# LECTURE 7

# LECTURE OUTLINE

- Partial Minimization
- Hyperplane separation
- Proper separation
- Nonvertical hyperplanes

**Reading:** Sections 3.3, 1.5

# PARTIAL MINIMIZATION

- Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider

$$f(x) = \inf_{z \in \Re^m} F(x, z)$$

- **1st fact:** If $F$ is convex, then $f$ is also convex.
- **2nd fact:**

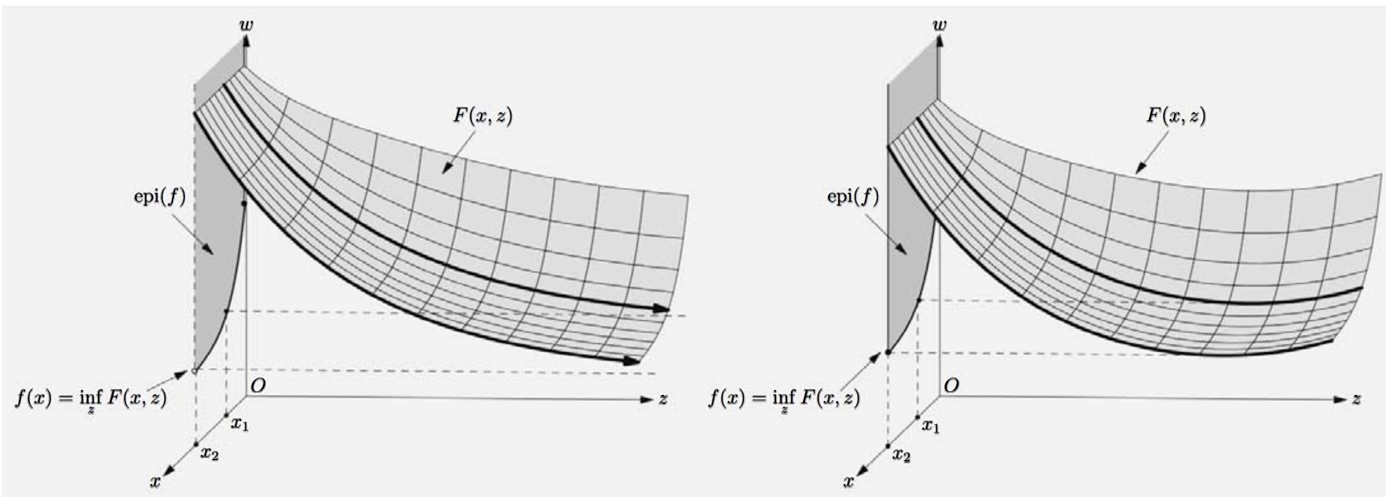
$$P\big(\text{epi}(F)\big) \subset \text{epi}(f) \subset \text{cl}\Big(P\big(\text{epi}(F)\big)\Big),$$

where $P(\cdot)$ denotes projection on the space of $(x, w)$, i.e., for any subset $S$ of $\Re^{n+m+1}$, $P(S) = \big\{(x, w) \mid (x, z, w) \in S\big\}$.

- Thus, if $F$ is closed and there is structure guaranteeing that the projection preserves closedness, then $f$ is closed.
- ... but convexity and closedness of $F$ does not guarantee closedness of $f$.

# PARTIAL MINIMIZATION: VISUALIZATION

- Connection of preservation of closedness under partial minimization and attainment of infimum over $z$ for fixed $x$.




- **Counterexample:** Let

$$F(x,z) = \begin{cases} e^{-\sqrt{xz}} & \text{if } x \geq 0,\ z \geq 0, \\ \infty & \text{otherwise.} \end{cases}$$

- $F$ convex and closed, but

$$f(x) = \inf_{z \in \Re} F(x,z) = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ \infty & \text{if } x < 0, \end{cases}$$

is not closed.

# PARTIAL MINIMIZATION THEOREM

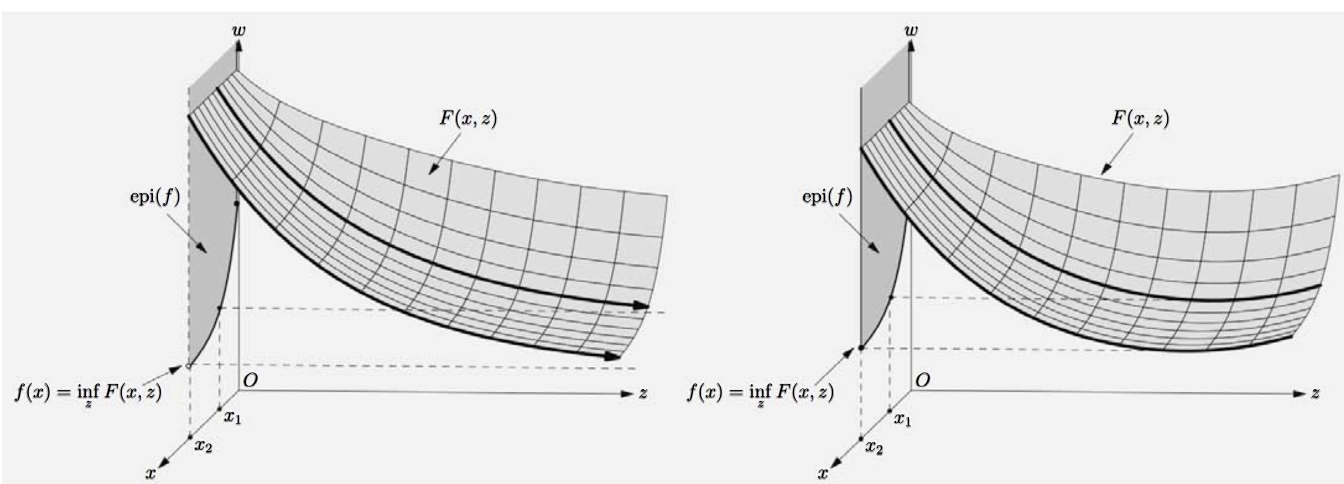
- Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider $f(x) = \inf_{z \in \Re^m} F(x, z)$.

- Every set intersection theorem yields a closedness result. The simplest case is the following:

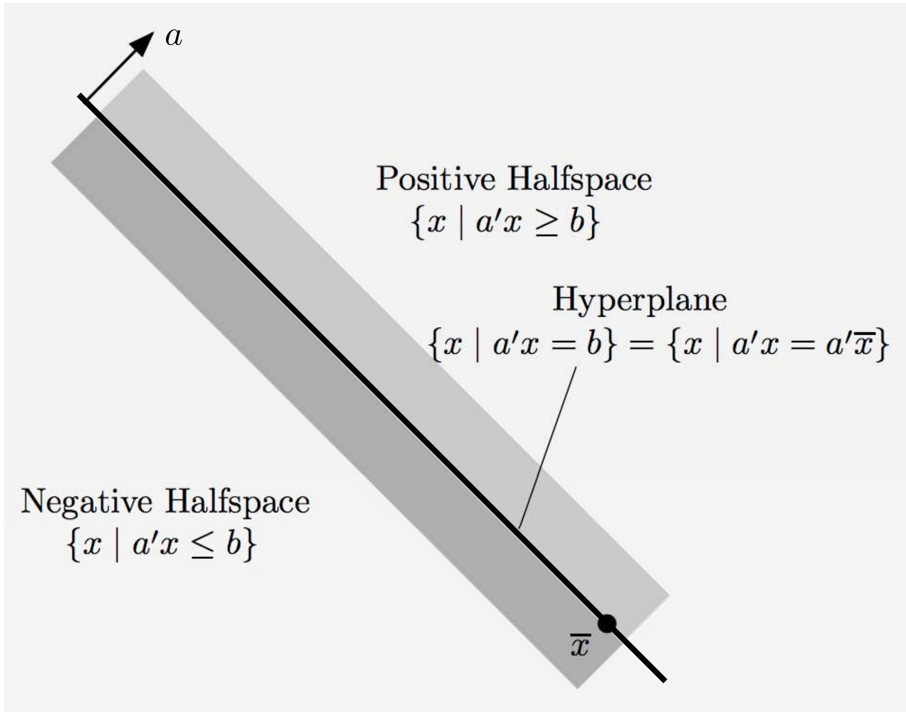
- **Preservation of Closedness Under Compactness:** If there exist $\overline{x} \in \Re^n$, $\overline{\gamma} \in \Re$ such that the set

$$\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\}$$

is nonempty and compact, then $f$ is convex, closed, and proper. Also, for each $x \in \text{dom}(f)$, the set of minima of $F(x, \cdot)$ is nonempty and compact.



# HYPERPLANES




- A *hyperplane* is a set of the form $\{x \mid a'x = b\}$, where $a$ is nonzero vector in $\Re^n$ and $b$ is a scalar.

- We say that two sets $C_1$ and $C_2$ are *separated by a hyperplane* $H = \{x \mid a'x = b\}$ if each lies in a different closed halfspace associated with $H$, i.e.,

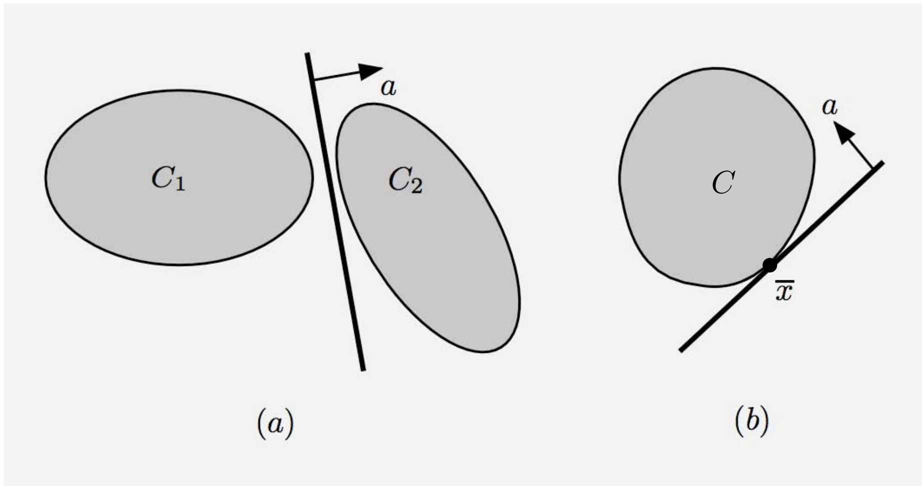
$$\text{either} \quad a'x_1 \leq b \leq a'x_2, \qquad \forall\, x_1 \in C_1, \forall\, x_2 \in C_2,$$

$$\text{or} \quad a'x_2 \leq b \leq a'x_1, \qquad \forall\, x_1 \in C_1, \ \forall\, x_2 \in C_2$$

- If $x$ belongs to the closure of a set $C$, a hyperplane that separates $C$ and the singleton set $\{x\}$ is said be *supporting* $C$ *at* $x$.

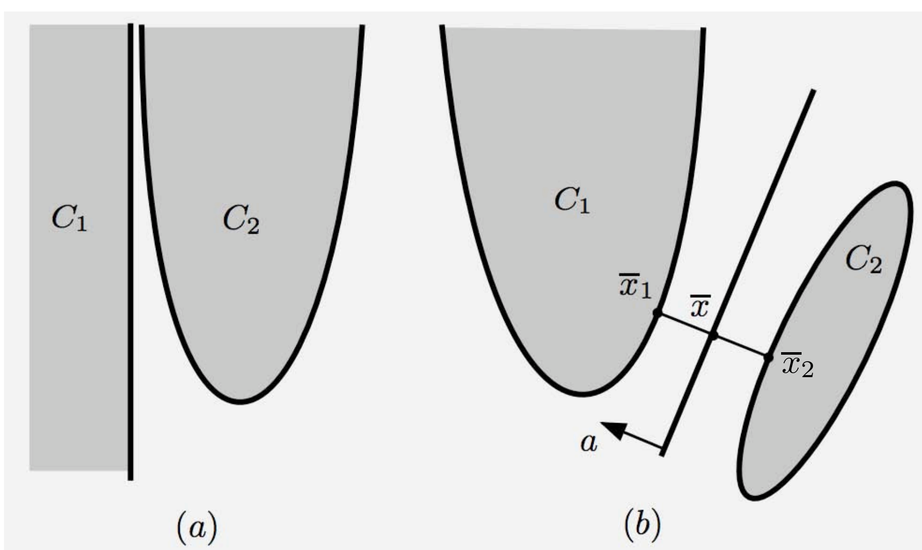
# VISUALIZATION

- Separating and supporting hyperplanes:



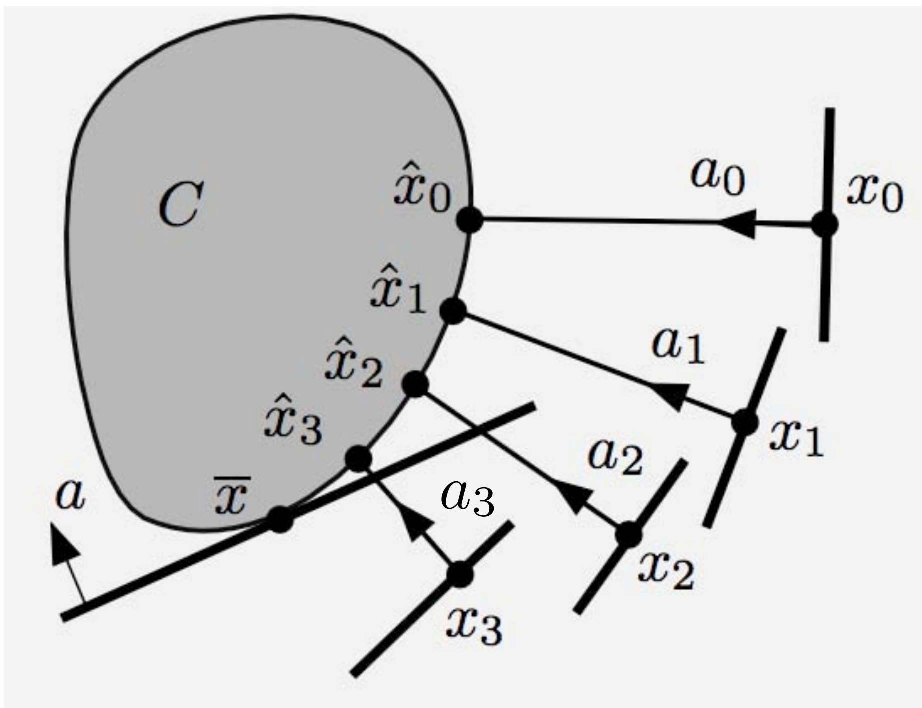

- A separating $\{x \mid a'x = b\}$ that is disjoint from $C_1$ and $C_2$ is called *strictly* separating:

$$a'x_1 < b < a'x_2, \qquad \forall\ x_1 \in C_1,\ \forall\ x_2 \in C_2$$



# SUPPORTING HYPERPLANE THEOREM

- Let $C$ be convex and let $x$ be a vector that is not an interior point of $C$. Then, there exists a hyperplane that passes through $x$ and contains $C$ in one of its closed halfspaces.




**Proof:** Take a sequence $\{x_k\}$ that does not belong to $\mathrm{cl}(C)$ and converges to $x$. Let $\hat{x}_k$ be the projection of $x_k$ on $\mathrm{cl}(C)$. We have for all $x \in \mathrm{cl}(C)$

$$a_k' x \geq a_k' x_k, \qquad \forall\, x \in \mathrm{cl}(C),\ \forall\, k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Let $a$ be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. **Q.E.D.**

# SEPARATING HYPERPLANE THEOREM

- Let $C_1$ and $C_2$ be two nonempty convex subsets of $\Re^n$. If $C_1$ and $C_2$ are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2.$$

**Proof:** Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1,\ x_2 \in C_2\}$$

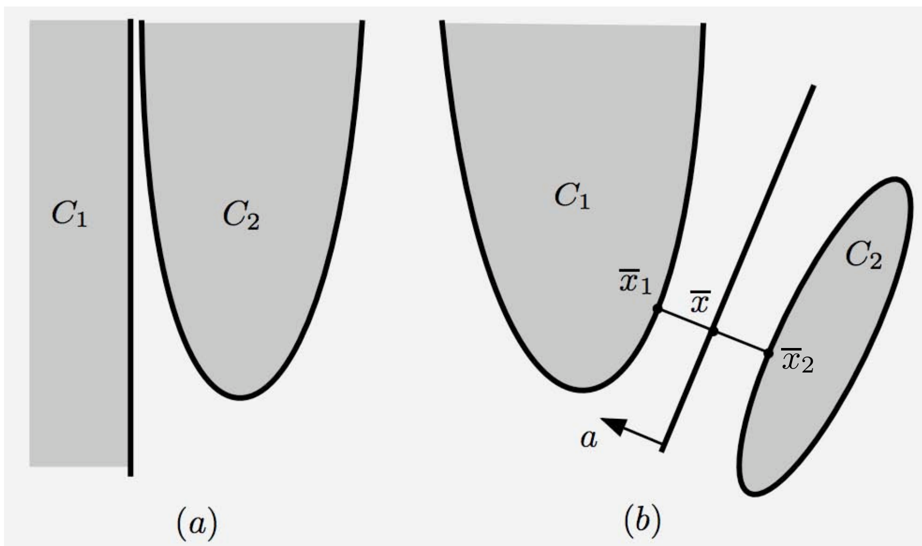
Since $C_1$ and $C_2$ are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\, x \in C_1 - C_2,$$

which is equivalent to the desired relation. **Q.E.D.**

# STRICT SEPARATION THEOREM

- **Strict Separation Theorem**: Let $C_1$ and $C_2$ be two disjoint nonempty convex sets. If $C_1$ is closed, and $C_2$ is compact, there exists a hyperplane that strictly separates them.



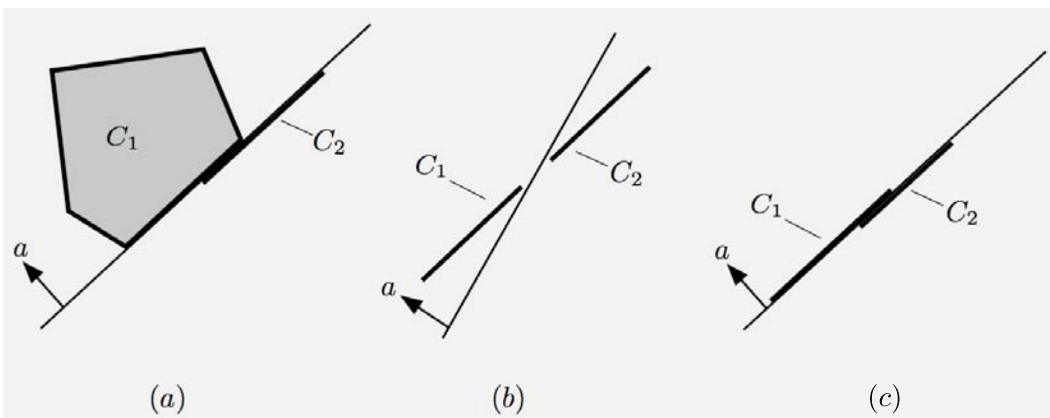

**Proof:** (Outline) Consider the set $C_1 - C_2$. Since $C_1$ is closed and $C_2$ is compact, $C_1 - C_2$ is closed. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Let $x_1 - x_2$ be the projection of 0 onto $C_1 - C_2$. The strictly separating hyperplane is constructed as in (b).

- **Note**: Any conditions that guarantee closedness of $C_1 - C_2$ guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without $C_1 - C_2$ being closed.

# ADDITIONAL THEOREMS

- **Fundamental Characterization**: The closure of the convex hull of a set $C \subset \Re^n$ is the intersection of the closed halfspaces that contain $C$. (Proof uses the strict separation theorem.)

- We say that a hyperplane *properly separates $C_1$ and $C_2$* if it separates $C_1$ and $C_2$ and does not fully contain both $C_1$ and $C_2$.




- **Proper Separation Theorem**: Let $C_1$ and $C_2$ be two nonempty convex subsets of $\Re^n$. There exists a hyperplane that properly separates $C_1$ and $C_2$ if and only if

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset$$

# PROPER POLYHEDRAL SEPARATION

- Recall that two convex sets $C$ and $P$ such that

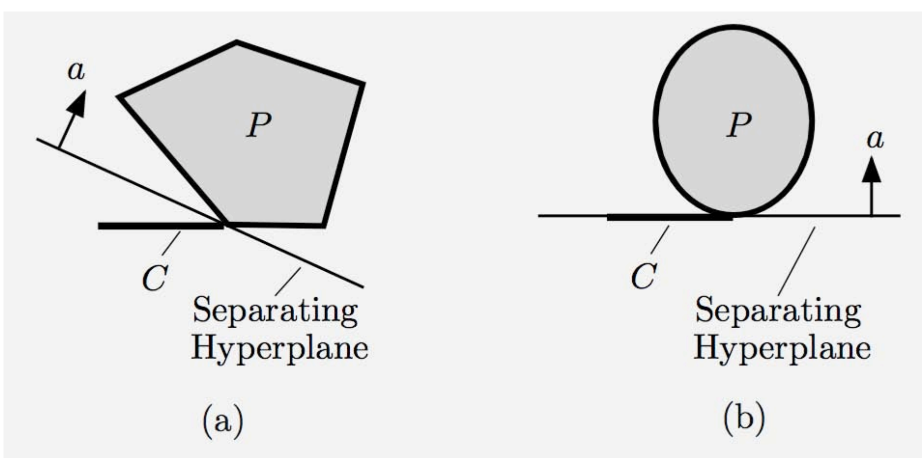
$$\mathrm{ri}(C) \cap \mathrm{ri}(P) = \emptyset$$

can be properly separated, i.e., by a hyperplane that does not contain both $C$ and $P$.

- If $P$ is polyhedral and the slightly stronger condition

$$\mathrm{ri}(C) \cap P = \emptyset$$

holds, then the properly separating hyperplane can be chosen so that it does not contain the non-polyhedral set $C$ while it may contain $P$.

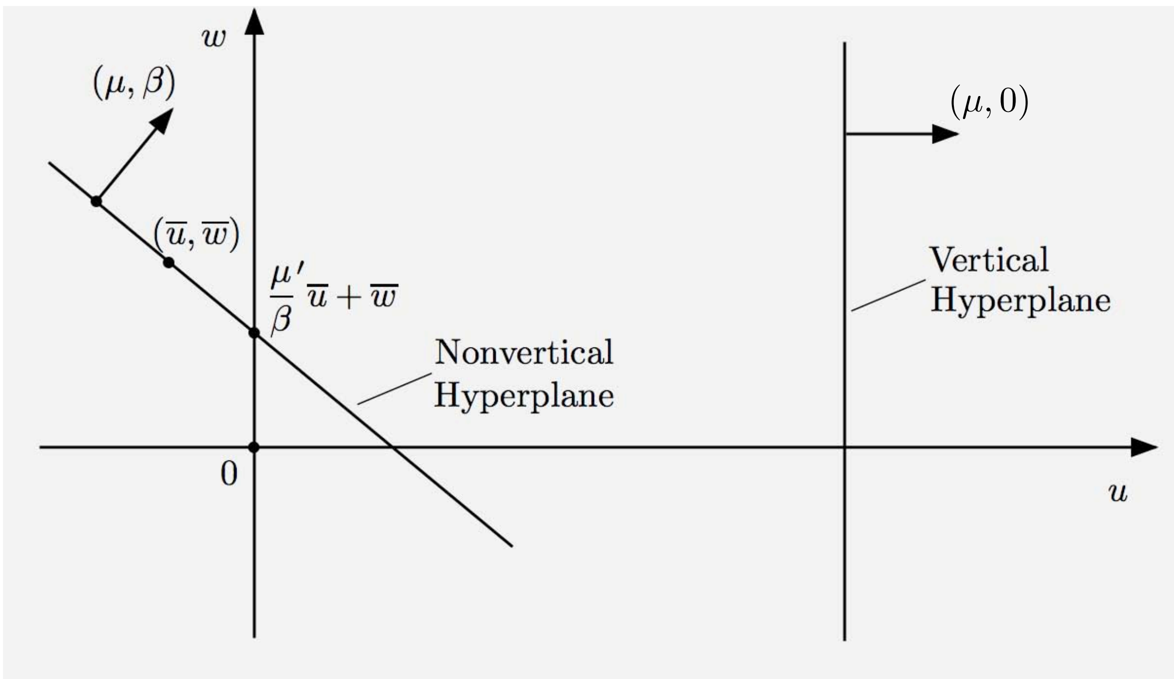



On the left, the separating hyperplane can be chosen so that it does not contain $C$. On the right where $P$ is not polyhedral, this is not possible.

# NONVERTICAL HYPERPLANES

- A hyperplane in $\Re^{n+1}$ with normal $(\mu,\beta)$ is nonvertical if $\beta \neq 0$.

- It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)'u+w$, where $(u, w)$ is any vector on the hyperplane.




- A nonvertical hyperplane that contains the epigraph of a function in its "upper" halfspace, provides lower bounds to the function values.

- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the "upper" halfspace of some nonvertical hyperplane.

# NONVERTICAL HYPERPLANE THEOREM

- Let $C$ be a nonempty convex subset of $\Re^{n+1}$ that contains no vertical lines. Then:

  (a) $C$ is contained in a closed halfspace of a nonvertical hyperplane, i.e., there exist $\mu \in \Re^n$, $\beta \in \Re$ with $\beta \neq 0$, and $\gamma \in \Re$ such that $\mu' u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(u, w) \notin \mathrm{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(u, w)$ and $C$.

**Proof:** Note that $\mathrm{cl}(C)$ contains no vert. line [since $C$ contains no vert. line, $\mathrm{ri}(C)$ contains no vert. line, and $\mathrm{ri}(C)$ and $\mathrm{cl}(C)$ have the same recession cone]. So we just consider the case: $C$ closed.

(a) $C$ is the intersection of the closed halfspaces containing $C$. If all these corresponded to vertical hyperplanes, $C$ would contain a vertical line.

(b) There is a hyperplane strictly separating $(u, w)$ and $C$. If it is nonvertical, we are done, so assume it is vertical. "Add" to this vertical hyperplane a small $\epsilon$-multiple of a nonvertical hyperplane containing $C$ in one of its halfspaces as per (a).

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