LECTURE 20

LECTURE OUTLINE

- Approximation methods
- Cutting plane methods
- Proximal minimization algorithm
- Proximal cutting plane algorithm
- Bundle methods

APPROXIMATION APPROACHES

- Approximation methods replace the original problem with an approximate problem.

- The approximation may be iteratively refined, for convergence to an exact optimum.

- A partial list of methods:
  - Cutting plane/outer approximation.
  - Simplicial decomposition/inner approximation.
  - Proximal methods (including Augmented Lagrangian methods for constrained minimization).
  - Interior point methods.

- A partial list of combination of methods:
  - Combined inner-outer approximation.
  - Bundle methods (proximal-cutting plane).
  - Combined proximal-subgradient (incremental option).

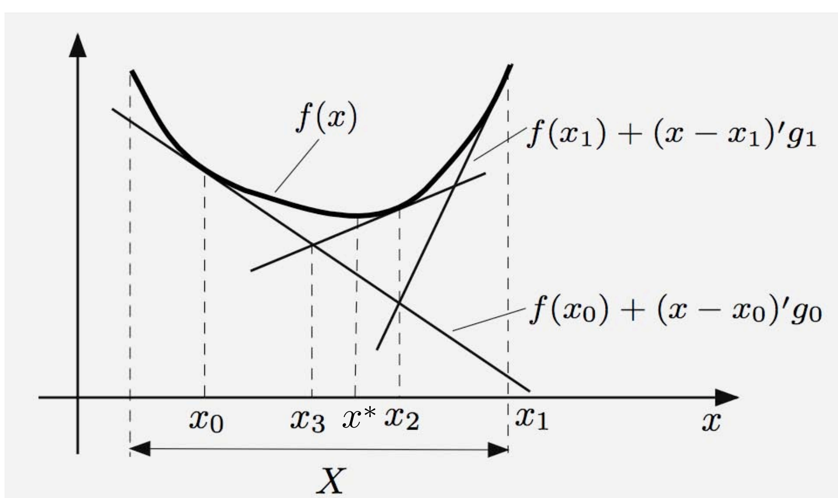
SUBGRADIENTS-OUTER APPROXIMATION

- Consider minimization of a convex function $f : \Re^n \mapsto \Re$, over a closed convex set X.
- We assume that at each $x \in X$, a subgradient g of f can be computed.
- We have

$$f(z) \geq f(x) + g'(z - x), \qquad \forall\, z \in \Re^n,$$

so each subgradient defines a plane (a linear function) that approximates f from below.

- The idea of the outer approximation/cutting plane approach is to build an ever more accurate approximation of f using such planes.



CUTTING PLANE METHOD

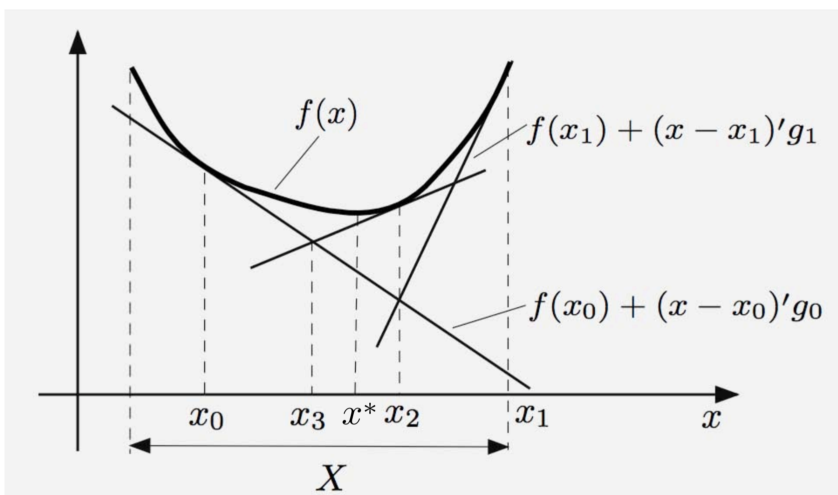
- Start with any $x_0 \in X$. For $k \geq 0$, set

$$x_{k+1} \in \arg\min_{x \in X} F_k(x),$$

where

$$F_k(x) = \max\left\{f(x_0)+(x-x_0)'g_0, \ldots, f(x_k)+(x-x_k)'g_k\right\}$$

and g_i is a subgradient of f at x_i.



- Note that $F_k(x) \leq f(x)$ for all x, and that $F_k(x_{k+1})$ increases monotonically with k. These imply that all limit points of x_k are optimal.

Proof: If $x_k \to x$ then $F_k(x_k) \to f(x)$, [otherwise there would exist a hyperplane strictly separating epi(f) and $(x, \lim_{k\to\infty} F_k(x_k))$]. This implies that $f(x) \leq \lim_{k\to\infty} F_k(x) \leq f(x)$ for all x. **Q.E.D.**

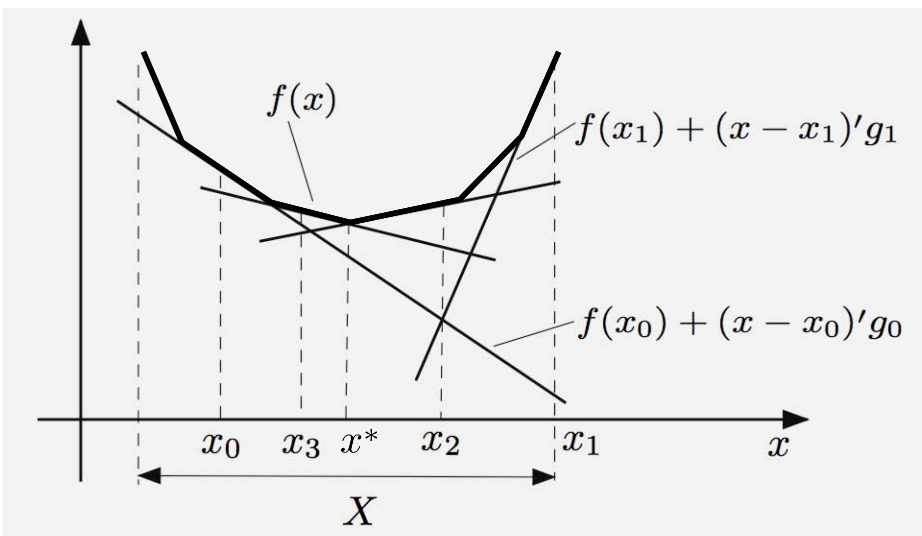
CONVERGENCE AND TERMINATION

- We have for all k

$$F_k(x_{k+1}) \leq f^* \leq \min_{i \leq k} f(x_i)$$

- Termination when $\min_{i\leq k} f(x_i) - F_k(x_{k+1})$ comes to within some small tolerance.

- For f polyhedral, we have finite termination with an exactly optimal solution.

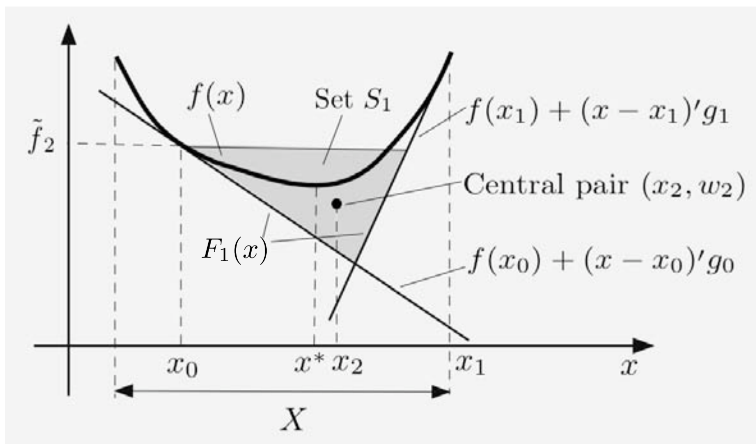


- **Instability problem:** The method can make large moves that deteriorate the value of f.

- Starting from the exact minimum it typically moves away from that minimum.

VARIANTS

- **Variant I:** Simultaneously with f, construct polyhedral approximations to X.
- **Variant II:** Central cutting plane methods



- **Variant III:** Proximal methods - to be discussed next.

PROXIMAL/BUNDLE METHODS

- Aim to reduce the instability problem at the expense of solving a more difficult subproblem.
- A general form:

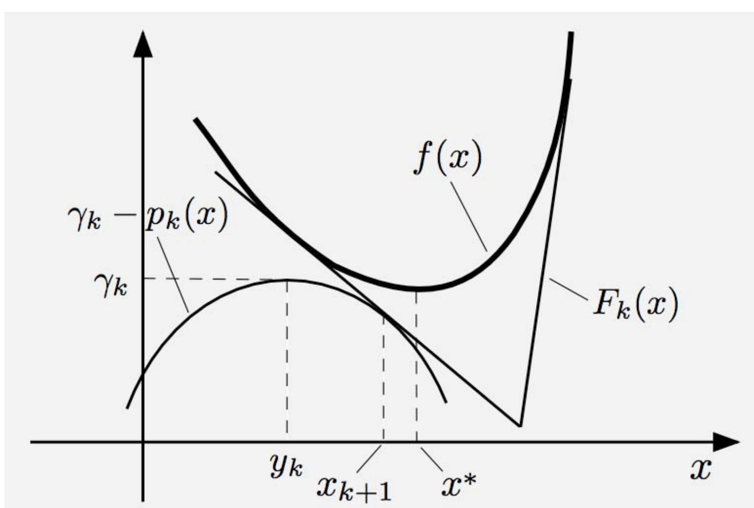
$$x_{k+1} \in \arg\min_{x \in X} \big\{ F_k(x) + p_k(x) \big\}$$

$$F_k(x) = \max\big\{ f(x_0) + (x - x_0)' g_0, \ldots, f(x_k) + (x - x_k)' g_k \big\}$$

$$p_k(x) = \frac{1}{2c_k} \|x - y_k\|^2$$

where c_k is a positive scalar parameter.

- We refer to $p_k(x)$ as the *proximal term*, and to its center y_k as the *proximal center*.

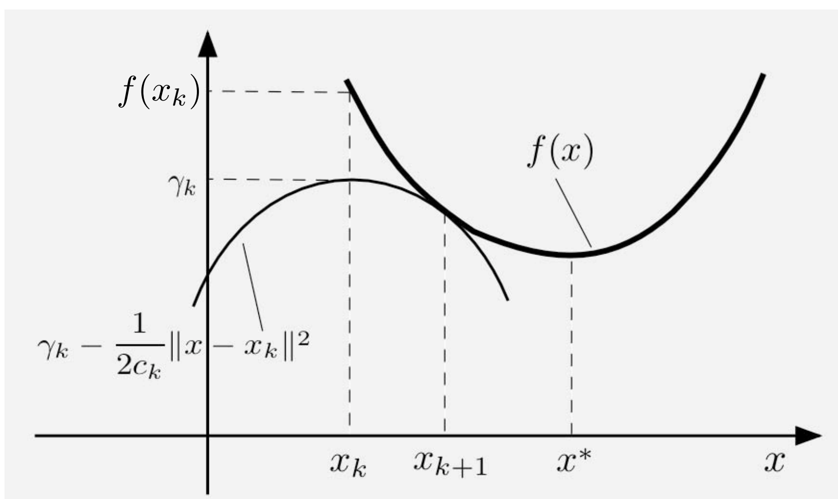


PROXIMAL MINIMIZATION ALGORITHM

- Starting point for analysis: A general algorithm for convex function minimization

$$x_{k+1} \in \arg \min_{x \in \Re^n} \left\{ f(x) + \frac{1}{2c_k} \|x - x_k\|^2 \right\}$$

  - $f : \Re^n \mapsto (-\infty, \infty]$ is closed proper convex
  - c_k is a positive scalar parameter
  - x_0 is arbitrary starting point



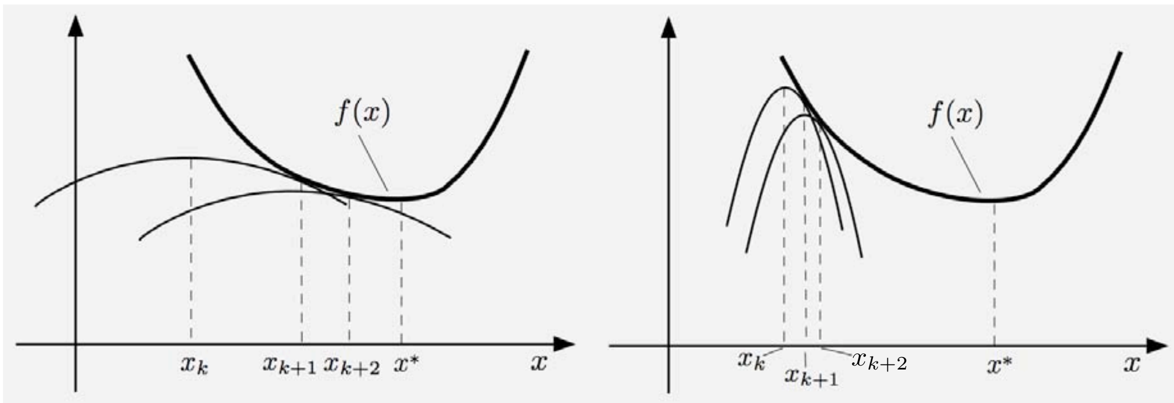
- Convergence mechanism:

$$\gamma_k = f(x_{k+1}) + \frac{1}{2c_k} \|x_{k+1} - x_k\|^2 < f(x_k).$$

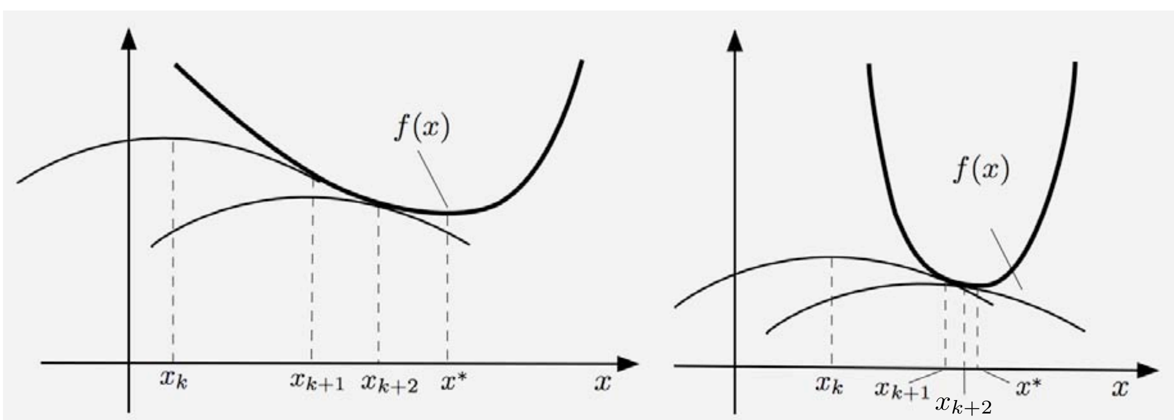
Cost improves by at least $\frac{1}{2c_k} \|x_{k+1} - x_k\|^2$, and this is sufficient to guarantee convergence.

RATE OF CONVERGENCE I

- Role of penalty parameter c_k:



- Role of growth properties of f near optimal solution set:



RATE OF CONVERGENCE II

- Assume that for some scalars $\beta > 0$, $\delta > 0$, and $\alpha \geq 1$,

$$f^* + \beta\big(d(x)\big)^\alpha \leq f(x), \quad \forall\, x \in \Re^n \ \text{ with } d(x) \leq \delta$$

where

$$d(x) = \min_{x^* \in X^*} \|x - x^*\|$$

i.e., **growth of order α from optimal solution set X^*.**

- If $\alpha = 2$ and $\lim_{k\to\infty} c_k = \bar{c}$, then

$$\limsup_{k\to\infty} \frac{d(x_{k+1})}{d(x_k)} \leq \frac{1}{1 + \beta\bar{c}}$$

linear convergence.

- If $1 < \alpha < 2$, then

$$\limsup_{k\to\infty} \frac{d(x_{k+1})}{\big(d(x_k)\big)^{1/(\alpha-1)}} < \infty$$

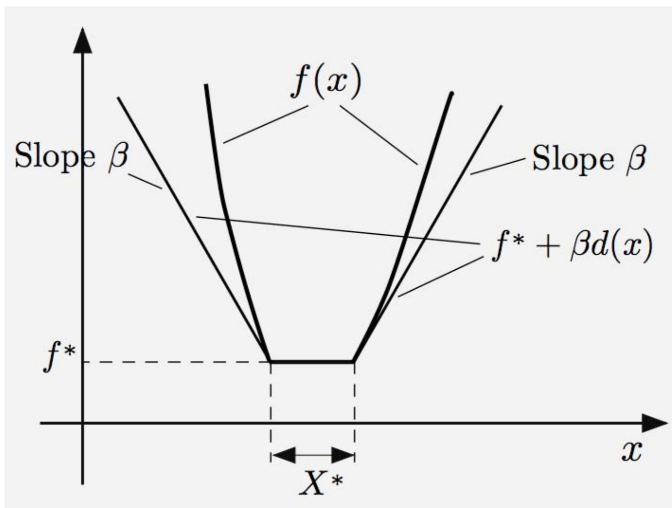
superlinear convergence.

FINITE CONVERGENCE

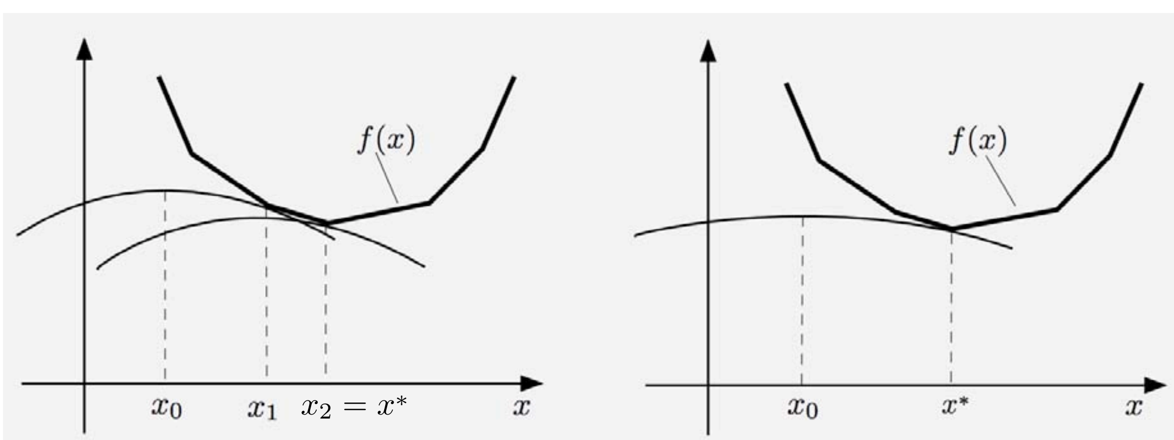
- Assume growth order $\alpha = 1$:

$$f^* + \beta d(x) \leq f(x), \qquad \forall\, x \in \Re^n,$$

e.g., f is polyhedral.



- Method converges finitely (in a single step for c_0 sufficiently large).



PROXIMAL CUTTING PLANE METHODS

- Same as proximal minimization algorithm, but f **is replaced by a cutting plane approximation** F_k**:**

$$x_{k+1} \in \arg\min_{x \in X} \left\{ F_k(x) + \frac{1}{2c_k} \|x - x_k\|^2 \right\}$$

where

$$F_k(x) = \max\left\{ f(x_0) + (x - x_0)' g_0, \ldots, f(x_k) + (x - x_k)' g_k \right\}$$

- Drawbacks:

  (a) **Hard stability tradeoff:** For large enough c_k and polyhedral X, x_{k+1} is the exact minimum of F_k over X in a single minimization, so it is identical to the ordinary cutting plane method. For small c_k convergence is slow.

  (b) **The number of subgradients used in** F_k **may become very large;** the quadratic program may become very time-consuming.

- These drawbacks motivate algorithmic variants, called *bundle methods*.

BUNDLE METHODS

- Allow a proximal center $y_k \neq x_k$:

$$x_{k+1} \in \arg\min_{x \in X} \left\{ F_k(x) + p_k(x) \right\}$$

$$F_k(x) = \max\left\{ f(x_0)+(x-x_0)'g_0, \ldots, f(x_k)+(x-x_k)'g_k \right\}$$

$$p_k(x) = \frac{1}{2c_k} \|x - y_k\|^2$$

- **Null/Serious test** for changing y_k: For some fixed $\beta \in (0,1)$

$$y_{k+1} = \begin{cases} x_{k+1} & \text{if } f(y_k) - f(x_{k+1}) \geq \beta\delta_k, \\ y_k & \text{if } f(y_k) - f(x_{k+1}) < \beta\delta_k, \end{cases}$$

$$\delta_k = f(y_k) - \Big(F_k(x_{k+1}) + p_k(x_{k+1}) \Big) > 0$$