Solutions to Homework 5
6.262 Discrete Stochastic Processes
MIT, Spring 2011

Solution to Exercise 2.28:

Suppose that the states are numbered so that state 1 to J_1 are in the recurrent class 1, $J_1 + 1$ to $J_1 + J_2$ in recurrent class 2, etc. Thus, $[P]$ has the following form:

$$[P] = \begin{bmatrix} [P_1] & 0 & 0 & 0 \\ 0 & [P_2] & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ [P_{t1}] & [P_{t2}] & \cdots & [P_{tt}] \end{bmatrix}$$

As in exercise 3.17 part (d), $[P_{tt}]$ has no eigenvalue of value 1, and thus in any left eigenvector $\pi = \left(\hat{\pi}^{(1)}, \hat{\pi}^{(2)}, \cdots, \hat{\pi}^{(r)}, \hat{\pi}^{(t)} \right)$ of eigenvalue 1, we have $\hat{\pi}^{(t)} = 0$. This means that $\hat{\pi}^{(i)}$ must be a left eigenvector of $[P_i]$ for each i, $1 \le i \le r$. Since $\hat{\pi}^{(i)}$ can be given an arbitrary scale factor for each i, we get exactly r independent left eigenvectors, each a probability vector as desired. We take these eigenvectors to be $\pi^{(i)} = \left(0, \cdots, 0, \hat{\pi}^{(i)}, 0, \cdots, 0 \right)$, i.e., zero outside of class i and $\hat{\pi}^{(i)}$ inside of class i.

Let $v_j^{(i)} = \Pr \{\text{recurrent class } i \text{ is ever reached} \mid X(0) = j \}$. Note that $v_j^{(i)} = 1$ for j in the i-th recurrent class and that $v_j^{(i)} = 0$ for j in any other recurrent class. For j in a transient class, we have the relation:

$$v_j^{(i)} = \sum_k p_{jk} v_k^{(i)}$$

In other words, the probability of going to class i is the sum, over all states, of the probability of going to state k at time 1 and then ever going to class i from state k. We can rewrite this as $v_j^{(i)} = \sum_{k \in \mathbb{T}} p_{jk} v_k^{(i)} + \sum_{k \in \text{ class } i} p_{jk}$. Since the matrix $[P_{tt}]$ has no eigenvalue equal to 1, $[I - P_{tt}]$ is non-singular, and this set of equations, over all transient j, has a unique solution. That solution, along with $v_j^{(i)}$ over the recurrent j yields a vector $\mathbf{v}^{(i)}$ that is a right eigenvector of $[P]$. There is one such eigenvector for each recurrent class and there can be no more independent right eigenvectors because there are no more left eigenvectors.

Exercise 3.14:

Answer the following questions for the following stochastic matrix $[P]$

$$[P] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Find $[P^n]$ in closed form for arbitrary $n > 1$.

Solution: There are several approaches here. We first give the brute-force solution of simply multiplying $[P]$ by itself multiple times (which is reasonable for a first look), and then give the elegant solution.

$$[P^2] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 2/4 & 1/4 \\ 0 & 1/4 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[P^3] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 2/4 & 1/4 \\ 0 & 1/4 & 3/4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/8 & 3/8 & 5/8 \\ 0 & 1/8 & 7/4 \\ 0 & 0 & 1 \end{bmatrix}.$$

We could proceed to $[P^4]$, but it is natural to stop and think whether this is telling us something. The bottom row of $[P^n]$ is clearly $(0,0,1)$ for all n, and we can easily either reason or guess that the first two main diagonal elements are 2^{-n}. The final column is whatever is required to make the rows sum to 1. The only questionable element is P_{12}^n. We guess that is $n2^{-n}$ and verify it by induction,

$$
\begin{aligned}
[P^{n+1}] &= [P][P^n] = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{-n} & n2^{-n} & 1-(n+1)2^{!-n} \\ 0 & 2^{-n} & 1-2^{-n} \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{-n-1} & (n+1)2^{-n-1} & 1-(n+3)2^{-n-1} \\ 0 & 2^{-n} & 1-2^{-n} \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

This solution is not very satisfying, first because it is tedious, second because it required a guess that was not very well motivated, and third because no clear rhyme or reason emerged.

The elegant solution, which can be solved with no equations, requires looking at the graph of the Markov chain,
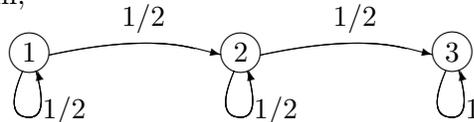
It is now clear that $P_{11}!^n = 2^{-n}$ is the probability of taking the lower loop for ! n successive steps starting in state 1. Similarly $P_{22}^n = 2^{-n}$ is the probability of taking the lower loop at state 2 for n successive steps.

Finally, P_{12}^n is the probability of taking the transition from state 1 to 2 exactly once out of the n transitions starting in state 1 and of staying in the same state for the other $n-1$ transitions. There are n such paths, corresponding to the n possible steps at which the $1 \to 2$ transition can occur, and each path has probability 2^{-n}. Thus $P_{12}^n = n2^{-n}$, and we 'see' why this factor of n appears. The transitions P_{i3}^n are then chosen to make the rows sum to 1, yielding the same solution as above..

b) Find all distinct eigenvalues and the multiplicity of each distinct eigenvalue for $[P]$.

Solution: Use the following equation to find the determinant of $[P - \lambda I]$ and note that the only permutation of the columns that gives a non-zero value is the main diagonal.

$$\det A = \sum_{\mu} \pm \prod_{i=1}^{J} A_{i,\mu(i)}$$

Thus $\det[P - \Lambda I] = (\frac{1}{2} - \lambda)^2 (1 - \lambda)$. It follows that $\lambda = 1$ is an eigenvalue of multiplicity 1 and $\lambda = 1/2$ is an eigenvalue of multiplicity 2.

c) Find a right eigenvector for each distinct eigenvalue, and show that the eigenvalue of multiplicity 2 does not have 2 linearly independent eigenvectors.

Solution: For any Markov chain, $\vec{e} = (1, 1, 1)^\mathsf{T}$ is a right eigenvector. This is unique here within a scale factor, since $\lambda = 1$ has multiplicity 1. For $\boldsymbol{\nu}$ to be a right eigenvector of eigenvalue $1/2$, it must satisfy

$$\frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 + 0\nu_3 = \frac{1}{2}\nu_1$$
$$0\nu_1 + \frac{1}{2}\nu_2 + \frac{1}{2}\nu_3 = \frac{1}{2}\nu_2$$
$$\nu_3 = \frac{1}{2}\nu_3$$

From the first equation, $\nu_2 = 0$ and from the third $\nu_3 = 0$, so $\nu_2 = 1$ is the right eigenvector, unique within a scale factor.

d) Use (c) to show that there is no diagonal matrix $[\Lambda]$ and no invertible matrix $[U]$ for which $[P][U] = [U][\Lambda]$.

Solution: Letting $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ be the columns of an hypothesized matrix $[U]$, we see that $[P][U] = [U][\Lambda]$ can be written out as $[P]\boldsymbol{\nu}_i = \lambda_1 \boldsymbol{\nu}_i$ for $i = 1, 2, 3$. For $[U]$ to be invertible, $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\nu}_3$ must be linearly independent eigenvectors of $[P]$. Part (c) however showed that 3 such eigenvectors do not exist.

e) Rederive the result of part d) using the result of a) rather than c).

Solution: If the $[U]$ and $[\Lambda]$ of part (d) exist, then $[P^n] = [U][\Lambda^n][U^{-1}]$. Then, as in (3.30) of the text, $[P^n] = \sum_{i=1}^{3} \lambda_i^n \boldsymbol{\nu}^{(i)} \boldsymbol{\pi}^{(i)}$ where $\boldsymbol{\nu}^{(i)}$ is the ith column of $[U]$ and $\boldsymbol{\pi}^{(i)}$ is the ith row of $[U^{-1}]$. Since $P_{12}^n = n(1/2)^n$, the factor of $n!$ means that it cannot have the form $a\lambda_1^n + b\lambda_2^n + c\lambda_3^n$ for any choice of $\lambda_1, \lambda_2, \lambda_3, a, b, c$.

Note that the argument here is quite general. If $[P^n]$ has any terms containing a polynomial in n times λ_i^n, then the eigenvectors can't span the space and a Jordan form decomposition is required.

Exercise 3.15:

a) Let $[J_i]$ be a 3 by 3 block of a Jordan form, i.e.,

$$[J_i] = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}.$$

Show that the nth power of $[J_i]$ is given by

$$[J_i^n] = \begin{bmatrix} \lambda_i^n & n\lambda_i^{n-1} & \binom{n}{2}\lambda_i^{n-2} \\ 0 & \lambda_i^n & n\lambda_i^{n-1} \\ 0 & 0 & \lambda_i^n \end{bmatrix}.$$

Hint: Perhaps the easiest way is to calculate $[J_i^2]$ and $[J_i^3]$ and then use iteration.

Solution: It is probably worthwhile to multiply $[J_i]$ by itself one or two times to gain familiarity with the problem, but note that the proposed formula for $[J_i^n]$ is equal to $[J_i]$ for $n = 1$, so we can use induction to show that it is correct for all larger n. That is, for each $n \geq 1$, we assume the given formula for $[J_i^n]$ is correct and demonstrate that $[J_i^n][J_i^n]$ is the given formula evaluated at $n + 1$. We suppress the subscript i in the evaluation.

We start with the elements on the first row, i.e.,

$$\sum_j J_{1j} J_{j1}^n = \lambda \cdot \lambda^n = \lambda^{n+1}$$

$$\sum_j J_{1j} J_{j2}^n = \lambda \cdot n\lambda^{n-1} + 1 \cdot \lambda^n = (n+1)\lambda^n$$

$$\sum_j J_{1j} J_{j3}^n = \binom{n}{2}\lambda \cdot \lambda^{n-2} + 1 \cdot n\lambda^{n-1} = \binom{n+1}{2}\lambda^{n-1}$$

In the last equation, we used

$$\binom{n}{2} + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

The elements in the second and third row can be handled the same way, although with slightly less work. The solution in par! t b) provides a more elegant and more general solution.

b) Generalize a) to a k by k block of a Jordan form J. Note that the nth power of an entire Jordan form is composed of these blocks along the diagonal of the matrix.

Solution: This is rather difficult unless looked at in just the right way, but the right way is quite instructive and often useful. Thus we hope you didn't spin your wheels too much on this, but hope you will learn from the solution. The idea comes from the elegant way to solve part a) of Exercise 3.14, namely taking a graphical approach rather than a matrix multiplication approach. Forgetting for the moment that J is not a stochastic matrix, we can draw its graph, i.e., the representation of the nonzero elements of the matrix, as
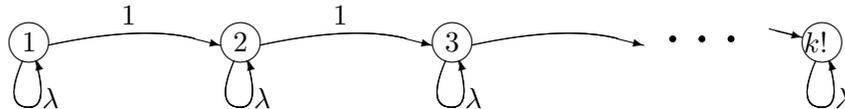
Each edge, say from $i \to j$ in the graph, represents a nonzero element of J and is labelled as J_{ij}. If we look at $[J^2]$, then $J_{ij}^2 = \sum_k J_{ik} J_{kj}$ is the sum over the walks of length 2 in the graph of the 'measure' of that walk, where the measure of a walk is the product of the labels on the edges of that walk. Similarly, we can see that J_{ij}^n is the sum over walks of

length n of the product of the labels for that walk. In short, J_{ij}^n can be found from the graph in the same way as for stochastic matrices; we simply ignore the fact that the out! going edges from each node do not sum to 1.

We can now see immediately how to evaluate these elements. For J_{ij}^n, we must look at all the walks of length n that go from i to j. For $i > j$, the number is 0, so we assume $j \geq i$. Each such walk must include $j - i$ rightward transitions and thus must include $n - j + i$ self loops. Thus the measure of each such walk is λ^{n-j+i}. The number of such walks is the number of ways that $j - i$ rightward transitions can be selected out the n transitions altogether, i.e., $\binom{n}{j-i}$. Thus, the general formula is

$$J_{ij}^n = \binom{n}{j - i} \lambda^{j-i} \qquad \text{for } j > i$$

c) Let $[P]$ be a stochastic matrix represented by a Jordan form $[J]$ as $[P] = U[J][U^{-1}]$ and consider $[U^{-1}][P][U] = [J]$. Show that any repeated eigenvalue of $[P]$ that is represented by a Jordan block of 2 by 2 or more must be strictly less than 1. Hint: Upper bound the elements of $[U^{-1}][P^n][U]$! by taking the magnitude of the elements of $[U]$ and $[U^{-!1}]$ an d upper bounding each element of a stochastic matrix by 1.

Solution: Each element of the matrix $[U^{-1}][P^n][U]$ is a sum of products of terms and the magnitude of that sum is upper bounded by the sum of products of the magnitudes of the terms. Representing the matrix whose elements are the magnitudes of a matrix $[U]$ as $[|U|]$, and recognizing that $[P^n] = [|P^n|]$ since its elements are nonnegative, we have

$$[|J^n|] \leq [|U^{-1}|][P^n][|U|] \leq [|U^{-1}|][M][|U|]$$

where $[M]$ is a matrix whose elements are all equal to 1.

The matrix on the right is independent of n, so each of its elements is simply a finite number. If $[P]$ has an eigenvalue λ of magnitude 1 whose multiplicity exceeds its number of linearly independent eigenvectors, then $[J^n]$ contains an element $n\lambda^{n-1}$ of magnitude n, and for large enough n, this is larger than the finite number above, yielding a contradiction. Thus, for any stochastic ! matrix, any repeated eigenvalue with a defective number of linearly independent eigenvectors has a magnitude strictly less than 1.

d) Let λ_s be the eigenvalue of largest magnitude less than 1. Assume that the Jordan blocks for λ_s are at most of size k. Show that each ergodic class of $[P]$ converges at least as fast as $n^k \lambda_s^k$.

Solution: There was a typo in the original problem statement; convergence as $n^k \lambda^k$ does not converge at all with increasing n. What was intended was $n^k \lambda_s^n$. What we want to show, then, is that $|[P^n] - \vec{e}\boldsymbol{\pi}| \leq bn^k |\lambda_s^n|$ for some sufficiently large constant b depending on $[P]$ but not n.

The states of an ergodic class have no transitions out, and for questions of convergence within the class we can ignore transitions in. Thus we will get rid of some excess notation by simply assuming an ergodic Markov chain. Thus there is a single eigenvalue equal to 1 and all other eigenvalues are strictly less than 1. The largest such in magnitude is denoted as λ_s. Assume that the eigenvalues are arranged in $[J]$ in decreasing magnitude, so that $J_{11} = 1$.

For all n, we have $[P^n] = [U][J^n][U^{-1}]$. Note that $\lim_{n\to\infty}[J^n]$ is a matrix with $J_{ij} = 0$ except for $i = j = 1$. Thus it can be seen that the first column of $[U]$ is $\vec{e}$ and the first row of $[U^{-1}]$ is $\boldsymbol{\pi}$. It follows from this that $[P^n] - \vec{e}\boldsymbol{\pi} = [U][\tilde{J}^n][U^{-1}]$ where $[\tilde{J}^n]$ is the same as $[J^n]$ except that the eigenvalue 1 has been replaced by 0. This means, however, that the magnitude of each element of $[\tilde{J}^n]$ is upper bounded by $n^k \lambda_s^{n-k}$. It follows that when the magnitude of the elements of $[U][\tilde{J}^n][U^{-1}]$ the resulting elements are at most $bn^k|\lambda_s|^n$ for large enough b (the value of b takes account of $[U], [U^{-1}]$, and λ_s^{-k}).

Exercise 3.22

a) Here state 1 is split into to states 1 and $1'$ where state 1 can be the initial state and state $1'$ is the trapping state. You observe that there is a transition from state 1 to state $1'$ in the modified Markov chain which corresponds to the self transition of state 1 in the original Markov chain. You should assign reward 1 to states 1 to 4 and reward 0 to state $1'$ in order to be able to find the expected first passage time.
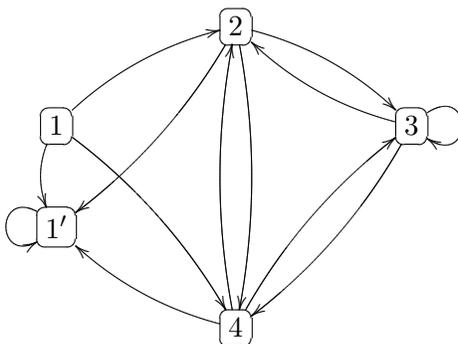
b) If the initial state is state 1, with probability p_{11} it takes one step to go into state 1 again. But with probability p_{1j}, the expected recurrence time will be $1 + v_j$. Thus the expected first recurrence time will be:

$$v_1 = p_{11} + \sum_{j=2}^{M} p_{1j}(1 + v_j) = 1 + \sum_{j=2}^{M} p_{1j}v_j$$

You can see the same thing in the modified Markov chain in which v_j for $j = 1, \cdots, M$ is defined as the expected first passage time from state j to state $1'$. Thus, for all $j = 1, \cdots, M$,

$$v_j = p_{j1'} + \sum_{k=2}^{M} p_{jk}(1 + v_j) = 1 + \sum_{k=2}^{M} p_{jk}v_k$$

So for $j = 1$,

$$v_1 = 1 + \sum_{j=2}^{M} p_{1j}v_j$$

Exercise 3.26

a) In this Markov reward system, state 1 is the trapping state, So the reward values will be: $\mathbf{r} = (r_1, r_2, r_3) = (0, 1, 1)$. Thus set of rewards will help in counting the number of transitions to finally get trapped in state 1 and then it will stay in state 1 forever and the reward will not increase anymore (because $r_1 = 0$).

We know that

$$
\begin{aligned}
v_i(n) &= \mathbb{E}[R(X_m) + R(X_{m+1}) + \cdots + R(X_{m+n-1})|X_m = i] \\
&= r_i + \sum_i p_{ij} r_j + \cdots + \sum_j p_{ij}^{n-1} r_j
\end{aligned}
$$

Note that each increase in n adds one additional term to this expression, and thus the expression can be written as:

$$
v_i(n) = v_i(n-1) + \sum_j p_{ij}^{n-1} r_j
$$

In vector form, this is

$$
\vec{v}(n) = \vec{v}(n-1) + [P^{n-1}]\vec{r},
$$

which gives the desired expression for $\vec{v}(n)$ in terms of $\vec{v}(n-1)$

An alternate, and perfectly acceptable approach, which gives an alternate expression for $\vec{v}$ in terms of $\vec{v}(n-1)$ is:

$$
\begin{aligned}
\vec{v}(n) &= \vec{r} + [P]\vec{r} + \cdots + [P^{n-1}]\vec{r} = \sum_{h=0}^{n-1} [P^h]\vec{r} \\
&= [P]\vec{v}(n-1) + \vec{r}
\end{aligned}
$$

b) We start by following the first approach in part a), which requires us to calculate $[P^n]$. Noting that state 2 can never lead to state 3, we note that $P_{12}^n = (1/2)^n$. Using this in finding the rest of $[P^n]$, some calculation leads to :

$$
[P] = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}
$$

$$[P^n] = \begin{bmatrix} 1 & 0 & 0 \\ 1 - \frac{1}{2^n} & \frac{1}{2^n} & 0 \\ 1 - \frac{2^{n+1}-1}{2^{2n}} & \frac{2^n-1}{2^{2n-1}} & \frac{1}{2^{2n}} \end{bmatrix}$$

For each $n \geq 1$, $\vec{v}(n)$ can be calculated:

$$\vec{v}(n) = \vec{v}(n-1) + \begin{bmatrix} 0 \\ \frac{1}{2^{n-1}} \\ \frac{2^n-1}{2^{2n-2}} \end{bmatrix}$$

We know that $\vec{v}(0) = \vec{0}$. Thus for $n \geq 1$,

$$\vec{v}(n) = \sum_{h=1}^{n} \begin{bmatrix} 0 \\ \frac{1}{2^{h-1}} \\ \frac{2^h-1}{2^{2h-2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_{h=1}^{n} \frac{1}{2^{h-1}} \\ \sum_{h=1}^{n} \frac{2^h-1}{2^{2h-2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 2(1 - \frac{1}{2^n}) \\ \frac{8}{3} - \frac{1}{2^{n-2}} + \frac{1}{3 \times 4^{n-1}} \end{bmatrix}$$

As $n \to \infty$, we get the $\lim_{n \to \infty} \vec{v}(n) = \begin{bmatrix} 0 \\ 2 \\ \frac{8}{3} \end{bmatrix}$.

An alternative way of looking at this vector is by studying the following equation:

$$\vec{v}(n) = [P]\vec{v}(n-1) + \vec{r}$$

Considering that for all n, $v_1(n) = 0$, we would have:

$$\vec{v}(n) = \begin{bmatrix} 0 \\ \frac{1}{2}v_2(n-1) + 1 \\ \frac{1}{2}v_2(n-1) + \frac{1}{4}v_3(n-1) + 1 \end{bmatrix}$$

Which gives the same result in closed form for $\vec{v(n)}$ with some tedious calculations.

Each element $v_j(n)$ is the expected first passage time of the Markov chain corresponding to the first time that the Markov chain ends in state 1 condition on the assumption that it is in state j at $n = 0$.

c) Eq. 3.32 in the text, $\vec{v} = \vec{t} + [P]\vec{v}$ is the same as the second alternative solution in part a), with the limit $n \to \infty$ already taken. The solution here shows that such a limit

actually exists. As a direct argument for this limit to exist, note that it should be a fixed point for the equation. Thus,

$$\mathbf{v} = [P]\mathbf{v} + \mathbf{r}$$

$$(I - P)\mathbf{v} = \mathbf{r}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Letting $v_1(n) = 0$, we would easily get $v_2 = 2$ and $v_3 = \frac{8}{3}$. You observe that the fixed point of the equation $\mathbf{v} = [P]\mathbf{v} + \mathbf{r}$ is the limit of the expected aggregate reward of the Markov chain.

Exercise 4.1

a) WLLN says that S_n converges in probability to $n\bar{X}$ meaning that for each $s > 0$

$$\lim_{n \to \infty} \Pr\{|S_n - n\bar{X}| > s\} = 0$$

.

$$\lim_{n \to \infty} \Pr\{n\bar{X} - s \le S_n \le n\bar{X} + s\} = 1$$

For any fixed $t > 0$ and $s > 0$, for large enough n, $t < n\bar{X} - s$ (assuming that $\bar{X} > 0$). Thus,

$$\lim_{n \to \infty} \Pr\{S_n \le t\} < \lim_{n \to \infty} [1 - \Pr\{n\bar{X} - s \le S_n \le n\bar{X} + s\}] = 0$$

b) We know that for each n and t, these events are equivalent to each other:

$$\{S_n \le t\} = \{N(t) \ge n\}$$

. It is also proved in Lemma 4.3.1 that $\lim_{t \to \infty} N(t) = \infty$ with probability 1. So with probability 1, $N(t)$ increases through all the nonnegative integers as t increase from 0 to ∞. So we can write:

$$\lim_{n \to \infty} \Pr\{N(t) \ge n\} = \lim_{t \to \infty} \Pr\{N(t) \ge n\} = \lim_{t \to \infty} \Pr\{S_n \le t\} = 0$$

. So, $\lim_{n \to \infty} \Pr\{N(t) < n\} = 1$ meaning that $N(t)$ is a non-defective random variable for each $t > 0$.

c) First, we can get some bound on the value of $\mathbb{E}[\check{X}_i] = \mathbb{E}[\min(X_i, b)]$:

$$\begin{aligned}
\mathbb{E}[\breve{x}] &= \int_0^\infty \breve{x}\,dF(\breve{x}) \\
&= \int_0^\infty \min(x,b)\,dF(\breve{x}) \\
&= \int_0^b \min(x,b)\,dF(x) + \int_b^\infty \min(x,b)\,dF(x) \\
&= \int_0^b x\,dF(x) + \int_b^\infty b\,dF(x) \\
&\leq b\int_0^b dF(x) + b\int_b^\infty dF(x) \\
&\leq b
\end{aligned}$$

We see that $\mathbb{E}[\breve{X}] \leq b$ is finite. As proved in part (a) and (b), the finiteness of $\mathbb{E}[\breve{X}_i]$ is enough for $\breve{N}(t)$ to be non-defective.

Having $\breve{X}_i = \min(X_i, b)$ for all i, we know that $\breve{X}_i \leq X_i$. Thus, for each n, $\breve{S}_n = \sum_{i=1}^n \breve{X}_i \leq \sum_{i=1}^n X_i = S_n$. So for each $n \geq 1$, $\breve{S}_n \leq S_n$. We can say that the following relations are held for these events for all n and t:

$$\{N(t) \geq n\} = \{S_n \leq t\} \subseteq \{\breve{S}_n \leq t\} = \{\breve{N}(t) \geq n\}$$

For each n and t, $\{N(t) \geq n\} \subseteq \{\breve{N}(t) \geq n\}$. This mens that for all t, $N(t) \leq \breve{N}(t)$. It also means that $\Pr\{N(t) \geq n\} \leq \Pr\{\breve{N}(t) \geq n\}$. So, $\lim_{n\to\infty} \Pr\{N(t) \geq n\} \leq \lim_{n\to\infty} \Pr\{\breve{N}(t) \geq n\} = 0$. Thus, $\breve{N}(t)$ is non-defective too and $\lim_{n\to\infty} \Pr\{N(t) \geq n\} = 0$.

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