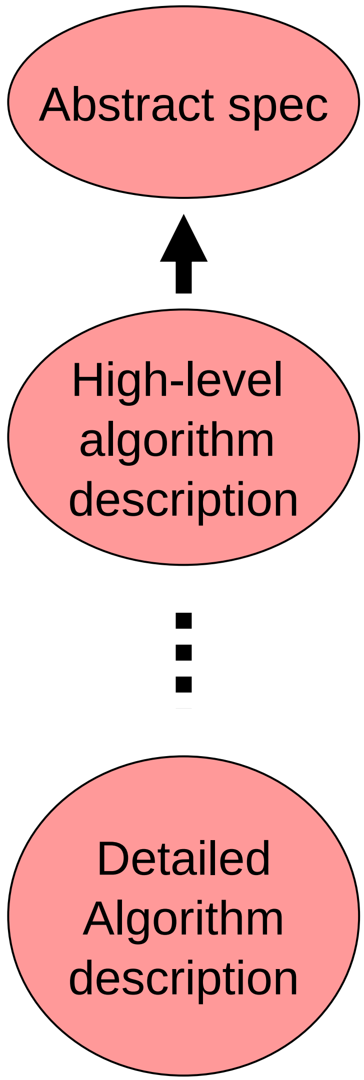
Hierarchical proofs

Hierarchical proofs

- Important strategy for proving correctness of complex asynchronous distributed algorithms.

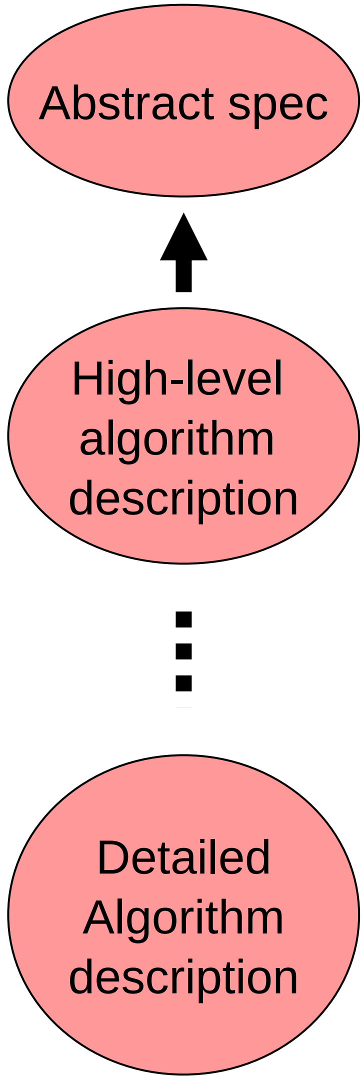
- Define a series of automata, each implementing the previous one (“successive refinement”).
- Highest-level = Problem specification.
- Then a high-level algorithm description.
- Then more and more detailed versions, e.g.:
  – High levels centralized, lower levels distributed.
  – High levels inefficient but simple, lower levels optimized and more complex.
  – High levels with large granularity steps, lower levels with finer granularity steps.

- Reason about lower levels by relating them to higher levels.
- Similar to what we did for synchronous algorithms.



Hierarchical proofs

- For synchronous algorithms (recall):
  - Optimized algorithm runs side-by-side with unoptimized version, and "invariant" proved to relate the states of the two algorithms.
  - Prove using induction.
- For asynchronous algorithms, it's harder:
  - Asynchronous model has more nondeterminism (in choice of new state, in order of steps).
  - So, harder to determine which execs to compare.
- One-way implementation is enough:
  - For each execution of the lower-level algorithm, there is a corresponding execution of the higher-level algorithm.
  - "Everything the algorithm does is allowed by the spec."
  - Don't need the other direction: doesn't matter if the algorithm does everything that is allowed.



Simulation relations

- Most common method of proving that one automaton implements another.
- Assume A and B have the same extsig, and R is a relation from states(A) to states(B).
- Then R is a simulation relation from A to B provided:
  - $s_A \in$ start(A) implies there exists $s_B \in$ start(B) such that s_A R s_B.
  - If s_A, s_B are reachable states of A and B, s_A R s_B and (s_A, π, s'_A) is a step, then there is an execution fragment β starting with s_B and ending with s'_B such that s'_A R s'_B and trace(β) = trace(π).

Simulation relations

$$
\begin{tikzcd}
s_B \arrow[r, dotted, "\beta"] \arrow[d, dash, "R"] & s'_B \arrow[d, dotted, dash, "R"] \\
s_A \arrow[r, "\pi"] & s'_A
\end{tikzcd}
$$

- R is a simulation relation from A to B provided:
  - $s_A \in$ start(A) implies $\exists s_B \in$ start(B) such that s_A R s_B.
  - If s_A, s_B are reachable states of A and B, s_A R s_B and (s_A, π, s'_A) is a step, then $\exists\beta$ starting with s_B and ending with s'_B such that s'_A R s'_B and trace(β) = trace(π).

Simulation relations

- Theorem: If there is a simulation relation from A to B then traces(A) $\subseteq$ traces(B).
- This means all traces of A, not just finite traces.
- Proof: Fix a trace of A, arising from a (possibly infinite) execution of A.
- Create a corresponding execution of B, using an iterative construction.

$$s_{0,A} \xrightarrow{\pi_1} s_{1,A} \xrightarrow{\pi_2} s_{2,A} \xrightarrow{\pi_3} s_{3,A} \xrightarrow{\pi_4} s_{4,A} \xrightarrow{\pi_5} s_{5,A}$$

Simulation relations

- Theorem: If there is a simulation relation from A to B then traces(A) $\subseteq$ traces(B).

$$\begin{tikzcd}
s_{0,B} \arrow[d, dash, dotted, "R"] & & & & & \\
s_{0,A} \arrow[r, "\pi_1"] & s_{1,A} \arrow[r, "\pi_2"] & s_{2,A} \arrow[r, "\pi_3"] & s_{3,A} \arrow[r, "\pi_4"] & s_{4,A} \arrow[r, "\pi_5"] & s_{5,A}
\end{tikzcd}$$

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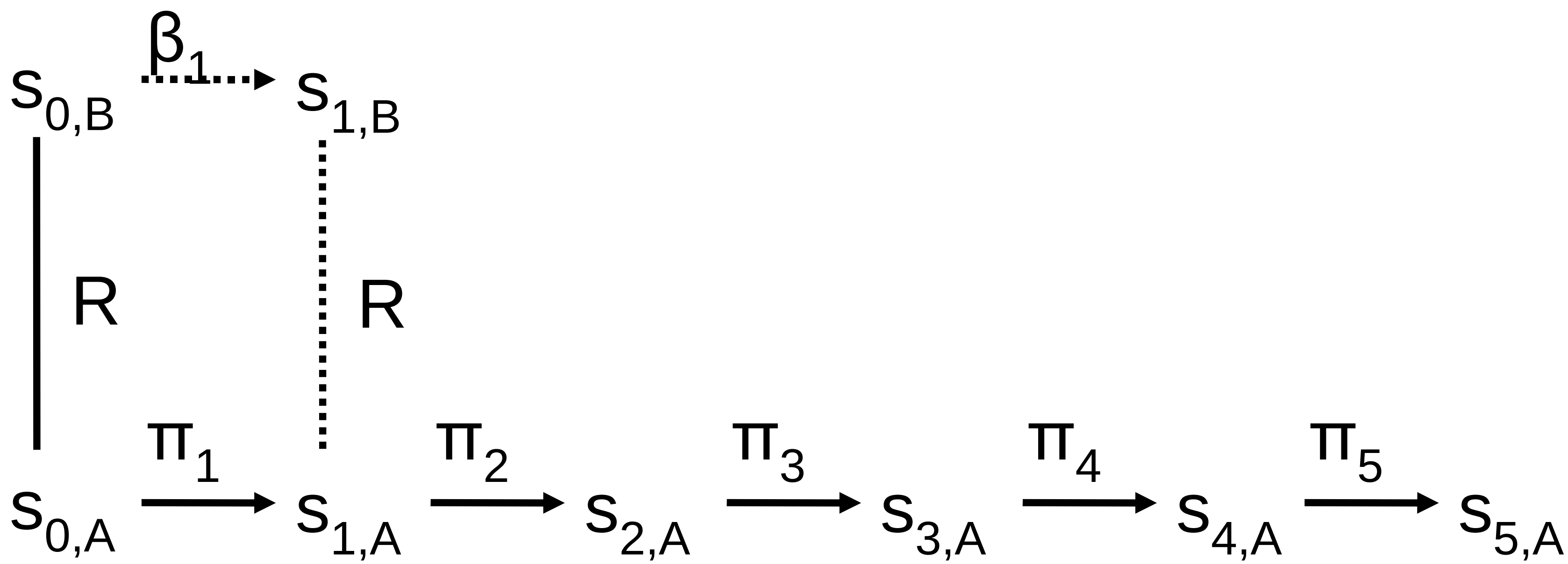

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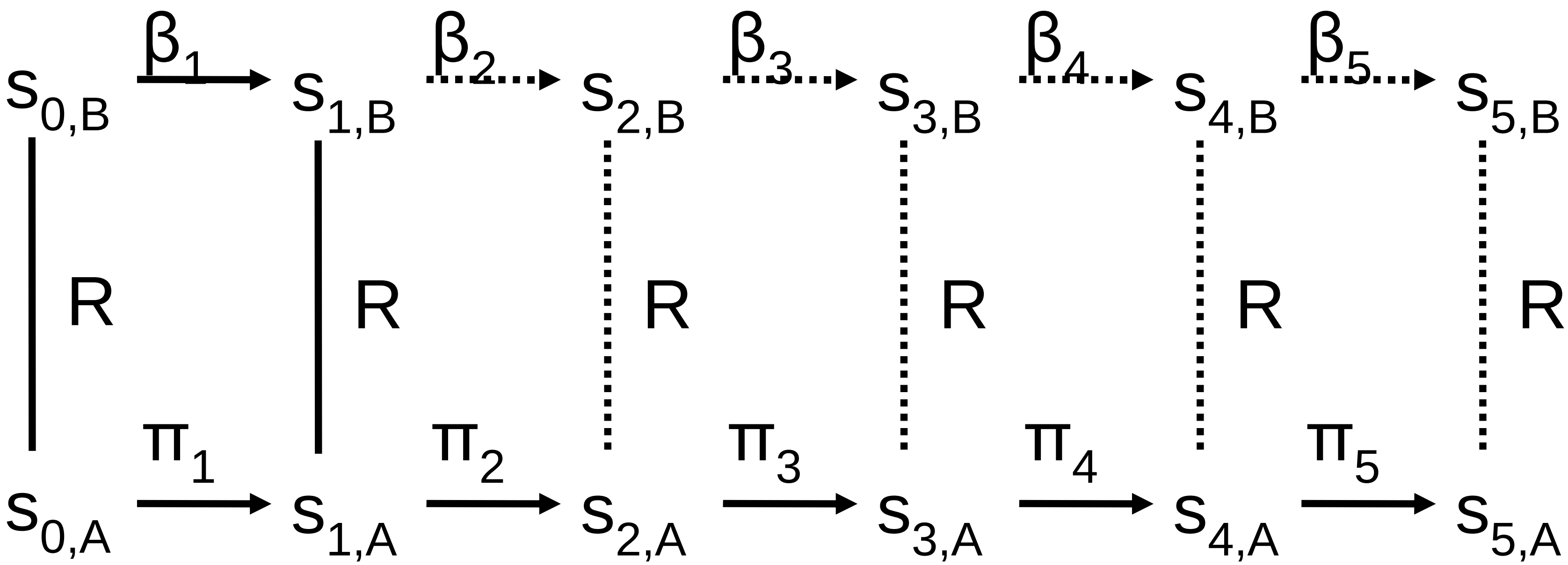
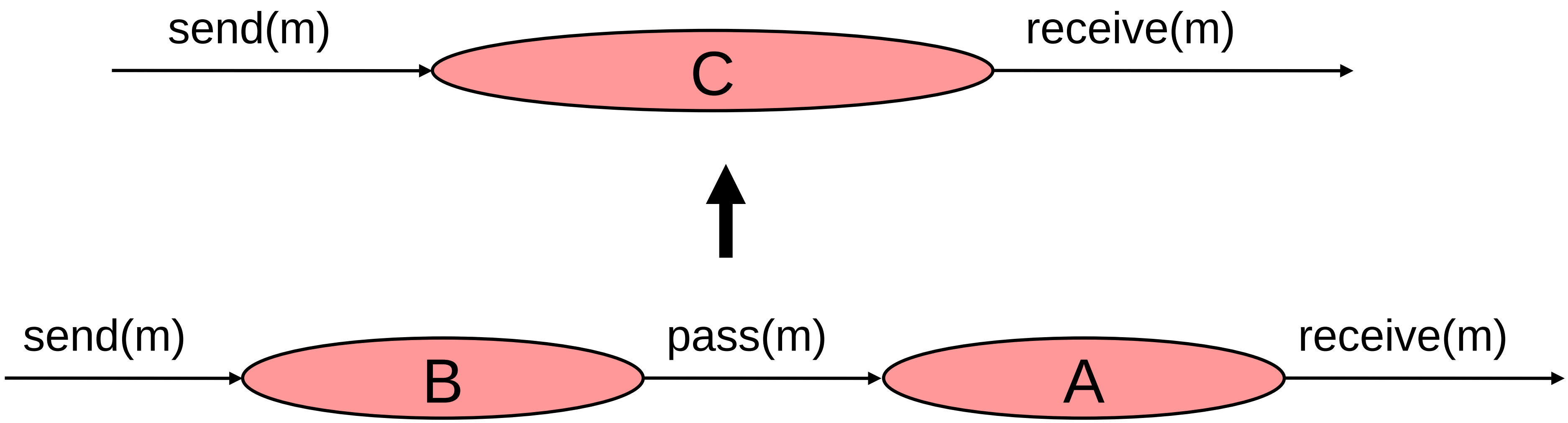


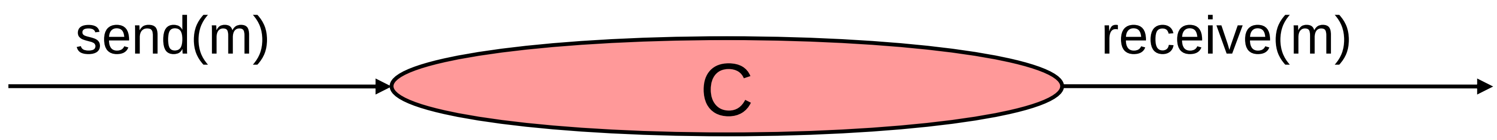
Example: Channels

- Show two channels implement one.



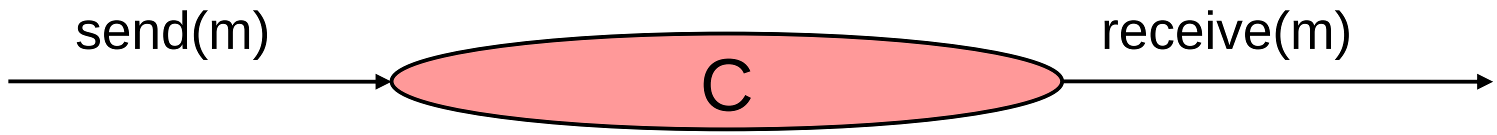
- Rename some actions.
- Claim that $D = \text{hide}_{\{pass(m)\}}\ A \times B$ implements C, in the sense that $traces(D) \subseteq traces(C)$.

Recall: Channel automaton



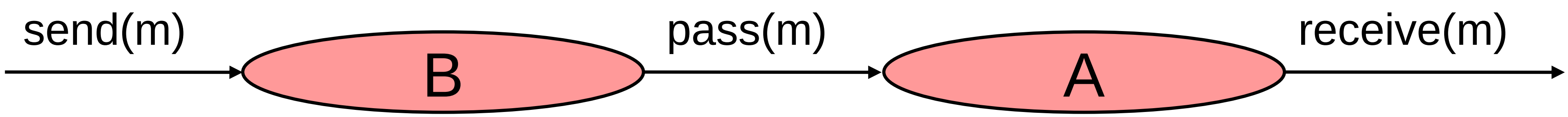
- Reliable unidirectional FIFO channel.
- signature
  - Input actions: send(m), m $\in$ M
  - output actions: receive(m), m $\in$ M
  - no internal actions
- states
  - queue: FIFO queue of M, initially empty

Channel automaton



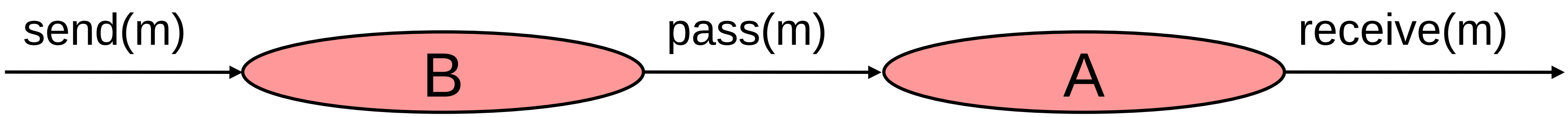
- trans
  - send(m)
    - effect: add m to queue
  - receive(m)
    - precondition: m = head(queue)
    - effect: remove head of queue
- tasks
  - All receive actions in one task

Composing two channel automata



- Output of B is input of A
  - Rename receive(m) of B and send(m) of A to pass(m).
- $D = \text{hide}_{\{\text{ pass(m) } \mid \text{ m} \in \text{M }\}} A \times B$ implements C
- Define simulation relation R:
  - For s $\in$ states(D) and u $\in$ states(C), s R u iff u.queue is the concatenation of s.A.queue and s.B.queue
- Proof that this is a simulation relation:
  - Start condition: All queues are empty, so start states correspond.
  - Step condition: Define "step correspondence":

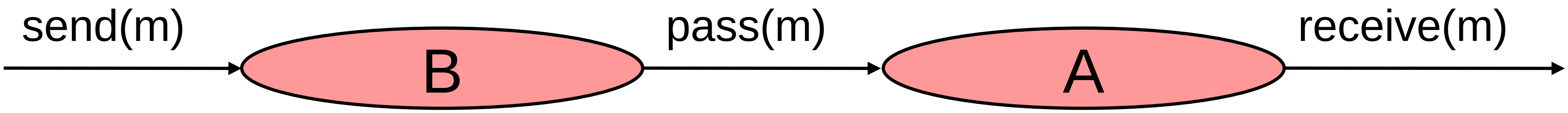
Composing two channel automata



s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Step correspondence:
  - For each step $(s, \pi, s') \in$ trans(D) and u such that s R u, define execution fragment β of C:
    - Starts with u, ends with u′ such that s′ R u′.
    - trace(β) = trace(π)
  - Here, actions in β happen to depend only on π, and uniquely determine post-state.
    - Same action if external, empty sequence if internal.

Composing two channel automata



s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Step correspondence:
  - π = send(m) in D corresponds to send(m) in C
  - π = receive(m) in D corresponds to receive(m) in C
  - π = pass(m) in D corresponds to λ in C
- Verify that this works:
  - Actions of C are enabled.
  - Final states related by relation R.
- Routine case analysis:

Showing R is a simulation relation

s R u iff u.queue is concatenation of s.A.queue and s.B.queue

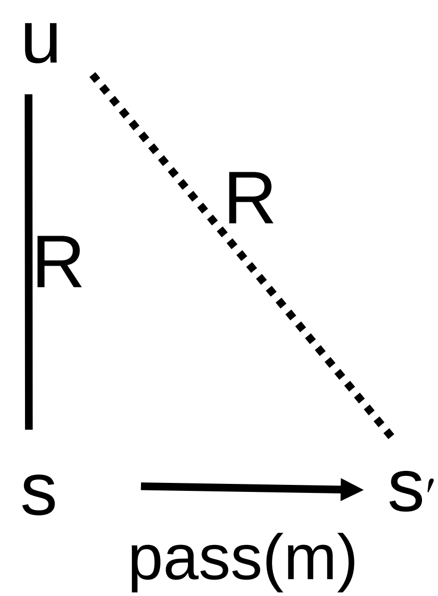
- Case: π = send(m)
  - No enabling issues (input).
  - Must check s′ R u′.
    - Since s R u, u.queue is the concatenation of s.A.queue and s.B.queue.
    - Adding the same m to the end of u.queue and s.B.queue maintains the correspondence.

- Case: π = receive(m)
  - Enabling: Check that receive(m), for the same m, is also enabled in u.
    - We know that m is first on s.A.queue.
    - Since s R u, m is first on u.queue.
    - So enabled in u.
  - s′ R u′: Since m removed from both s.A.queue and u.queue.

Showing R is a simulation relation

s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Case: π = pass(m)
  - No enabling issues (since no high-level steps are involved).
  - Must check s′ R u:
    - Since s R u, u.queue is the concatenation of s.A.queue and s.B.queue.
    - Concatenation is unchanged as a result of this step, so also u.queue is the concatenation of s′.A.queue and s′.B.queue.



Safety and liveness properties

Specifications

- Trace property:
  - Problem specification in terms of external behavior.
  - (sig(P), traces(P))
- Automaton A satisfies trace property P if extsig(A) = sig(P) and (two different notions, depending on whether we're interested in liveness or not):
  - traces(A) $\subseteq$ traces(P), or
  - fairtraces(A) $\subseteq$ traces(P).

- All the problems we'll consider for asynchronous systems can be formulated as trace properties.
- And we'll usually be concerned about liveness, so will use the second notion.

Safety property S

- traces(S) are nonempty, prefix-closed, and limit-closed.
- “Something bad” never happens.
- Violations occur at some finite point in the sequence.

- Examples (we’ll see all these later):
  - Consensus: Agreement, validity
    - Describe as set of sequences of init and decide actions in which we never disagree, or never violate validity.
  - Graph algorithms: Correct shortest paths, correct minimum spanning trees,…
    - Outputs do not yield any incorrect answers.
  - Mutual exclusion: No two grants without intervening returns.

Proving a safety property

- That is, prove that all traces of A satisfy S.
- By limit-closure, it's enough to prove that all finite traces satisfy S.
- Can do this by induction on length of trace.
- Using invariants:
  - For most trace safety properties, can find a corresponding invariant.
  - Example: Consensus
    - Record decisions in the state.
    - Express agreement and validity in terms of recorded decisions.
  - Then prove the invariant as usual, by induction.

Liveness property L

- Every finite sequence over sig(L) has some extension in traces(L).
- Examples:
  - Temination: No matter where we are, we could still terminate in the future.
  - Some event happens infinitely often.

- Proving liveness properties:
  - Measure progress toward goals, using progress functions.
  - Intermediate milestones.
  - Formal reasoning using temporal logic.
  - Methods less well-established than those for safety properties.

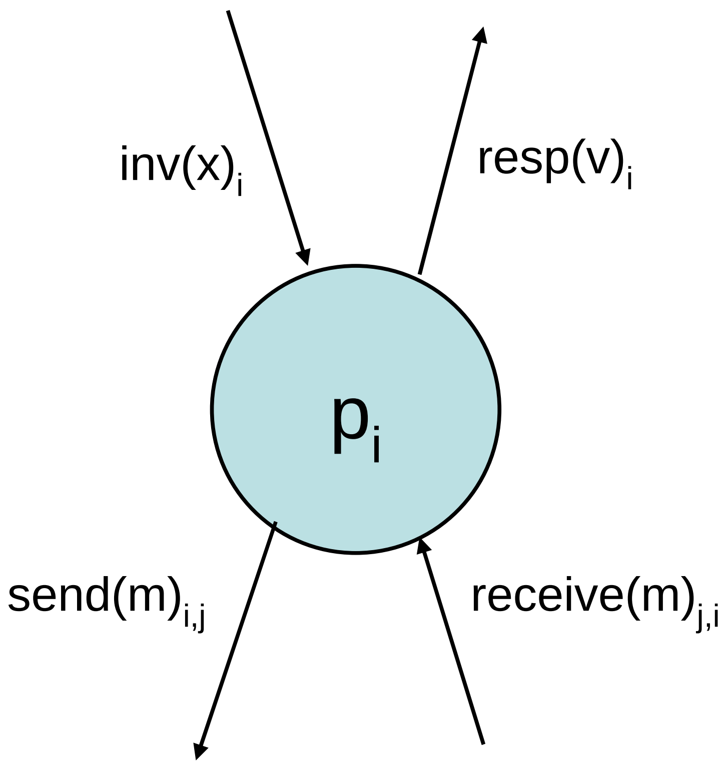
Safety and liveness

- Theorem: Every trace property can be expressed as the intersection of a safety and a liveness property.
- So, to specify a property, it’s enough to specify safety requirements and liveness requirements separately.
- Typical specifications of problems for asynchronous systems consist of:
  - A list of safety properties.
  - A list of liveness properties.
  - Nothing else.

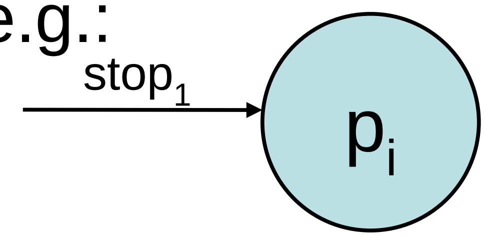
Asynchronous network model

Send/receive systems

- Digraph G = (V,E), with:
  - Process automata associated with nodes, and
  - Channel automata associated with directed edges.
- Model processes and channels as automata, compose.
- Processes

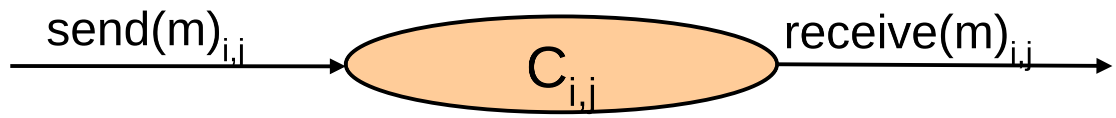


- User interface: inv, resp.
- Problems specified in terms of allowable traces at user interface
  - Hide send/receive actions
- Failure modeling, e.g.:



- Having explicit stop actions in external interface allows problems to be stated in terms of occurrence of failures.

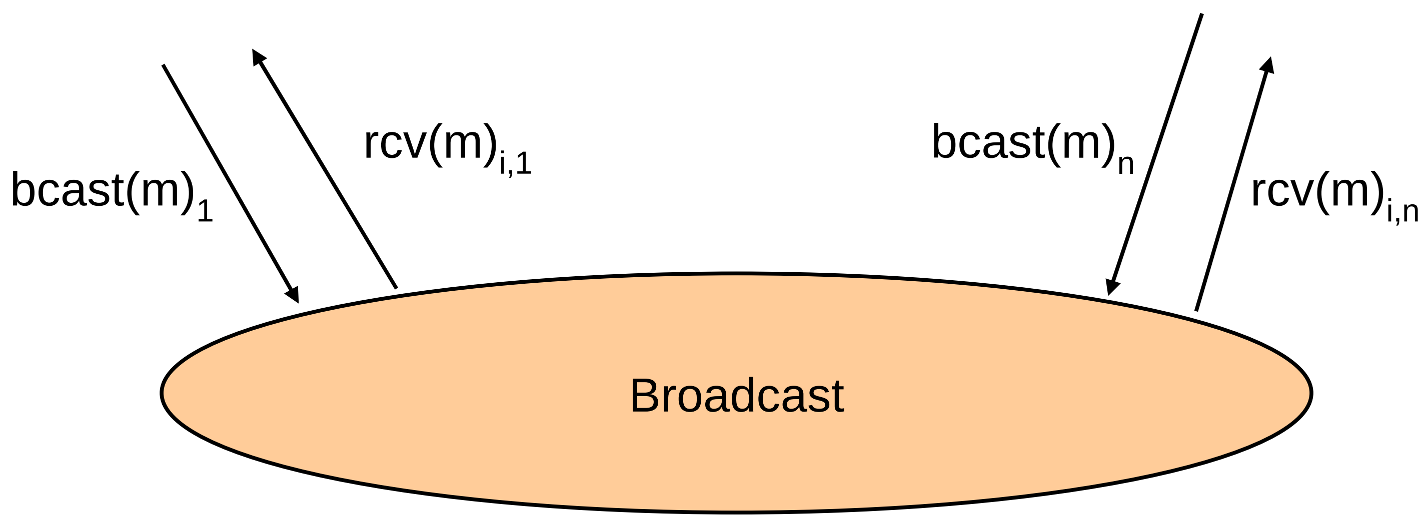
Channel automata



- Different kinds of channel with this interface:
  - Reliable FIFO, as before.
  - Weaker guarantees: Lossy, duplicating, reordering
- Can define channels by trace properties, using a "cause" function mapping receives to sends.
  - Integrity: Cause function preserves message.
  - No loss: Function is onto (surjective).
  - No duplicates: Function is 1-1 (injective).
  - No reordering: Function is order-preserving.
- Reliable channel satisfies all of these; weaker channels satisfy Integrity but weaken some of the other properties.

Broadcast and multicast

- Broadcast
  - Reliable FIFO between each pair.
  - Different processes can receive msgs from different senders in different orders.
  - Model using separate queues for each pair.
- Multicast: Processes designate recipients.
- Also consider bcast, mcast with failures, and/or with additional consistency conditions.



Asynchronous network algorithms

Asynchronous network algorithms

- Assume reliable FIFO point-to-point channels
- Revisit problems we considered in synchronous networks:
  - Leader election:
    - In a ring.
    - In general undirected networks.
  - Spanning tree
  - Breadth-first search
  - Shortest paths
  - Minimum spanning tree
- How much carries over?
  - Where did we use synchrony assumption?

Leader election in a ring

- Assumptions:
  - G is a ring, unidirectional or bidirectional communication
  - Local names for neighbors, UIDs
- LeLann-Chang-Roberts (AsynchLCR)
  - Send UID clockwise around ring (unidirectional).
  - Discard UIDs smaller than your own.
  - Elect self if your UID comes back.
  - Correctness: Basically the same as for synchronous version, with a few complications:
    - Finer granularity, consider individual steps rather than entire rounds.
    - Must consider messages in channels.

AsynchLCR, process i

- Signature
  - ***in*** $rcv(v)_{i-1,i}$, v is a UID
  - ***out*** $send(v)_{i,i+1}$, v is a UID
  - ***out*** $leader_i$
- State variables
  - **u**: UID, initially i's UID
  - **send**: FIFO queue of UIDs, initially containing i's UID
  - **status**: unknown, chosen, or reported, initially unknown
- Tasks
  - { $send(v)_{i,i+1}$ | v is a UID } and { $leader_i$ }

Transitions

- $send(v)_{i,i+1}$
  pre: v = head(**send**)
  eff: remove head of **send**
- $receive(v)_{i-1,i}$
  eff:
  if v = **u** then **status** := chosen
  if v > **u** then add v to **send**
- $leader_i$
  pre: **status** = chosen
  eff: **status** := reported

AsynchLCR properties

- Safety: No process other than i_{max} ever performs $leader_i$.
- Liveness: i_{max} eventually performs $leader_i$.

Safety proof

- Safety: No process other than i_{max} ever performs $leader_i$.

- Recall synchronous proof, based on showing invariant of global states, after any number of rounds:
  - If $i \neq i_{max}$ and $j \in [i_{max}, i)$ then u_i not in $send_j$.
- Can use a similar invariant for the asynchronous version.
- But now the invariant must hold after any number of steps:
  - If $i \neq i_{max}$ and $j \in [i_{max}, i)$ then u_i not in $send_j$ or in $queue_{j,j+1}$.
- Prove by induction on number of steps.
  - Use cases based on type of action.
  - Key case: $receive(v)_{imax-1,\ imax}$
    - Argue that if $v \neq u_{max}$ then v gets discarded.

Liveness proof

- Liveness: i_{max} eventually performs $leader_i$.

- Synchronous proof used an invariant saying exactly where the max is after r rounds.
- Now no rounds, need a different proof.
- Can establish intermediate milestones:
  - For $k \in [0,n-1]$, u_{max} eventually in $send_{imax+k}$
  - Prove by induction on k; use fairness for process and channel to prove inductive step.

Complexity

- Msgs: $O(n^2)$, as before.

- Time: $O(n(l+d))$
  - l is an upper bound on local step time for each process (that is, for each process task).
  - d is an upper bound on time to deliver first message in each channel (that is, for each channel task).
  - Measuring real time here (not counting rounds).
  - Only upper bounds, so does not restrict executions.
  - Bound still holds in spite of the possibility of “pileups” of messages in channels and send buffers.
    - Pileups can be interpreted as meaning that some tokens have sped up.
    - See analysis in book.

Reducing the message complexity

- Hirschberg-Sinclair:
  - Sending in both directions, to successively doubled distances.
  - Extends immediately to asynchronous model.
  - O(n log n) messages.
  - Use bidirectional communication.
- Peterson's algorithm:
  - O(n log n) messages
  - Unidirectional communication
  - Unknown ring size
  - Comparison-based

Peterson's algorithm

- Proceed in asynchronous "phases" (may execute concurrently).
- In each phase, each process is active or passive.
  - Passive processes just pass messages along.
- In each phase, at least half of the active processes become passive; so at most log n phases until election.
- Phase 1:
  - Send UID two processes clockwise; collect two UIDs from predecessors.
  - Remain active iff the middle UID is max.
  - In this case, adopt middle UID (the max one).
  - Some process remains active (assuming $n \geq 2$), but no more than half.
- Later phases:
  - Same, except that the passive processes just pass messages on.
  - No more than half of those active at the beginning of the phase remain active.
- Termination:
  - If a process sees that its immediate predecessor's UID is the same as its own, elects itself the leader (knows it's the only active process left).

PetersonLeader

- Signature
  - ***in*** receive(v)$_{i-1,i}$, v is a UID
  - ***out*** send(v)$_{i,i+1}$, v is a UID
  - ***out*** leader$_i$
  - ***int*** get-second-uid$_i$
  - ***int*** get-third-uid$_i$
  - ***int*** advance-phase$_i$
  - ***int*** become-relay$_i$
  - ***int*** relay$_i$
- State variables
  - **mode**: active or relay, initially active
  - **status**: unknown, chosen, or reported, initially unknown
  - **uid1**; initially i's UID
  - **uid2**; initially null
  - **uid3**; initially null
  - **send**: FIFO queue of UIDs; initially contains i's UID
  - **receive**: FIFO queue of UIDs

PetersonLeader

- get-second-uid$_i$
  pre: **mode** = active
  receive is nonempty
  uid2 = null
  eff: **uid2** := head(**receive**)
  remove head of **receive**
  add **uid2** to **send**
  if **uid2** = **uid1** then
  status := chosen

- get-third-uid$_i$
  pre: **mode** = active
  receive is nonempty
  uid2 ≠ null
  uid3 = null
  eff: **uid3** := head(**receive**)
  remove head of **receive**

- advance-phase$_i$
  pre: **mode** = active
  uid3 ≠ null
  uid2 > max(**uid1**, **uid3**)
  eff: **uid1** := **uid2**
  uid2 := null
  uid3 := null
  add **uid1** to **send**

- become-relay$_i$
  pre: **mode** = active
  uid3 ≠ null
  uid2 ≤ max(**uid1**, **uid3**)
  eff: **mode** := relay

- relay$_i$
  pre: **mode** = relay
  receive is nonempty
  eff: move head(**receive**) to **send**

PetersonLeader

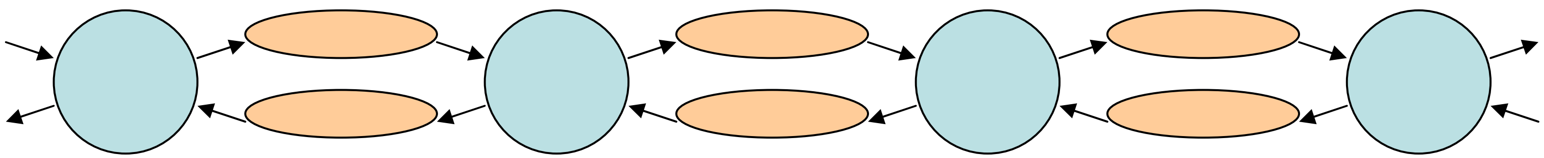
- Tasks:
  - { $\text{send}(v)_{i,i+1}$ | v is a UID }
  - { get-second-uid_i, get-third-uid_i, advance-phase_i, become-relay_i, relay_i }
  - { leader_i }
- Number of phases is O(log n)
- Complexity
  - Messages: O(n log n)
  - Time: O(n(l+d))

Leader election in a ring

- Can we do better than O(n log n) message complexity?
  - Not with comparison-based algorithms. (Why?)
  - Not at all: Can prove a lower bound.

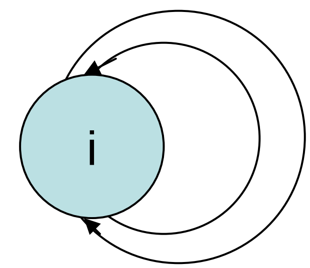
$\Omega(n \log n)$ lower bound

- Lower bound for leader election in asynchronous network.
- Assume:
  - Ring size n is unknown (algorithm must work in arbitrary size rings).
  - UIDS:
    - Chosen from some infinite set.
    - No restriction on allowable operations.
    - All processes identical except for UIDs.
  - Bidirectional communication allowed.
- Consider combinations of processes to form:
  - Rings, as usual.
  - Lines, where nothing is connected to the ends and no input arrives there.
  - Ring looks like line if communication delayed across ends.

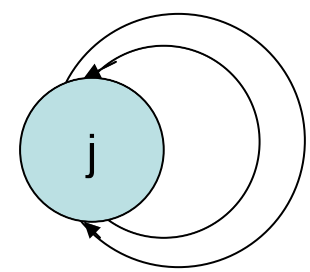


Ω(n log n) lower bound

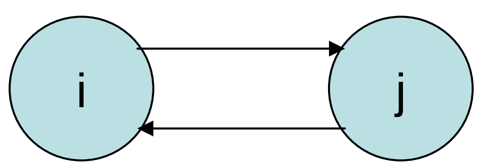
- Lemma 1: There are infinitely many process automata, each of which can send at least one message without first receiving one (in some execution).
- Proof:
  - If not, there are two processes i,j, neither of which ever sends a message without first receiving one.
  - Consider 1-node ring:
    - i must elect itself, with no messages sent or received.



  - Consider:
    - j must elect itself, with no messages sent or received.



  - Now consider:
    - Both i and j elect themselves, contradiction.

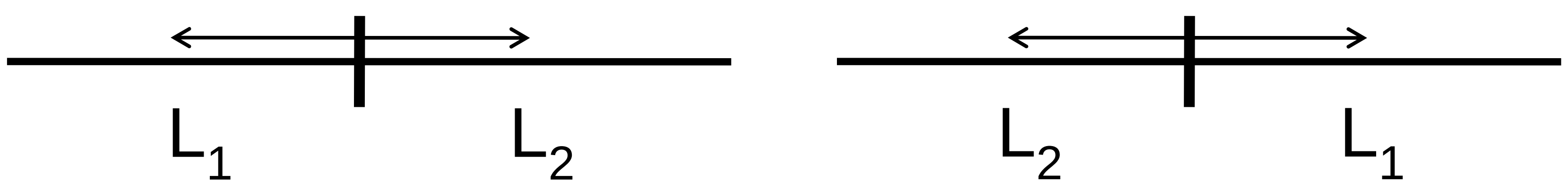


Ω(n log n) lower bound

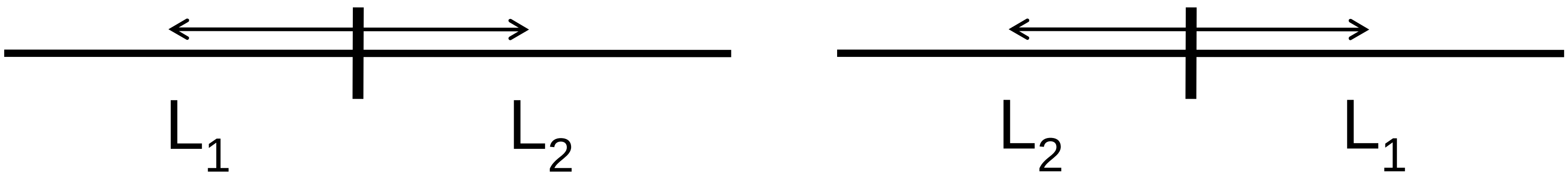
- C(L) = maximum (actually, supremum) of the number of messages that are sent in a single input-free execution of line L.

- Lemma 2: If L_1, L_2, L_3 are three line graphs of even length l such that $C(L_i) \geq k$ for i = 1, 2, 3, then $C(L_i \text{ join } L_j) \geq 2k + l/2$ for some $i \neq j$

- Proof:
  - Suppose not.
  - Consider two lines, L_1 join L_2 and L_2 join L_1.



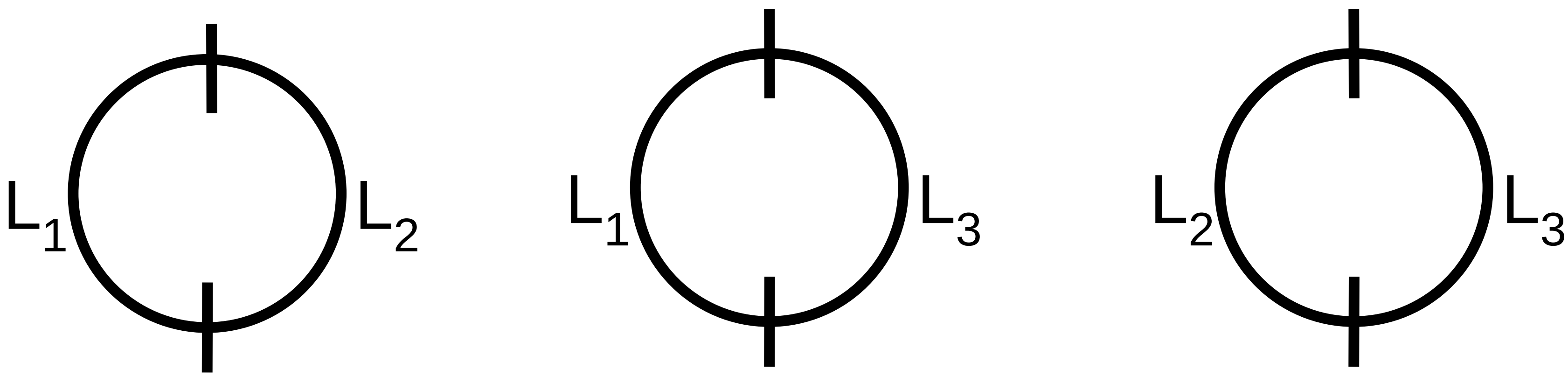
Proof of Lemma 2



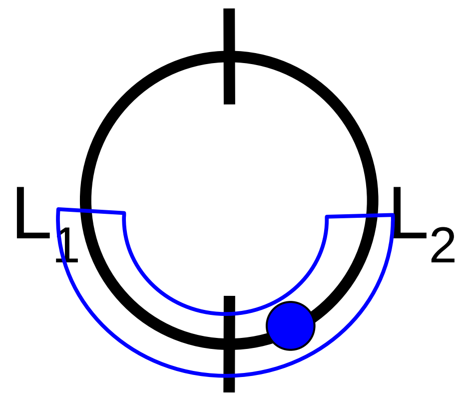
- Let α_i be finite execution of L_i with ≥ k messages.
- Run α_1 then α_2 then $\alpha_{1,2}$, an execution fragment of L_1 join L_2 beginning with messages arriving across the join boundary.
- By assumption, fewer than l/2 additional messages are sent in $\alpha_{1,2}$.
- So, the effects of the new inputs don't cross the middle edges of L_1 and L_2 before the system quiesces (no more messages sent).
- Similarly for $\alpha_{2,1}$, an execution of L_2 join L_1.

Proof of Lemma 2

- Now consider three rings:

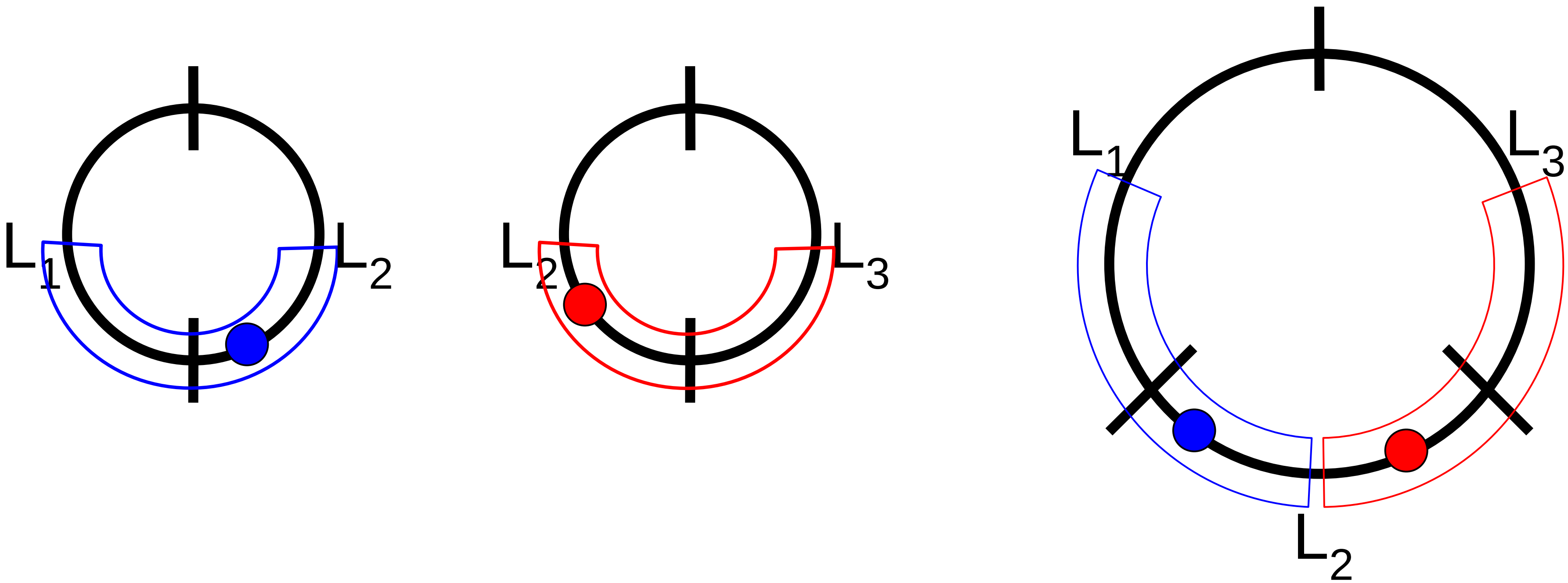


Proof of Lemma 2

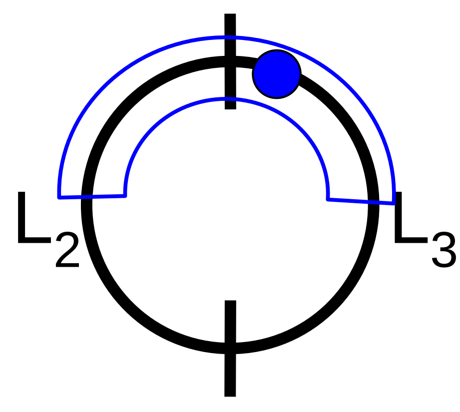


- Connect both ends of L_1 and L_2 .
  - Right neighbor in line is clockwise around ring.
- Run α_1 then α_2 then $\alpha_{1,2}$ then $\alpha_{2,1}$.
  - No interference between $\alpha_{1,2}$ and $\alpha_{2,1}$.
  - Quiesces: Eventually no more messages are sent.
  - Must elect leader (possibly in extension, but without any more messages).
- Assume WLOG that elected leader is in “bottom half”.

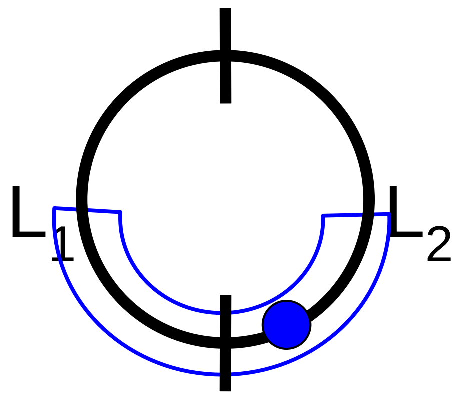
Proof of Lemma 2

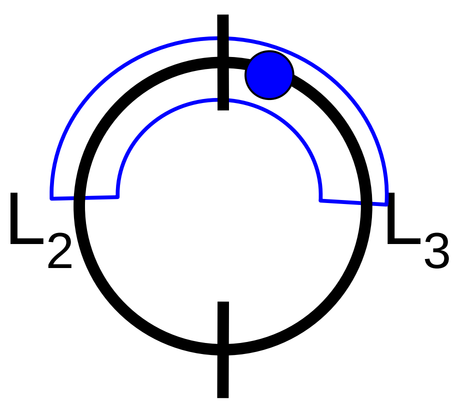


- Same argument for ring constructed from L_2 and L_3.
- Can leader be in bottom half?
- No!
- So must be in top half.

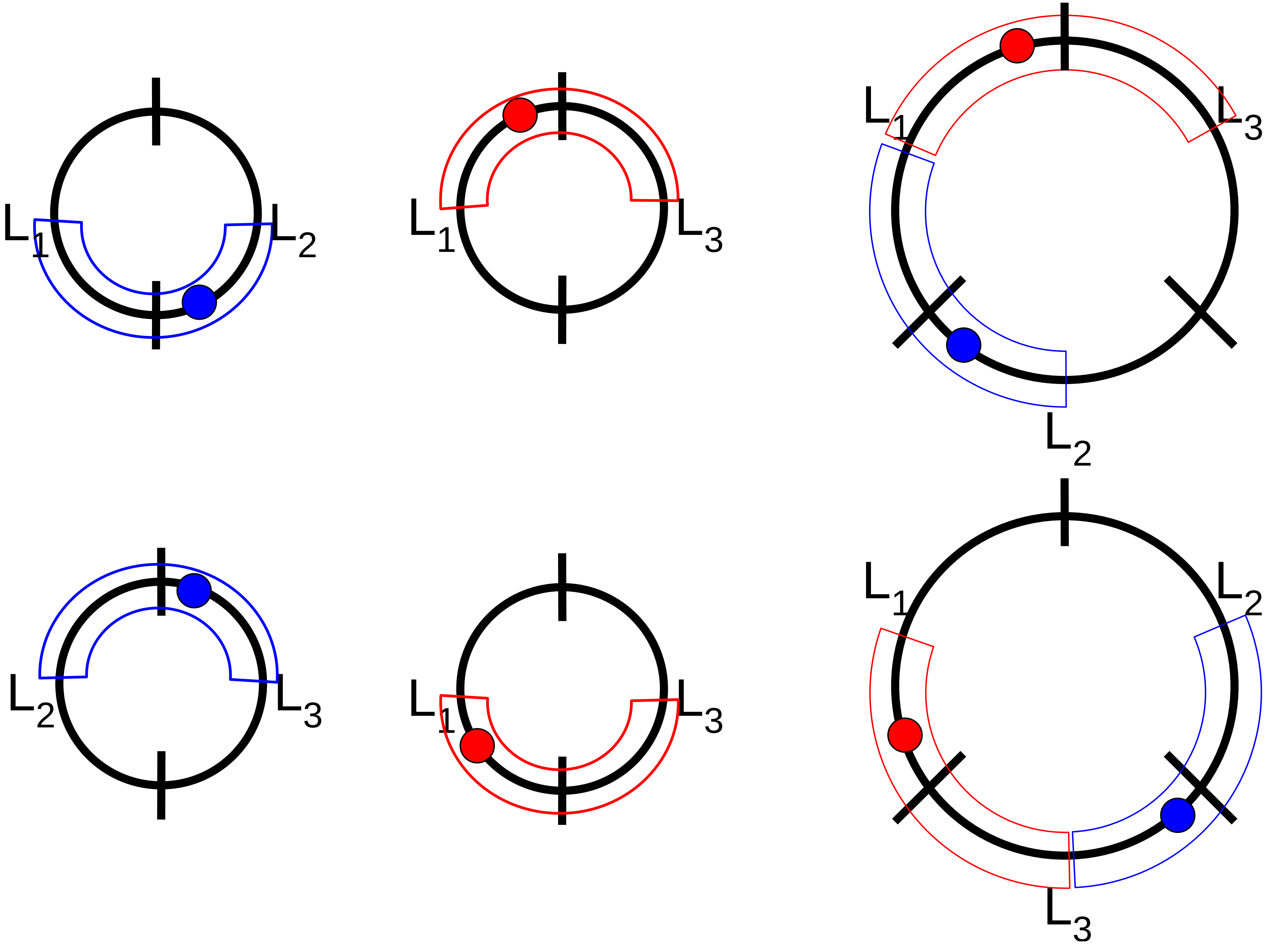


Proof of Lemma 2





Proof of Lemma 2



Lower bound, cont'd

- Summarizing, we have:
- Lemma 1: There are infinitely many process automata, each of which can send at least one message without first receiving one.
- Lemma 2: If L_1, L_2, L_3 are three line graphs of even length l such that $C(L_i) \geq k$ for all i, then $C(L_i \text{ join } L_j) \geq 2k + l/2$ for some $i \neq j$.
- Now combine:
- Lemma 3: For any $r \geq 0$, there are infinitely many disjoint line graphs L of length 2^r such that $C(L) \geq r\, 2^{r-2}$.
  - Base (r = 0): Trivial claim.
  - Base (r = 1): Use Lemma 1
    - Just need length-2 lines sending at least one message.
  - Inductive step ($r \geq 2$):
    - Choose L_1, L_2, L_3 of length 2^{r-1} with $C(L_i) \geq (r-1)\, 2^{r-3}$.
    - By Lemma 2, for some i,j, $C(L_i \text{ join } L_j) \geq 2(r-1)2^{r-3} + 2^{r-1}/2 = r\, 2^{r-2}$.

Lower bound, cont'd

- Lemma 3: For any $r \geq 0$, there are infinitely many disjoint line graphs L of length 2^r such that $C(L) \geq r\,2^{r-2}$.

- Theorem: For any $r \geq 0$, there is a ring R of size $n = 2^r$ such that $C(R) = \Omega(n \log n)$.
  - Choose L of length 2^r such that $C(L) \geq r\,2^{r-2}$.
  - Connect ends, but delay communication across boundary.

- Corollary: For any $n \geq 0$, there is a ring R of size n such that $C(R) = \Omega(n \log n)$.

Leader election in general networks

- Undirected graphs.
- Can get asynchronous version of synchronous FloodMax algorithm:
  - Simulate rounds with counters.
  - Need to know diameter for termination.
- We'll see better asynchronous algorithms later:
  - Don't need to know diameter.
  - Lower message complexity.
- Depend on techniques such as:
  - Breadth-first search
  - Convergecast using a spanning tree
  - Synchronizers to simulate synchronous algorithm
  - Consistent global snapshots to detect termination.

Next lecture

- More asynchronous network algorithms
  - Constructing a spanning tree
  - Breadth-first search
  - Shortest paths
  - Minimum spanning tree (GHS)
- Reading: Section 15.3-15.5, [Gallager, Humblet, Spira]

6.852J / 18.437J Distributed Algorithms
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