LECTURE 14

LECTURE OUTLINE

- Min-Max Duality
- Existence of Saddle Points

Given $\phi : X \times Z \mapsto \Re$, where $X \subset \Re^n$, $Z \subset \Re^m$ consider

$$\begin{aligned} &\text{minimize} \ \sup_{z \in Z} \phi(x, z) \\ &\text{subject to} \ x \in X \end{aligned}$$

and

$$\begin{aligned} &\text{maximize} \ \inf_{x \in X} \phi(x, z) \\ &\text{subject to} \ z \in Z. \end{aligned}$$

REVIEW

- **Minimax inequality** (holds always)

$$\sup_{z\in Z}\inf_{x\in X}\phi(x,z)\le\inf_{x\in X}\sup_{z\in Z}\phi(x,z)$$

Important issue is whether minimax *equality* holds.

- **Definition:** (x^*, z^*) is called a *saddle point* of ϕ if

$$\phi(x^*,z)\le\phi(x^*,z^*)\le\phi(x,z^*),\quad\forall\,x\in X,\ \forall\,z\in Z$$

- **Proposition**: (x^*, z^*) is a saddle point if and only if the minimax equality holds and

$$x^*\in\arg\min_{x\in X}\sup_{z\in Z}\phi(x,z),\quad z^*\in\arg\max_{z\in Z}\inf_{x\in X}\phi(x,z)$$

- **Connection w/ constrained optimization:**
  - Strong duality is equivalent to

$$\inf_{x\in X}\sup_{\mu\ge 0}L(x,\mu)=\sup_{\mu\ge 0}\inf_{x\in X}L(x,\mu)$$

    where L is the Lagrangian function.
  - Optimal primal-dual solution pairs (x^*, μ^*) are the saddle points of L.

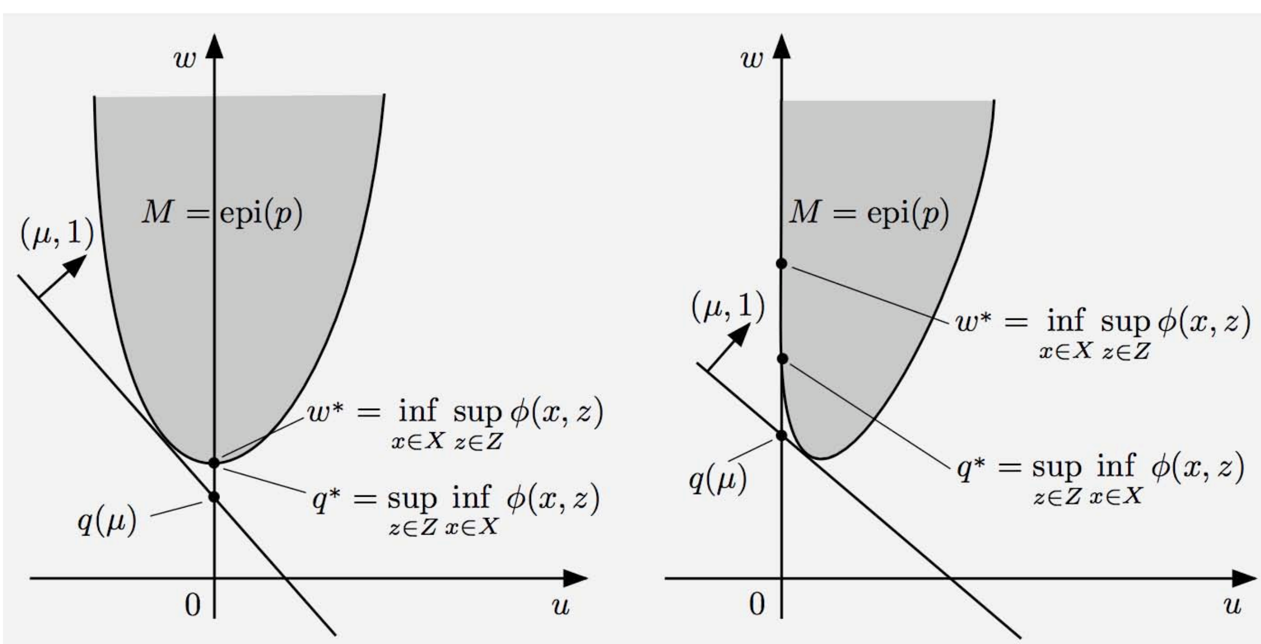
MC/MC FRAMEWORK FOR MINIMAX

- Use MC/MC with $M = \text{epi}(p)$ where $p : \Re^m \mapsto [-\infty, \infty]$ is the perturbation function

$$p(u) = \inf_{x\in X} \sup_{z\in Z} \{\phi(x,z) - u'z\}, \qquad u \in \Re^m$$

- **Important fact:** p is obtained by partial min.
- Note that $w^* = p(0) = \inf \sup \phi$ and $\phi(\cdot, z)$: convex for all z implies that M is convex.
- If $-\phi(x, \cdot)$ is closed and convex, the dual function in MC/MC is

$$q(z) = \inf_{x\in X} \phi(x,z), \qquad q^* = \sup \inf \phi$$



MINIMAX THEOREM I

Assume that:

(1) X and Z are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \Re$ is closed and convex.

Then, the minimax equality holds if and only if the function p is lower semicontinuous at $u = 0$.

Proof: The convexity/concavity assumptions guarantee that the minimax equality is equivalent to $q^* = w^*$ in the min common/max crossing framework. Furthermore, $w^* < \infty$ by assumption, and the set M [equal to M and epi(p)] is convex.

By the 1st Min Common/Max Crossing Theorem, we have $w^* = q^*$ iff for every sequence $\{(u_k, w_k)\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k \to \infty} w_k$. This is equivalent to the lower semicontinuity assumption on p:

$$p(0) \leq \liminf_{k \to \infty} p(u_k), \quad \text{for all } \{u_k\} \text{ with } u_k \to 0$$

MINIMAX THEOREM II

Assume that:

(1) X and Z are convex.

(2) $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) > -\infty$.

(3) For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.

(4) For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \Re$ is closed and convex.

(5) 0 lies in the relative interior of $\mathrm{dom}(p)$.

Then, the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$. [Also the set of z where the sup is attained is compact if 0 is in the interior of $\mathrm{dom}(p)$.]

Proof: Apply the 2nd Min Common/Max Crossing Theorem.

- Counterexamples of strong duality and existence of solutions/saddle points can be constructed from corresponding constrained min examples.

EXAMPLE I

- Let $X = \{(x_1, x_2) \mid x \geq 0\}$ and $Z = \{z \in \Re \mid z \geq 0\}$, and let

$$\phi(x, z) = e^{-\sqrt{x_1 x_2}} + z x_1,$$

which satisfy the convexity and closedness assumptions. For all $z \geq 0$,

$$\inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = 0,$$

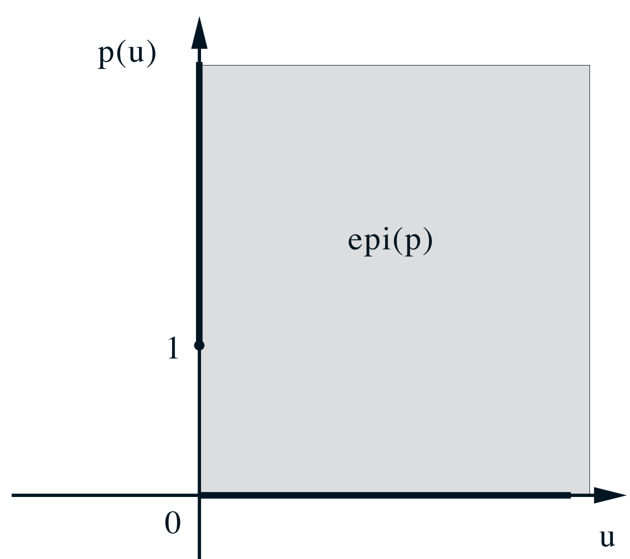
so $\sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0$. Also, for all $x \geq 0$,

$$\sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z x_1 \right\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0, \end{cases}$$

so $\inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1$.

- Here

$$p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + z(x_1 - u) \right\}$$



EXAMPLE II

- Let $X = \Re$, $Z = \{z \in \Re \mid z \geq 0\}$, and let

$$\phi(x, z) = x + zx^2,$$

which satisfy the convexity and closedness assumptions. For all $z \geq 0$,

$$\inf_{x \in \Re} \{x + zx^2\} = \begin{cases} -1/(4z) & \text{if } z > 0, \\ -\infty & \text{if } z = 0, \end{cases}$$

so $\sup_{z \geq 0} \inf_{x \in \Re} \phi(x, z) = 0$. Also, for all $x \in \Re$,

$$\sup_{z \geq 0} \{x + zx^2\} = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise}, \end{cases}$$

so $\inf_{x \in \Re} \sup_{z \geq 0} \phi(x, z) = 0$. However, the sup is not attained, i.e., there is no saddle point.

- Here

$$\begin{aligned} p(u) &= \inf_{x \in \Re} \sup_{z \geq 0} \{x + zx^2 - uz\} \\ &= \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases} \end{aligned}$$

SADDLE POINT ANALYSIS

- The preceding analysis indicates the importance of the perturbation function

$$p(u) = \inf_{x \in \Re^n} F(x, u),$$

where

$$F(x, u) = \begin{cases} \sup_{z \in Z} \{\phi(x, z) - u'z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

It suggests a two-step process to establish the minimax equality and the existence of a saddle point:

(1) **Show that p is closed and convex**, thereby showing that the minimax equality holds by using the first minimax theorem.

(2) **Verify that the inf of** $\sup_{z \in Z} \phi(x, z)$ **over** $x \in X$**, and the sup of** $\inf_{x \in X} \phi(x, z)$ **over** $z \in Z$ **are attained**, thereby showing that the set of saddle points is nonempty.

SADDLE POINT ANALYSIS (CONTINUED)

- Step (1) requires two types of assumptions:
  (a) Convexity/concavity/semicontinuity conditions of Minimax Theorem I (so the MC/MC framework applies).
  (b) Conditions for preservation of closedness by the partial minimization in

$$p(u) = \inf_{x \in \Re^n} F(x, u)$$

  e.g., for some u, the nonempty level sets

$$\{x \mid F(x, u) \leq \gamma\}$$

  are compact.

- Step (2) requires that either Weierstrass' Theorem can be applied, or else one of the conditions for existence of optimal solutions developed so far is satisfied.

CLASSICAL SADDLE POINT THEOREM

- Assume convexity/concavity/semicontinuity of ϕ and that X and Z are compact. Then the set of saddle points is nonempty and compact.

- **Proof:** F is convex and closed by the convexity/concavity/semicontinuity of ϕ, so p is also convex. Using the compactness of Z, F is real-valued over $X \times \Re^m$, and from the compactness of X, it follows that p is also real-valued and therefore continuous. Hence, the minimax equality holds by the first minimax theorem.

  The function $\sup_{z \in Z} \phi(x, z)$ is equal to $F(x, 0)$, so it is closed, and the set of its minima over $x \in X$ is nonempty and compact by Weierstrass' Theorem. Similarly the set of maxima of the function $\inf_{x \in X} \phi(x, z)$ over $z \in Z$ is nonempty and compact. Hence the set of saddle points is nonempty and compact. **Q.E.D.**

ANOTHER THEOREM

- Use the theory of preservation of closedness under partial minimization.

- Assume convexity/concavity/semicontinuity of ϕ. Consider the functions

$$t(x) = F(x,0) = \begin{cases} \sup_{z\in Z} \phi(x,z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

and

$$r(z) = \begin{cases} -\inf_{x\in X} \phi(x,z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z. \end{cases}$$

- If the level sets of t are compact, the minimax equality holds, and the min over x of

$$\sup_{z\in Z} \phi(x,z)$$

[which is $t(x)$] is attained. (Take $u = 0$ in the partial min theorem to show that p is closed.)

- If the level sets of t and r are compact, the set of saddle points is nonempty and compact.

- Various extensions: Use conditions for preservation of closedness under partial minimization.

SADDLE POINT THEOREM

Assume the convexity/concavity/semicontinuity conditions, and that any *one* of the following holds:

(1) X and Z are compact.

(2) Z is compact and there exists a vector $z \in Z$ and a scalar γ such that the level set $\{x \in X \mid \phi(x,z) \le \gamma\}$ is nonempty and compact.

(3) X is compact and there exists a vector $x \in X$ and a scalar γ such that the level set $\{z \in Z \mid \phi(x,z) \ge \gamma\}$ is nonempty and compact.

(4) There exist vectors $x \in X$ and $z \in Z$, and a scalar γ such that the level sets

$$\{x \in X \mid \phi(x,z) \le \gamma\}, \quad \{z \in Z \mid \phi(x,z) \ge \gamma\},$$

are nonempty and compact.

Then, the minimax equality holds, and the set of saddle points of ϕ is nonempty and compact.