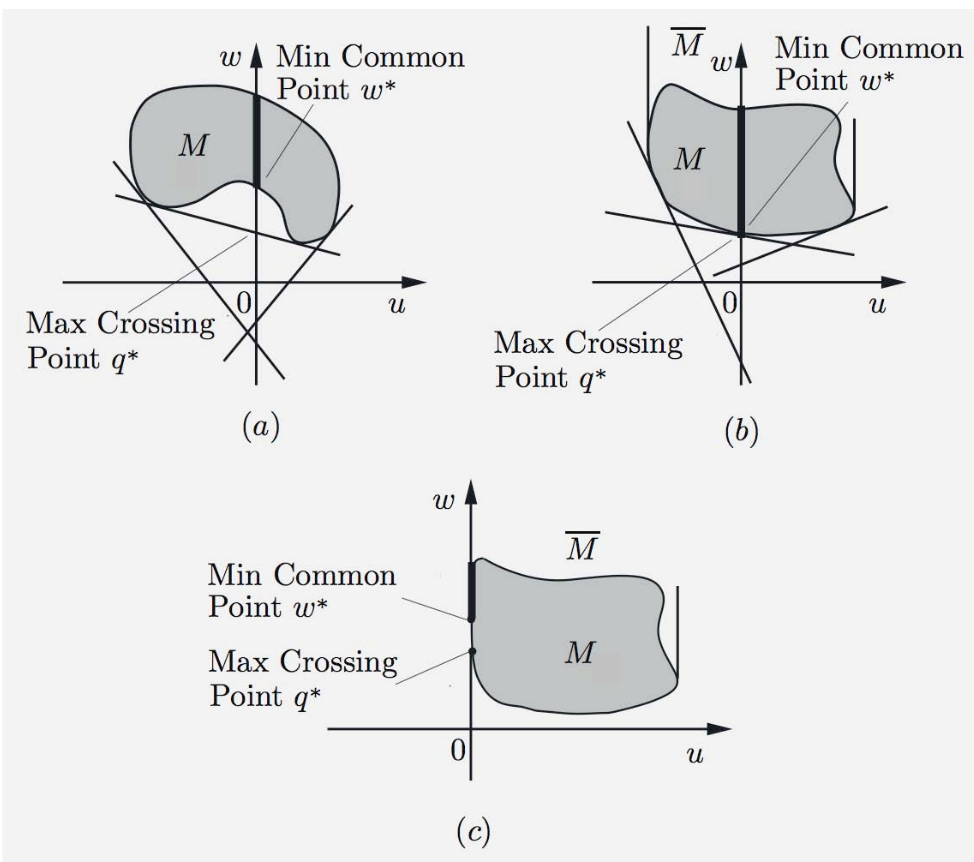
LECTURE 10

LECTURE OUTLINE

- Min Common / Max Crossing duality theorems
- Strong duality conditions
- Existence of dual optimal solutions
- Nonlinear Farkas' lemma

Reading: Sections 4.3, 4.4, 5.1



DUALITY THEOREMS

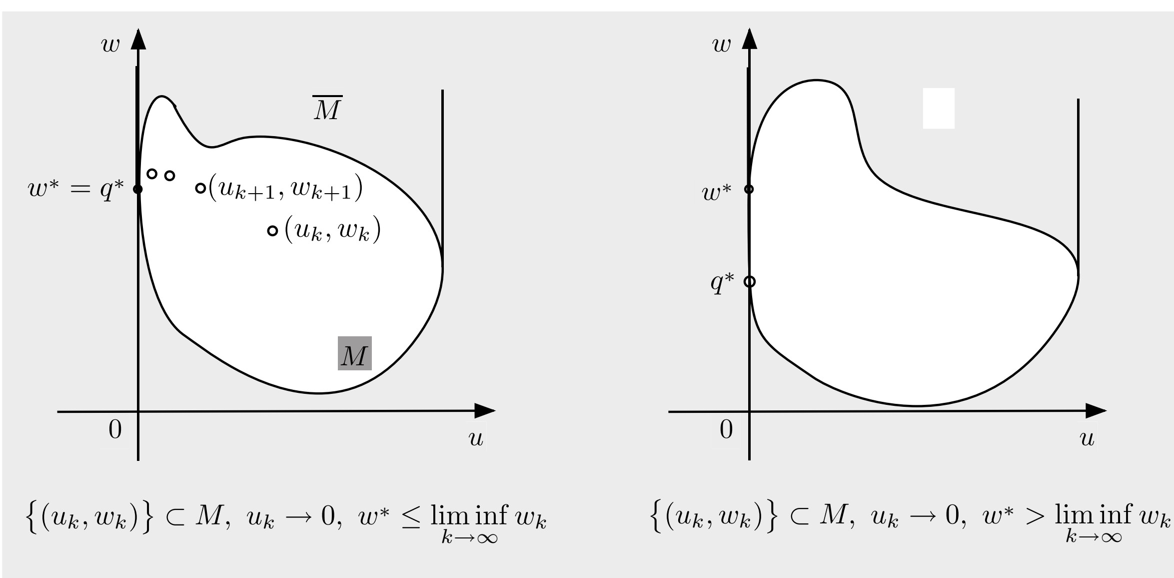
- Assume that $w^* < \infty$ and that the set

$$M = \Big\{ (u, w) \mid \text{there exists } w \text{ with } w \le w \text{ and } (u, w) \in M \Big\}$$

is convex.

- **Min Common/Max Crossing Theorem I:** We have $q^* = w^*$ if and only if for every sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$, there holds

$$w^* \le \liminf_{k \to \infty} w_k.$$



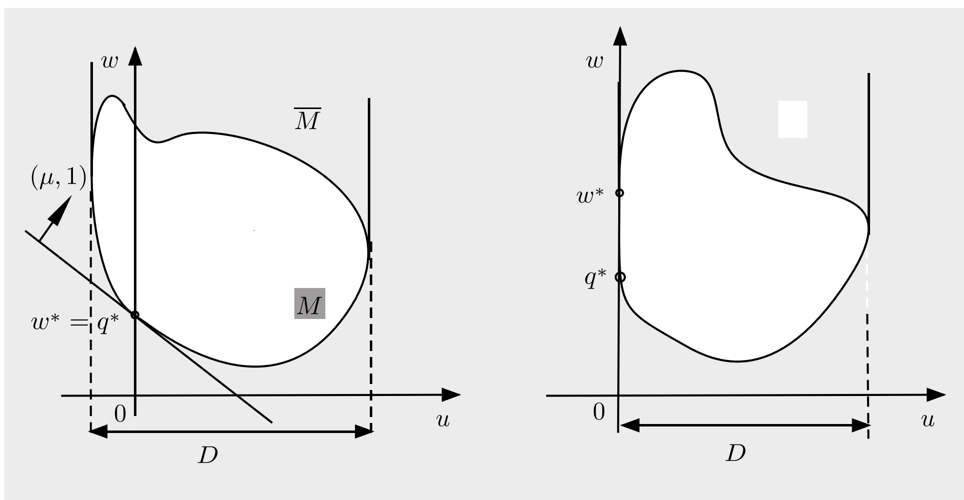
- **Corollary:** If $M = \text{epi}(p)$ where p is closed proper convex and $p(0) < \infty$, then $q^* = w^*$.)

DUALITY THEOREMS (CONTINUED)

- **Min Common/Max Crossing Theorem II:** Assume in addition that $-\infty < w^*$ and that

$$D = \{u \mid \text{there exists } w \in \Re \text{ with } (u, w) \in M\}$$

contains the origin in its relative interior. Then $q^* = w^*$ and there exists μ such that $q(\mu) = q^*$.



- Furthermore, the set $\{\mu \mid q(\mu) = q^*\}$ is nonempty and compact if and only if D contains the origin in its interior.

- **Min Common/Max Crossing Theorem III:** Involves polyhedral assumptions, and will be developed later.

PROOF OF THEOREM I

- Assume that $q^* = w^*$. Let $\big\{(u_k, w_k)\big\} \subset M$ be such that $u_k \to 0$. Then,

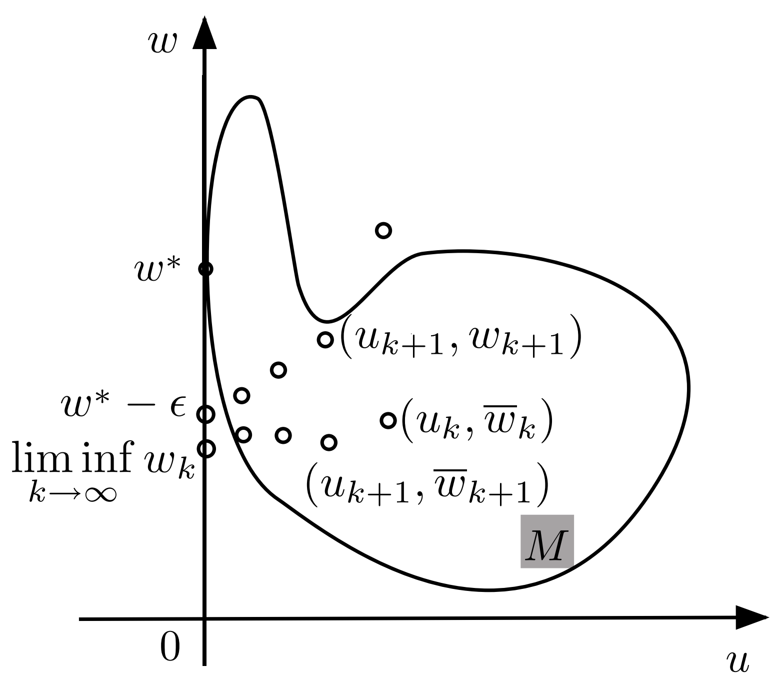
$$q(\mu) = \inf_{(u,w)\in M} \{w+\mu' u\} \leq w_k+\mu' u_k, \quad \forall\, k,\, \forall\, \mu \in \Re^n$$

Taking the limit as $k \to \infty$, we obtain $q(\mu) \leq \liminf_{k\to\infty} w_k$, for all $\mu \in \Re^n$, implying that

$$w^* = q^* = \sup_{\mu\in\Re^n} q(\mu) \leq \liminf_{k\to\infty} w_k$$

Conversely, assume that for every sequence $\big\{(u_k, w_k)\big\} \subset M$ with $u_k \to 0$, there holds $w^* \leq \liminf_{k\to\infty} w_k$. If $w^* = -\infty$, then $q^* = -\infty$, by weak duality, so assume that $-\infty < w^*$. Steps:

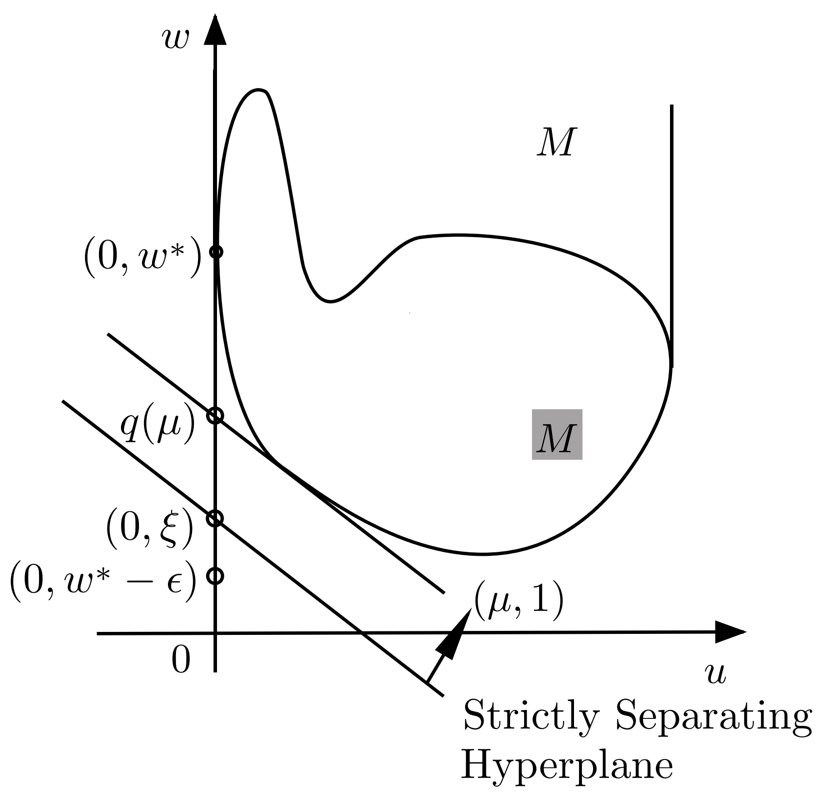
- **Step 1:** $(0, w^* - \epsilon) \notin \mathrm{cl}(M)$ for any $\epsilon > 0$.



PROOF OF THEOREM I (CONTINUED)

- **Step 2:** M does not contain any vertical lines. If this were not so, $(0, -1)$ would be a direction of recession of $\text{cl}(M)$. Because $(0, w^*) \in \text{cl}(M)$, the entire halfline $\{(0, w^* - \epsilon) \mid \epsilon \geq 0\}$ belongs to $\text{cl}(M)$, contradicting Step 1.

- **Step 3:** For any $\epsilon > 0$, since $(0, w^* - \epsilon) \notin \text{cl}(M)$, there exists a nonvertical hyperplane strictly separating $(0, w^* - \epsilon)$ and M. This hyperplane crosses the $(n+1)$st axis at a vector $(0, \xi)$ with $w^* - \epsilon \leq \xi \leq w^*$, so $w^* - \epsilon \leq q^* \leq w^*$. Since ϵ can be arbitrarily small, it follows that $q^* = w^*$.



PROOF OF THEOREM II

- Note that $(0, w^*)$ is not a relative interior point of M. Therefore, by the Proper Separation Theorem, there is a hyperplane that passes through $(0, w^*)$, contains M in one of its closed halfspaces, but does not fully contain M, i.e., for some $(\mu,\beta) \neq (0, 0)$

$$\beta w^* \leq \mu' u + \beta w, \qquad \forall\, (u, w) \in M,$$

$$\beta w^* < \sup_{(u,w)\in M} \{\mu' u + \beta w\}$$

Will show that the hyperplane is nonvertical.

- Since for any $(u, w) \in M$, the set M contains the halfline $\{(u, w) \mid w \leq w\}$, it follows that $\beta \geq 0$. If $\beta = 0$, then $0 \leq \mu' u$ for all $u \in D$. Since $0 \in \mathrm{ri}(D)$ by assumption, we must have $\mu' u = 0$ for all $u \in D$ a contradiction. Therefore, $\beta > 0$, and we can assume that $\beta = 1$. It follows that

$$w^* \leq \inf_{(u,w)\in M} \{\mu' u + w\} = q(\mu) \leq q^*$$

Since the inequality $q^* \leq w^*$ holds always, we must have $q(\mu) = q^* = w^*$.

NONLINEAR FARKAS' LEMMA

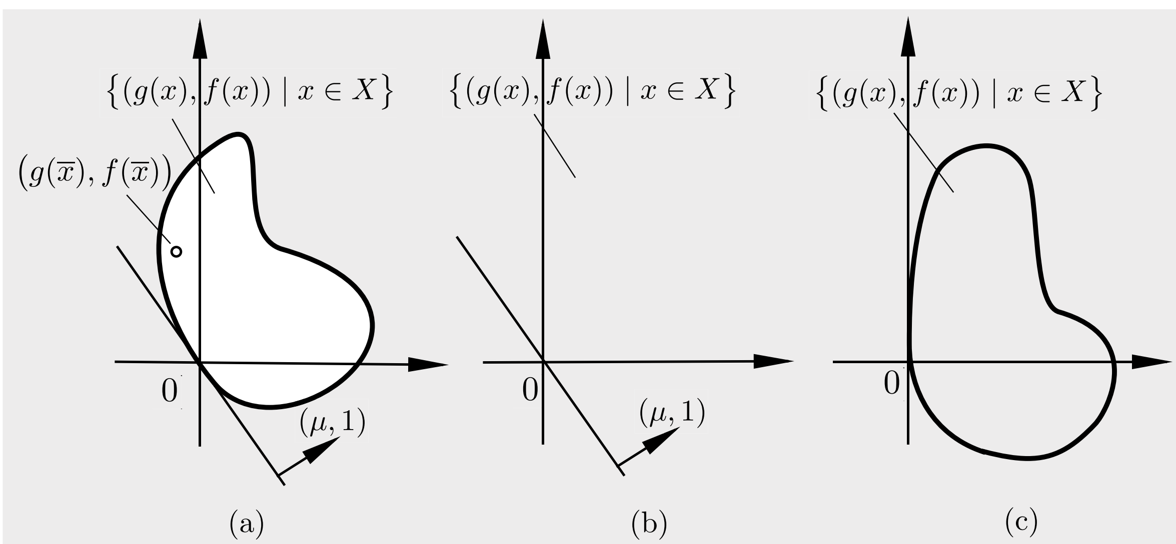
- Let $X \subset \Re^n$, $f : X \mapsto \Re$, and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, be convex. Assume that

$$f(x) \geq 0, \qquad \forall\, x \in X \text{ with } g(x) \leq 0$$

Let

$$Q^* = \big\{\mu \mid \mu \geq 0,\ f(x) + \mu' g(x) \geq 0,\ \forall\, x \in X\big\}.$$

Then Q^* is nonempty and compact if and only if there exists a vector $x \in X$ such that $g_j(x) < 0$ for all $j = 1, \ldots, r$.



- The lemma asserts the existence of a nonvertical hyperplane in $\Re^{r+1}$, with normal $(\mu, 1)$, that passes through the origin and contains the set

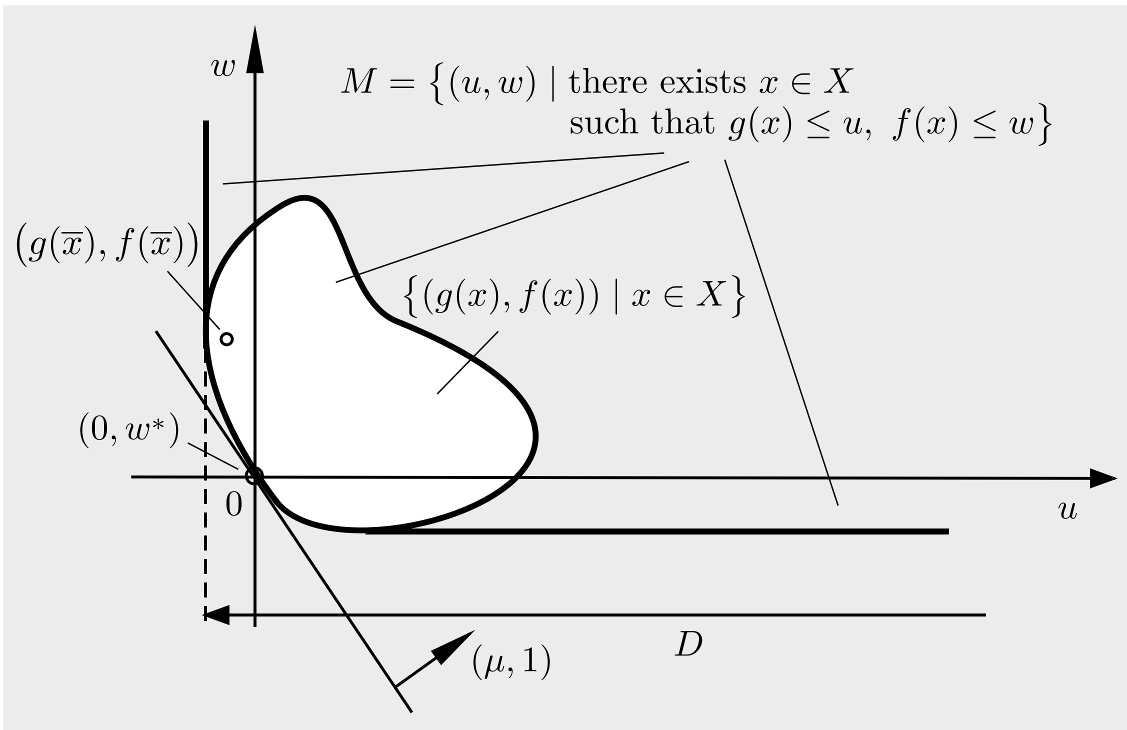
$$\big\{\big(g(x), f(x)\big) \mid x \in X\big\}$$

in its positive halfspace.

PROOF OF NONLINEAR FARKAS' LEMMA

- Apply MC/MC to

$M = \Big\{(u,w) \mid \text{there is } x \in X \text{ s. t. } g(x) \le u,\ f(x) \le w\Big\}$



- M is equal to M and is formed as the union of positive orthants translated to points $\big(g(x), f(x)\big)$, $x \in X$.

- The convexity of X, f, and g_j implies convexity of M.

- MC/MC Theorem II applies: we have

$$D = \big\{u \mid \text{there exists } w \in \Re \text{ with } (u,w) \in M\big\}$$

and $0 \in \text{int}(D)$, because $\big((g(x), f(x)\big) \in M$.