MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.341 DISCRETE-TIME SIGNAL PROCESSING
Fall 2004

FINAL EXAM
Monday, December 13, 2004

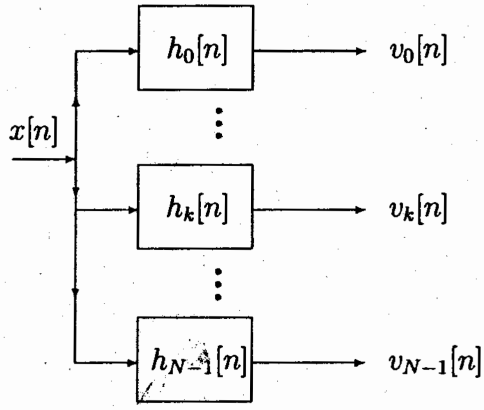
NAME: Solutions

Problem	Grade	Points	Grader
1			
2			
3			
4			
5 (a)			
5 (b)			
6			
7			
8 (a)			
8 (b)			
8 (c)			
8 (d)			
Total			

COURSE GRADE:

Problem 1 [10%]

The system in figure 1-1, uses a modulated filter bank for spectral analysis. The lowpass impulse response $h_0[n]$ is sketched in figure 1-2.



$$h_k[n] = e^{j\omega_k n} h_0[n], \qquad \omega_k = \frac{2\pi k}{N}, \qquad \text{where } k = 0,\ 1,\ \ldots,\ N-1$$

$$h_0[n] = \text{lowpass prototype filter}, \qquad H_k(z) = H_0(e^{-j\frac{2\pi k}{N}} z)$$

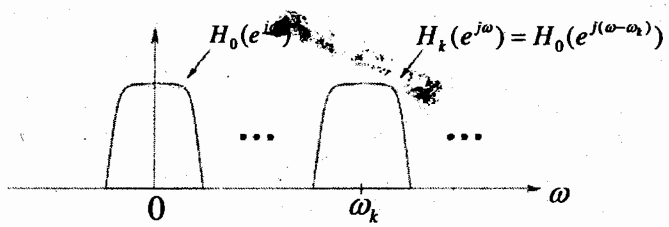
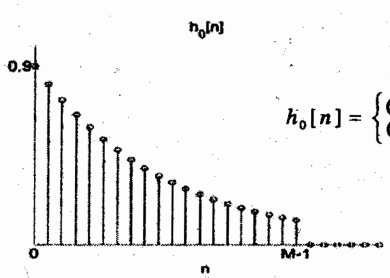


Figure 1-1:



$$h_0[n] = \begin{cases} 0.9^n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

where, $M < N$

Figure 1-2:

Name: ______________________________

An alternative system for spectral analysis is shown in figure 1-3. Determine $w[n]$ so that $G[k] = v_k[0]$, for $k = 0, 1, ..., N-1$.

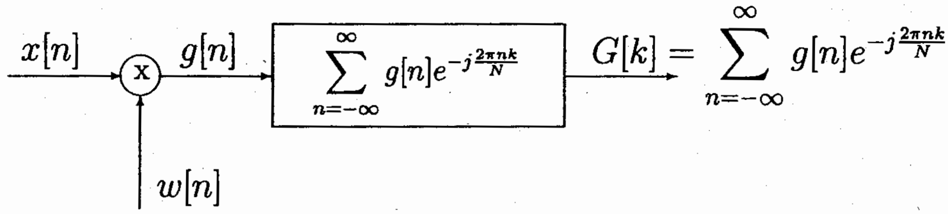


Figure 1-3

Work to be looked at and answer:
Note: As with all the problems, a correct answer without explanation and related work will not guarantee full credit.

$$G[k] = \sum_{n=-\infty}^{+\infty} g[n] e^{-j\frac{2\pi nk}{N}} = \sum_{n=-\infty}^{+\infty} x[n] w[n] e^{-j\frac{2\pi nk}{N}} \quad \text{——— } (1)$$

$$H_k(z) = H_0(e^{-j\frac{2\pi k}{N}} z) \Rightarrow h_k[n] = h_0[n] e^{j\frac{2\pi nk}{N}}$$

$$v_k[m] = \sum_{n=-\infty}^{+\infty} x[m] h_k[m-n] = \sum_{n=-\infty}^{+\infty} x[m] h_0[m-n] e^{j\frac{2\pi(m-n)k}{N}}$$

$$\Rightarrow v_k[0] = \sum_{n=-\infty}^{+\infty} x[n] h_0[-n] e^{-j\frac{2\pi nk}{N}} = G[k] \quad \text{——— } (2)$$

$$(1) \ \& \ (2) \Rightarrow w[n] = h_0[-n] = \begin{cases} 0.9^{-n} & -M+1 \le n \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 2 [8%]

Consider the stable LTI system with system function

$$H(z) = \frac{1 + 4z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

The system function $H(z)$ can be factored such that

$$H(z) = H_{\min}(z)H_{ap}(z)$$

where $H_{\min}(z)$ is a minimum phase system, and $H_{ap}(z)$ is an allpass system, i.e.,

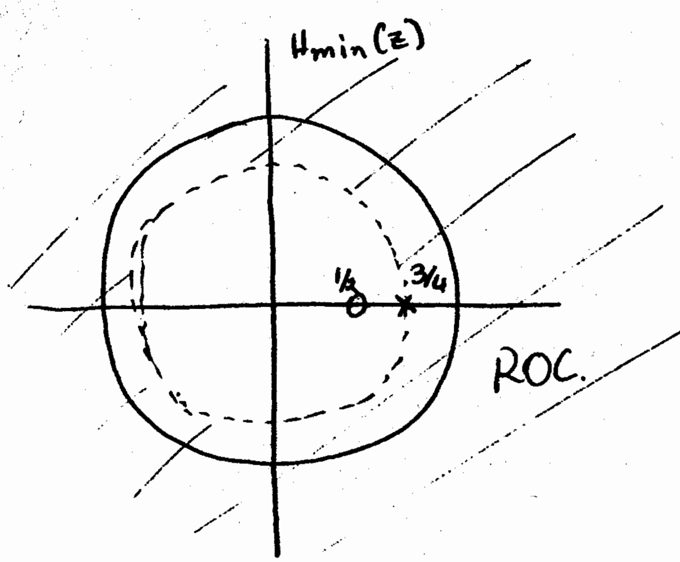
$$|H_{ap}(e^{j\omega})| = 1$$

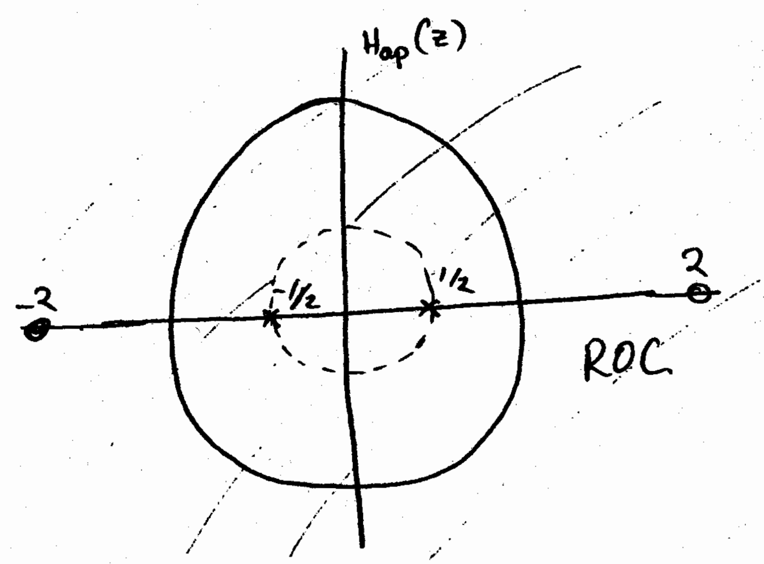
Sketch the pole-zero diagrams for $H_{\min}(z)$ and $H_{ap}(z)$. Be sure to label the positions of all the poles and zeros. Also, indicate the region of convergence for $H_{\min}(z)$ and $H_{ap}(z)$.

Work to be looked at and answer:

$$H(z) = \frac{(1-2z^{-1})(1+2z^{-1})}{(1-\frac{3}{4}z^{-1})(1+\frac{1}{2}z^{-1})} = \underbrace{\frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{3}{4}z^{-1})}}_{H_{min}(z)} \cdot \underbrace{\frac{(1-2z^{-1})(1+2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}}_{H_{ap}(z)}$$

← Both need to be stable





ROC: $|z| > 3/4$

ROC: $|z| > 1/2$.

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Problem 3 [8%]

The block diagram in figure 3-1 represents a system that we would like to implement. Determine a block diagram of an equivalent system consisting of a cascade of LTI systems, compressor blocks, and expander blocks which results in the minimum number of multiplications per output sample.

Note: By "equivalent system" we mean that it produces the same output sequence for any given input sequence.

$$x[n] \rightarrow \boxed{\uparrow 3} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow 2} \rightarrow \boxed{\uparrow 3} \rightarrow y[n]$$

Figure 3-1

$$H(z) = \frac{z^{-6}}{7 + z^{-6} - 2z^{-12}}$$

Work to be looked at and answer:

$\boxed{\uparrow 3}$ and $\boxed{\downarrow 2}$ can be exchanged $\Rightarrow$ We need to decimate as much as possible, filter, and then expand.

$$x[n] \rightarrow \underbrace{\boxed{\uparrow 3} \rightarrow \boxed{\downarrow 2}}_{\downarrow \text{ exchange}} \rightarrow \underbrace{\boxed{H(z)} \rightarrow \boxed{\downarrow 2}}_{\downarrow \text{ use Noble downsampling identity}} \rightarrow \boxed{\uparrow 3} \rightarrow y[n]$$

$$H_1(z) = \frac{z^{-3}}{7 + z^{-3} - 2z^{-6}}$$

$$x[n] \rightarrow \boxed{\downarrow 2} \rightarrow \underbrace{\boxed{\uparrow 3} \rightarrow \boxed{\downarrow 2}}_{\text{exchange } \downarrow} \rightarrow \boxed{H_1(z)} \rightarrow \boxed{\uparrow 3} \rightarrow y[n]$$

$$x[n] \rightarrow \underbrace{\boxed{\downarrow 2} \rightarrow \boxed{\downarrow 2}}_{\text{combine } \downarrow} \rightarrow \underbrace{\boxed{\uparrow 3} \rightarrow \boxed{H_1(z)} \rightarrow \boxed{\uparrow 3}}_{\text{Noble upsampling identity } \downarrow} \rightarrow y[n]$$

& combine $\rightarrow \boxed{\uparrow 3} \rightarrow \boxed{\uparrow 3} \rightarrow$

$$x[n] \rightarrow \boxed{\downarrow 4} \rightarrow \boxed{H_2(z)} \rightarrow \boxed{\uparrow 9} \rightarrow y[n] \qquad H_2(z) = \frac{z^{-1}}{7 + z^{-1} - 2z^{-2}}$$

2 mutiplies

Total: $\frac{2}{9}$ multiplications/output sample

Problem 4 [8%]

Consider a colored wide sense stationary stochastic signal $s[n]$ which we desire to whiten using the system in figure 4-1:

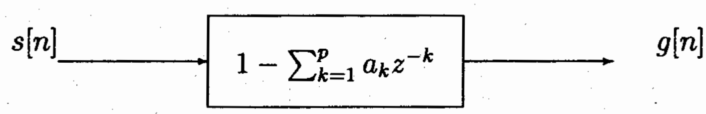

Figure 4-1

In designing the optimal whitening filter for a given order p, we pick $a_k^{(p)}$, $k = 1, ..., p$ that solve the following equations, where $\phi_s[m]$ is the autocorrelation of $s[n]$.

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \cdots & \phi_s[p-1] \\ \phi_s[1] & \phi_s[0] & \cdots & \\ \vdots & & & \\ \phi_s[p-1] & & \cdots & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1^{(p)} \\ a_2^{(p)} \\ \vdots \\ a_p^{(p)} \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \vdots \\ \phi_s[p] \end{bmatrix}$$

It is known that the optimal 2^{nd} order whitening filter for $s[n]$ is $H_2(z) = 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}$, (i.e. $a_1^{(2)} = -\frac{1}{4}$, $a_2^{(2)} = \frac{1}{8}$), which we implement in the 2^{nd} order lattice structure in figure 4-2:

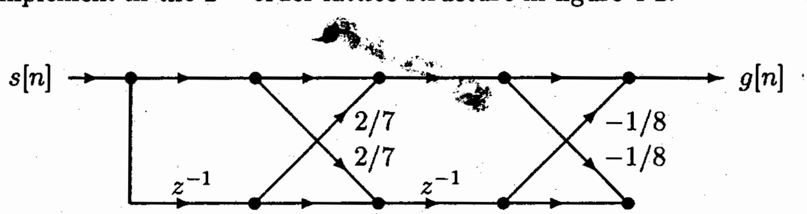

Figure 4-2: Lattice Structure for 2^{nd} Order System

We decided that a second order implementation is not sufficient for our application, and we would like to use a 4^{th} order system, with transfer function

$$H_4(z) = 1 - \sum_{k=1}^{4} a_k^{(4)} z^{-k}$$

We implement this system with the lattice structure in figure 4-3:

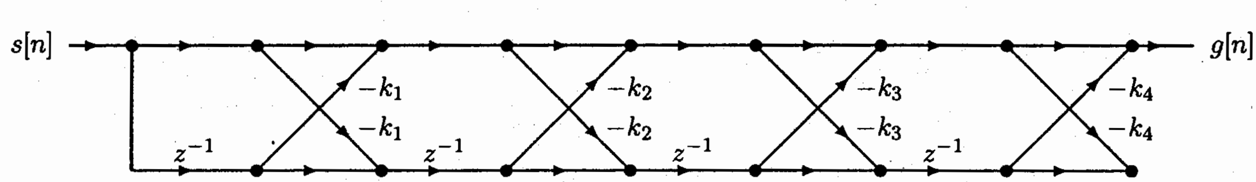

Figure 4-3: Lattice Structure for 4^{th} Order System

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Determine which, if any of $H_4(z)$, k_1, k_2, k_3, k_4 can be exactly determined from the information given above. Explain why you cannot determine the remaining, if any, parameters.

Note: For this problem you may find useful the lecture slide which we have reproduced on page 20.

Work to be looked at and answer:

To solve for the 4^{th} order system we extend the 2^{nd} order one using the Levinson recursion.

The resulting lattice will have the same k_1 and k_2.

However we need to know $\phi_s[m]$ to determine k_3 and k_4.

To determine $H_4(z)$, i.e. $a_i^{(4)}$ we need to know all the k_i's.

Therefore $k_1 = -2/7$, $k_2 = 1/8$, and the remaining parameters cannot be determined from the information provided.

Problem 5 [16%]

Each part of this problem may be solved independently. All parts use the signal $x[n]$ shown in figure 5-1.

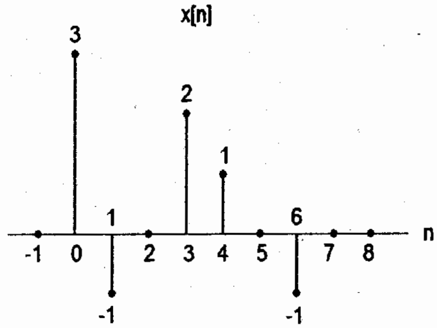


Figure 5-1:

[%8] (a) Let $X\left(e^{j\omega}\right)$ be the DTFT of $x[n]$. Define

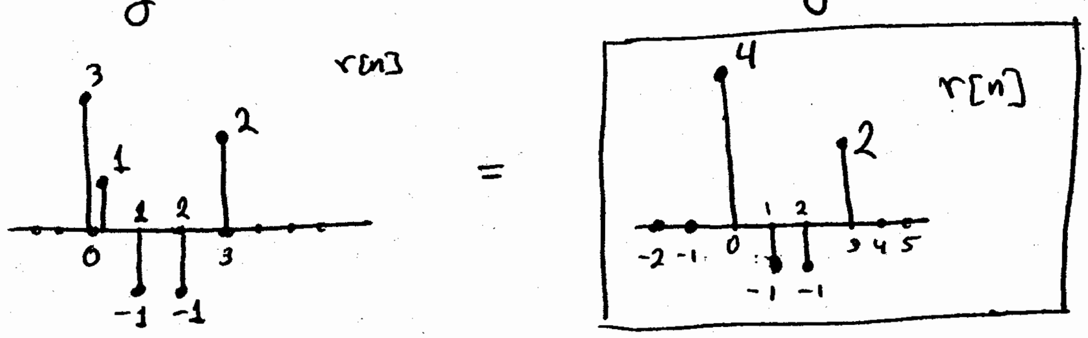
$$R[k] = X\left(e^{j\omega}\right)\big|_{\omega=\frac{2\pi k}{4}}, \qquad 0 \le k \le 3$$

Sketch the signal $r[n]$ which is the four-point inverse DFT of $R[k]$.

Work to be looked at and answer: (Not necessary to derive the result mathematically, but explain your result.)

R[k] is the 4-pt DFT of x[n]

Inverting R[k] creates time-aliasing:



Name: ______________________________

[%8] (b) Let $X[k]$ be the eight-point DFT of $x[n]$, and let $H[k]$ be the eight-point DFT of the impulse response $h[n]$ shown in figure 5-2. Define $Y[k] = X[k]H[k]$ for $0 \le k \le 7$. Sketch $y[n]$, the eight-point DFT of $Y[k]$.

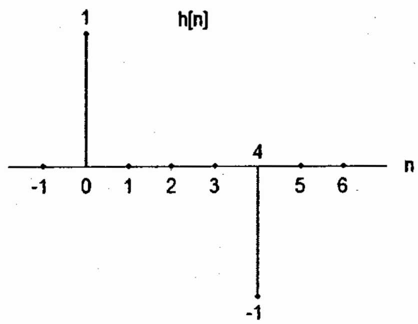
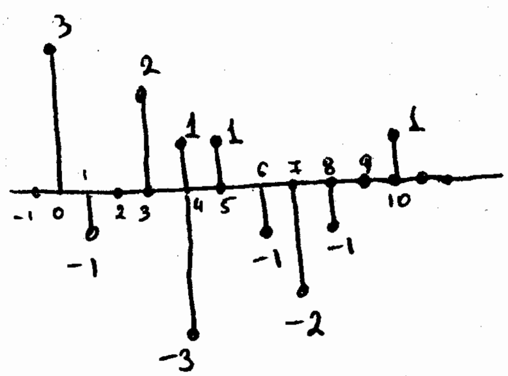


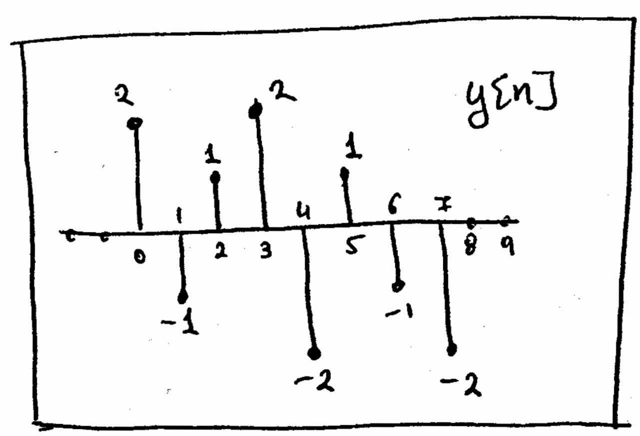
Figure 5-2:

Work to be looked at and answer:

The product of the DFTs corresponds to circular convolution, i.e. linear convolution followed by time-aliasing:



time aliasing $\longrightarrow$



Problem 6 [10%]

Consider a time-limited continuous-time signal $x_c(t)$ whose duration is $100ms$. Assume that this signal has a bandlimited Fourier transform such that $X_c(j\Omega) = 0$ for $|\Omega| \geq 2\pi(10,000)$rad/s; i.e., assume that aliasing is negligible. We want to compute samples of $X_c(j\Omega)$ with $5Hz$ spacing over the interval $0 \leq \Omega \leq 2\pi(10,000)$. This can be done with a 4000-point DFT. Specifically, we want to obtain a 4000-point sequence $x[n]$ for which the 4000-point DFT is related to $X_c(j\Omega)$ by:

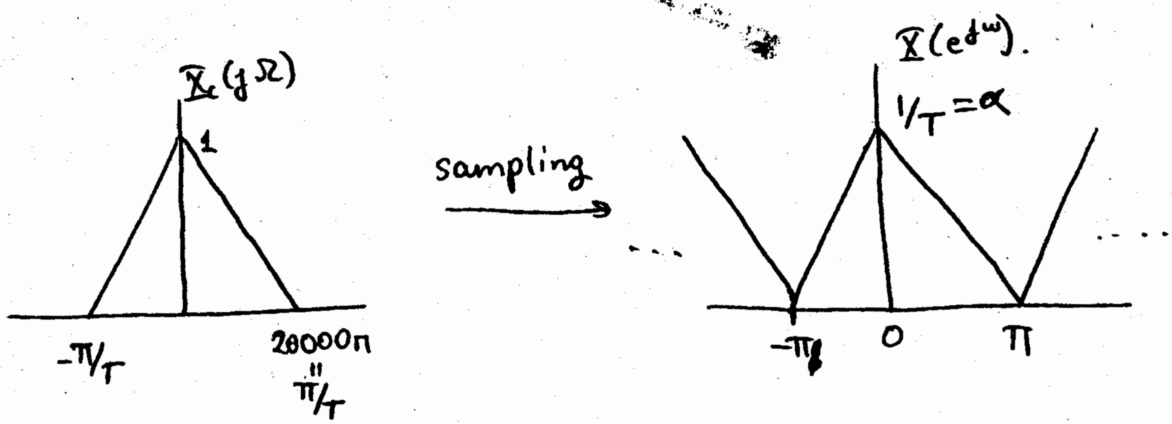
$$X[k] = \alpha X_c(j2\pi \cdot 5 \cdot k), \qquad k = 0, 1, \ldots 1999, \tag{1}$$

where α is a known scale factor. The following method is proposed to obtain a 4000-point sequence whose DFT gives the desired samples of $X_c(j\Omega)$. $x_c(t)$ is sampled with a sampling period of $T = 50\mu s$. The resulting 2000-point sequence is used to form the sequence $\hat{x}[n]$ as follows:

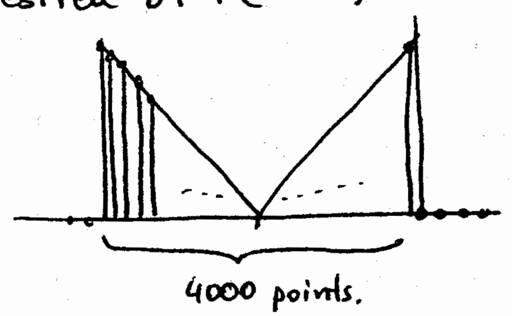
$$\hat{x}[n] = \begin{cases} x_c(nT), & 0 \leq n \leq 1999, \\ x_c((n-2000)T), & 2000 \leq n \leq 3999, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The 4000-point DFT $\hat{X}[k]$ of this sequence is computed. For this method, determine how $\hat{X}[k]$ is related to $X_c(j\Omega)$. Indicate this relationship in a sketch for a "typical" Fourier transform $X_c(j\Omega)$. Explicitly state whether or not $\hat{X}[k]$ is the desired result, i.e. whether $\hat{X}[k]$ equals $X[k]$ as specified in eqn(1).

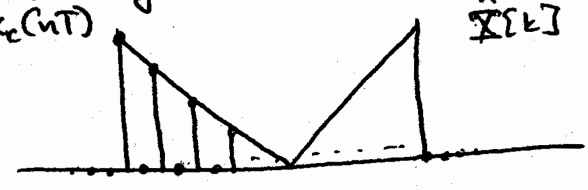
Work to be looked at and answer:
Note: As with all the problems, a correct answer without explanation and related work will not guarantee full credit.



Desired DFT($X[k]$).



But eq. (2) is equivalent to the expansion by 2 of the 2000pt DFT of $x_c(nT)$
i.e



$$\Rightarrow \hat{X}[k] = \begin{cases} X[k] & k \text{ even} \\ 0. & k \text{ odd.} \end{cases}$$

Name: ______________________________

Work to be looked at and answer for problem 6:
Note: This space may or may not be needed, but is not to be used for any other problem.

Problem 7 [10%]

The system in figure 7-1 computes an N-point (where N is an even number) DFT $X[k]$ of an N-point sequence $x[n]$ by decomposing $x[n]$ into two $N/2$-point sequences $g_1[n]$ and $g_2[n]$, computing the $N/2$-point DFT's $G_1[k]$ and $G_2[k]$, and then combining these to form $X[k]$.

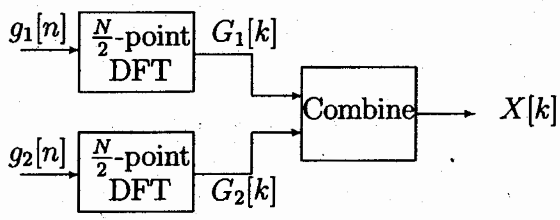


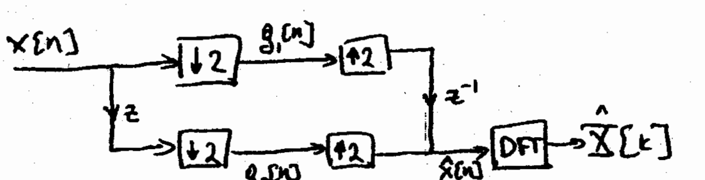
Figure 7-1

If $g_1[n]$ is the even-indexed values of $x[n]$ and $g_2[n]$ is the odd-indexed values of $x[n]$ i.e. $g_1[n] = x[2n]$ and $g_2[n] = x[2n+1]$ then $X[k]$ will be the DFT of $x[n]$.

In using the system in figure 7-1 an error is made in forming $g_1[n]$ and $g_2[n]$, such that $g_1[n]$ is **incorrectly** chosen as the odd-indexed values and $g_2[n]$ as the even indexed values but $G_1[k]$ and $G_2[k]$ are still combined as in figure 7-1 and the incorrect sequence $\hat{X}[k]$ results. Express $\hat{X}[k]$ in terms of $X[k]$

Work to be looked at and answer:

Swapping $g_1[n]$ and $g_2[n]$ corresponds to the following system:



$$G_1(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}-\pi)})\right]$$

$$G_2(e^{j\omega}) = \frac{1}{2}\left[e^{j\frac{\omega}{2}}X(e^{j\frac{\omega}{2}}) + e^{j(\frac{\omega}{2}-\pi)}X(e^{j(\frac{\omega}{2}-\pi)})\right]$$

$$= \frac{e^{j\frac{\omega}{2}}}{2}\left[X(e^{j\frac{\omega}{2}}) - X(e^{j(\frac{\omega}{2}-\pi)})\right].$$

$$\Rightarrow \hat{X}(e^{j\omega}) = \frac{e^{-j\omega}}{2}\left[X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right] + \frac{e^{j\omega}}{2}\left[X(e^{j\omega}) - X(e^{j(\omega-\pi)})\right].$$

$$= \frac{1}{2}\left[(e^{-j\omega} + e^{j\omega})X(e^{j\omega}) + (e^{-j\omega} - e^{j\omega})X(e^{j(\omega-\pi)})\right]$$

$$\Rightarrow \hat{X}[k] = \frac{1}{2}\left[(W_N^{+k} + W_N^{-k})X[k] + (W_N^{+k} - W_N^{-k})X[((k - N/2))_N]\right]$$

where $W_N^k = e^{-j\frac{2\pi k}{N}}$

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Problem 8 [30%] **Note:** each part of this problem is independent of the others.

The current CD technology, from production to playback, can be approximated with the block diagram in figure 8-1.

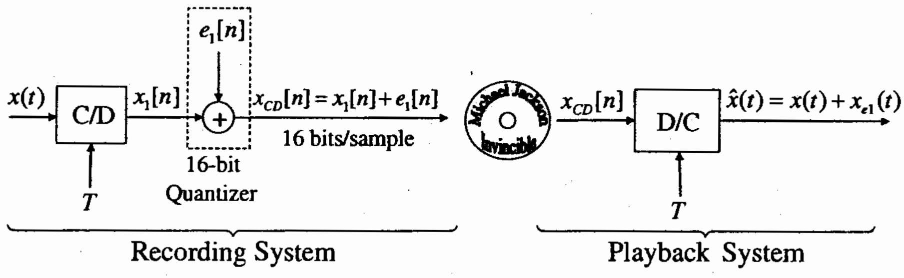


Figure 8-1:

A C/D is defined through the following relationships between the input $x_c(t)$ and the output $x_d[n]$:

$$x_d[n] = x_c(nT) \text{ and } R_{x_d x_d}[m] = R_{x_c x_c}(mT)$$

A D/C is defined through the following relationships between the input $x_d[n]$ and the output $x_c(t)$:

$$x_c(t) = \sum_{n=-\infty}^{+\infty} x_d[n] \frac{sin[\pi(t-nT)/T]}{\pi(t-nT)/T} \text{ and } R_{x_c x_c}(\tau) = \sum_{n=-\infty}^{+\infty} R_{x_d x_d}[m] \frac{sin[\pi(\tau-mT)/T]}{\pi(\tau-mT)/T}$$

$x(t)$ is a signal bandlimited to $\pm\pi/T$, i.e. $X(j\Omega) = 0$ for $|\Omega| \geq \pi/T$.
Assume the additive noise model for a quantizer holds, i.e. $e_1[n]$ in figure 8-1 is zero-mean, white, has variance σ_{e1}^2 and is uncorrelated with $x_1[n]$.

[6%] (a) Determine $E\{x_{e1}^2(t)\}$, the power of the quantization noise at the output of the playback system.

Work to be looked at and answer:

$$E\{x_{e1}^2(t)\} = R_{x_{e1}x_{e1}}(0) = R_{e_1e_1}[0] = E\{e_1^2[n]\} = \sigma_e^2.$$

Sony and Philips offer a new CD format called Superaudio CD. The new format can be approximately described with the block diagram in figure 8-2.

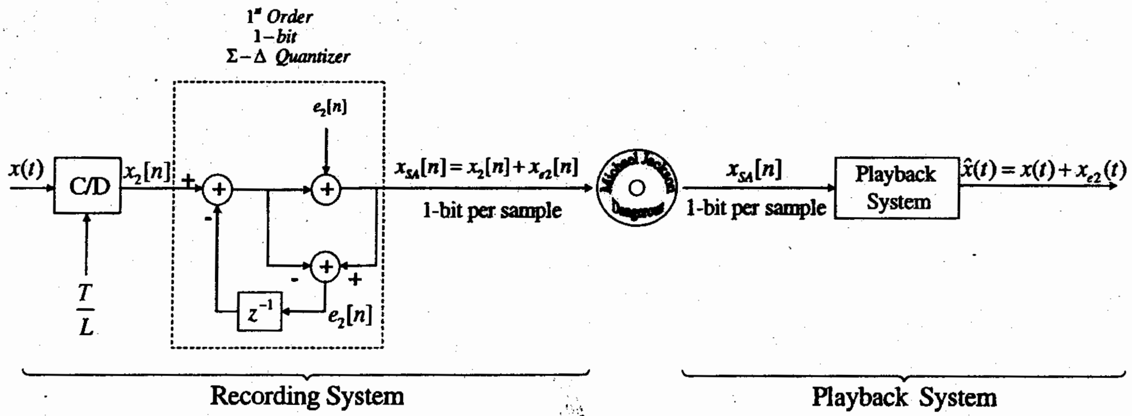


Figure 8-2:

where $e_2[n]$ in figure 8-2 is zero-mean, white, has variance σ_{e2}^2 and is uncorrelated with $x_1[n]$. The power spectral density of $x(t)$, $P_x(j\Omega)$, is as shown in figure 8-3.

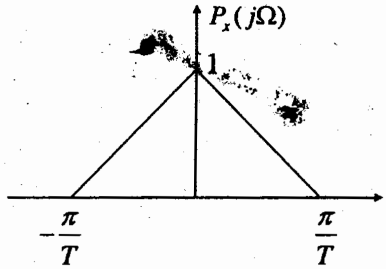


Figure 8-3:

[8%] (b) In the absence of quantization (i.e. if $e_2[n] = 0$), the playback system in figure 8-2 should reconstruct $x(t)$ exactly. In the presence of quantization, the overall system should minimize $E\{x_{e2}^2(t)\}$, the quantization noise power at the output. Using the components shown in figure 8-4, design the minimum cost playback system to reconstruct $x(t)$ from $x_{SA}[n]$ that satisfies these requirements. Make sure you specify the values for all the parameters of all the components you include in your design.

Note: In the components table all the components have inputs and outputs of arbitrary precision, and can be used with any input. The only exception is with the component named "1-bit D/C converter" that can only be used with a bitstream input, i.e. a signal which has already been quantized to 1-bit per sample. This component cannot be used with any other kind of input.

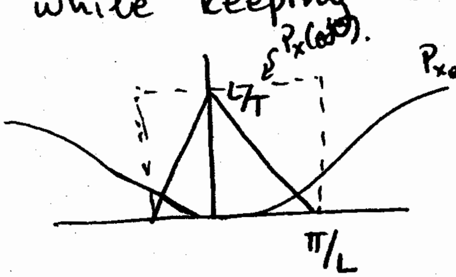
Name: ______________________________

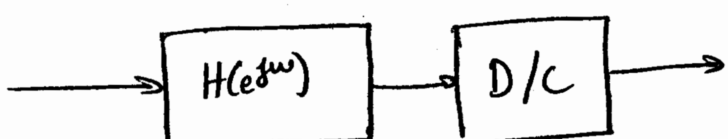
	Component	Parameters	Cost in $
$\uparrow R$	Expander	Ratio: R	1
$\downarrow R$	Compressor	Ratio: R	1
$H(e^{j\omega})$, G, ω_0	Discrete-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: ω_0	1
1-bit D/C Converter, T_0	1-bit D/C Converter	Rate: T_0	1
D/C Converter, T_0	D/C Converter	Rate: T_0	15
$H(j\Omega)$, G, Ω_0	Continuous-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: Ω_0	20

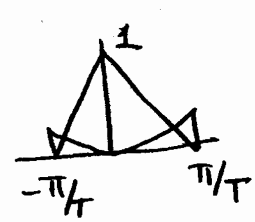
Figure 8-4:

Work to be looked at and answer for part (b):

The system should remove all the out-of-band noise, while keeping all the in-band frequency components intact. In this part, doing so in discrete-time is cheaper.







Discrete-Time LPF.
Gain = 1
$\omega_0 = \pi/L$

$T_0 = T/L$

Cost: $16

[8%] (c) Repeat part (b) assuming the cost of an analog lowpass filter is now \$1. Figure 8-4 is repeated here with the new cost for the analog filter.

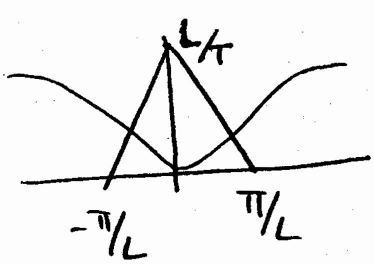
Again note that: In the components table all the components have inputs and outputs of arbitrary precision, and can be used with any input. The only exception is with the component named "1-bit D/C converter" that can only be used with a bitstream input, i.e. a signal which has already been quantized to 1-bit per sample. This component cannot be used with any other kind of input.

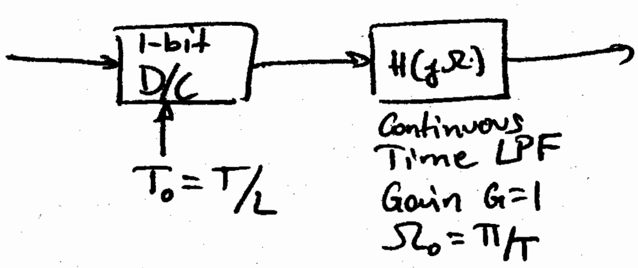
	Component	Parameters	Cost in \$
$\uparrow R$	Expander	Ratio: R	1
$\downarrow R$	Compressor	Ratio: R	1
$H(e^{j\omega})$, G, ω_0	Discrete-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: ω_0	1
1-bit D/C Converter, T_0	1-bit D/C Converter	Rate: T_0	1
D/C Converter, T_0	D/C Converter	Rate: T_0	15
$H(j\Omega)$, G, Ω_0	Continuous-Time Ideal Lowpass Filter	Gain: G Cutoff Frequency: Ω_0	1

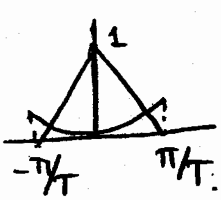
Figure 8-5:

Work to be looked at and answer:

In this case the continuous-time LPF is cheaper:







Name: ____________________

[8%] (d) With present technology, reconstructing the signal from traditional CDs (i.e. the reconstruction system in figure 8-1) consumes 2 Watts, while the Superaudio CD (i.e. the reconstruction system in figure 8-2) consumes 1 Watts (1 Watt = 1 Joule/sec). However, the energy consumption of reading 1000 bits from the disc is the same.

(i) If $T = 1/40kHz$ and $L = 64$, determine how many bits per second each format requires to store $x(t)$.

Traditional CD: 16 bits/sample × 40000 samples/sec = 640000 bits/sec.

Superaudio CD: 1 bit/sample × 64×40000 samples/sec = 2560000 bits/sec.

(ii) If $T = 1/40kHz$ and $L = 64$, as above, determine the energy consumption in Joules per 1000 bits of storage, below which the Superaudio CD format is more power efficient than the traditional CD format in playing back the signal.

Let C = consumption in J/1000 bits.
For 1sec of playback:
Traditional CD: 2 Joules + 640C

Superaudio CD: 1 Joule + 2560C.

$1 + 2560 < 2 + 640c \Rightarrow 1920c < 1 \Rightarrow \boxed{c < \frac{1}{1920} \text{ Joules/1000 bits}}$

END OF EXAM

YOU CAN USE THIS PAGE AS SCRATCH PAPER BUT NOTHING ON THIS PAGE WILL BE CONSIDERED DURING GRADING

Levinson-Durbin Recursion

$$\sum_{k=1}^{p} a_k \phi_s[i-k] = \phi_s[i] \qquad i = 1,2,\ldots,p$$

$$T_p = \begin{bmatrix} \phi_s[0] & \phi_s[1] \cdots & \phi_s[p-1] \\ \phi_s[1] & \phi_s[0] \cdots & \\ \vdots & & \\ \phi_s[p-1] & \cdots & \phi_s[0] \end{bmatrix}$$

$$\alpha_p = [a_1, a_2 \ldots a_p]^T \qquad r_p = [\phi_s[1]\phi_s[2]\ldots\phi_s[p]]^T$$

$$\beta_p = [a_p, a_{p-1} \ldots a_1]^T \qquad \rho_p = [\phi_s[p]\phi_s[p-1]\ldots\phi_s[1]]^T$$

$$k_{p+1} = \frac{\phi_s[p+1] - (\rho_p)^T \alpha_p}{\phi_s[0] - (r_p)^T \alpha_p}$$

$$\varepsilon_p = -k_{p+1}\beta_p$$

$$\alpha_{p+1} = \begin{bmatrix} \alpha_p \\ \hline 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_p \\ \hline k_{p+1} \end{bmatrix} = \begin{bmatrix} a_1^{(p)} \\ a_2^{(p)} \\ \vdots \\ a_p^{(p)} \\ 0 \end{bmatrix} - k_{p+1} \begin{bmatrix} a_p^{(p)} \\ a_{p-1}^{(p)} \\ \vdots \\ a_1^{(p)} \\ -1 \end{bmatrix} \qquad (7)$$

$a_k^{(p)}$ is the $a_k th$ coefficient for the pth order filter.