Bandlimited communication systems

Lecture 3

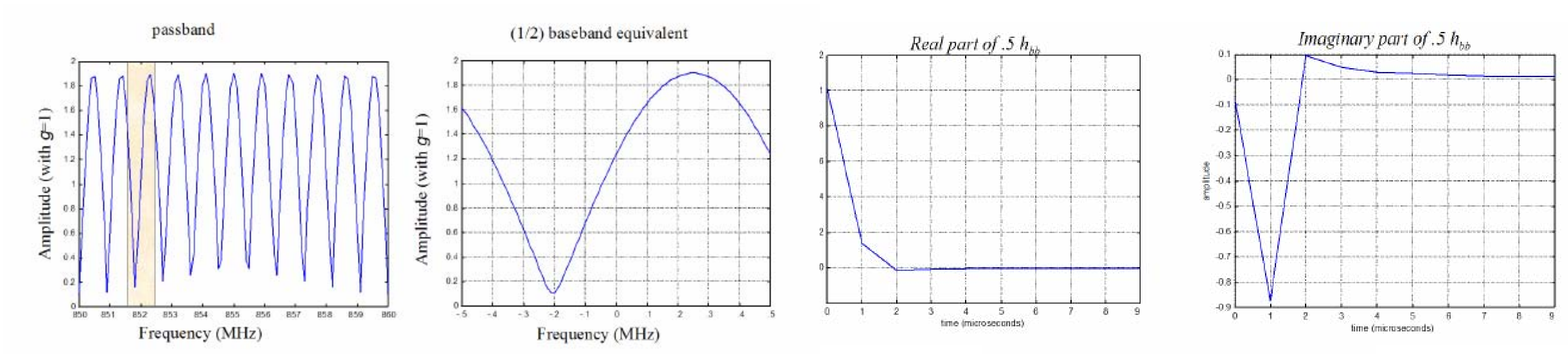
Vladimir Stojanović

6.973 Communication System Design – Spring 2006

Massachusetts Institute of Technology

Passband channel example

- Two-ray wireless channel (multi-path – 1+0.9D)



$$H(f) = g\left(1 - .9e^{-j2\pi f \tau}\right)$$

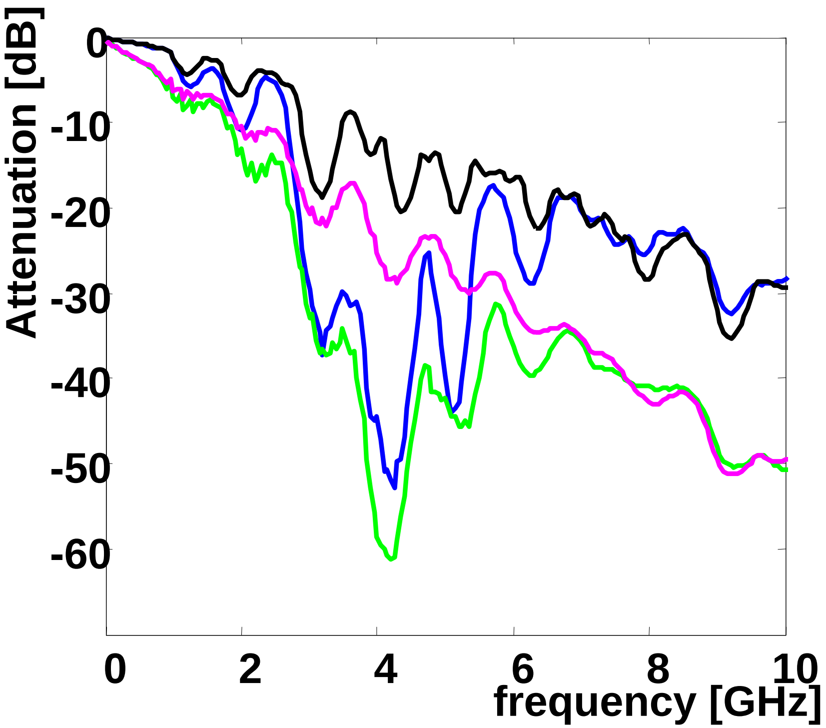
$$h(t) = g\left(\delta(t) - .9\delta(t-\tau)\right)$$

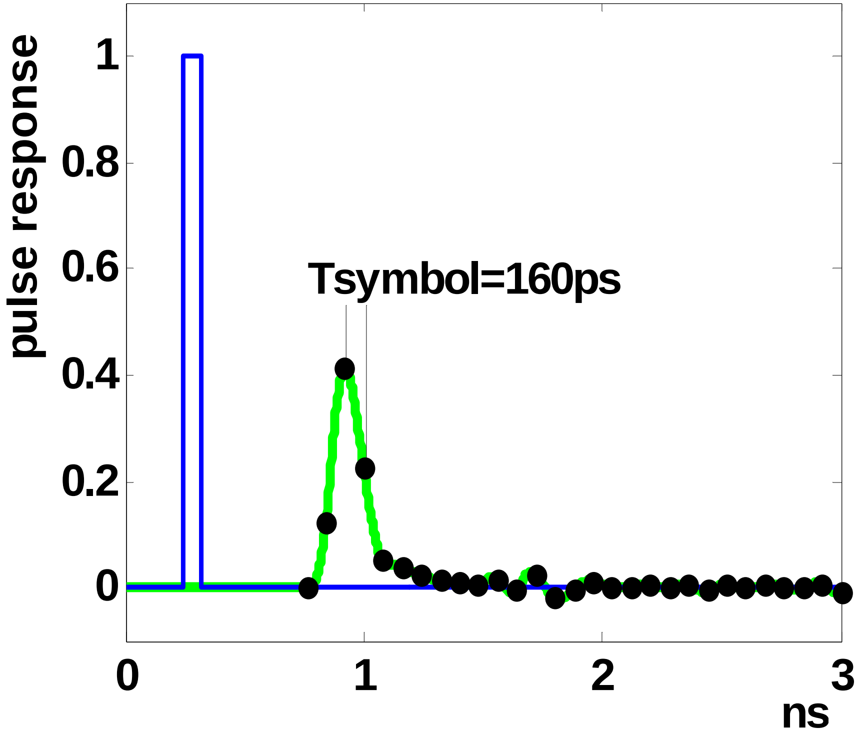
$$\frac{1}{2}H_{bb}(f) = H(f+f_c) = g\left(1 - .9e^{-j2\pi(f+f_c)\tau}\right)$$

$$\frac{1}{2}h_{bb}(t) = g\left(\delta(t) - .9\delta(t-\tau)\right) \cdot e^{-j2\pi f_c t}$$

- Multi-path creates notching in frequency domain
- Just slide the frequency window to bb
  - Add single-sided noise

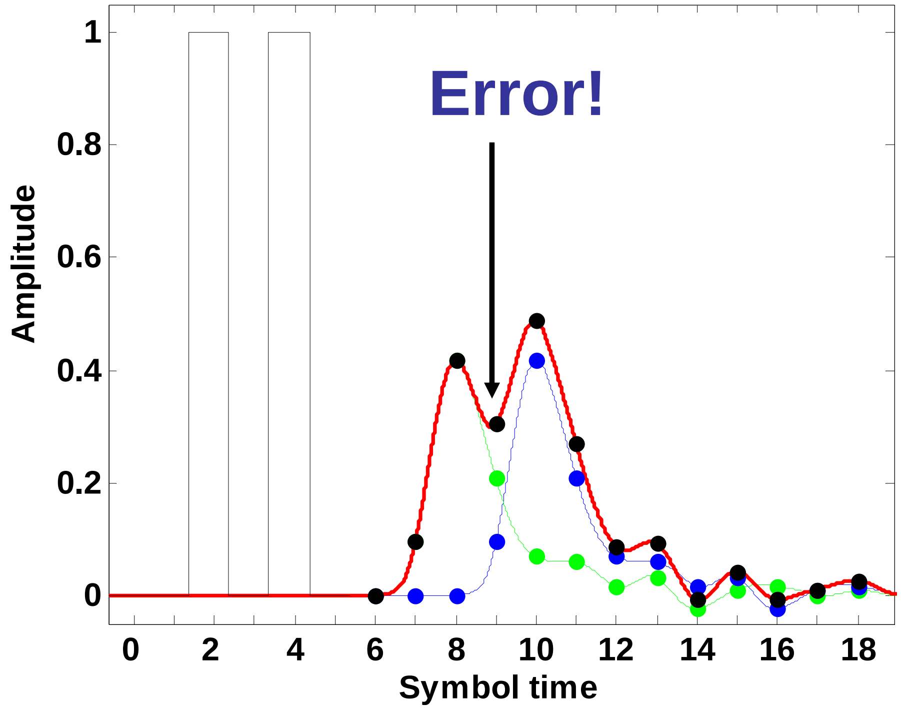
Bandlimited channel example





- Low-pass channel causes pulse attenuation and dispersion
- Notches cause ripples in time domain
- Makes it hard to transmit successive messages

Inter-Symbol Interference (ISI)



- Middle sample is corrupted by 0.2 trailing ISI (from the previous symbol), and 0.1 leading ISI (from the next symbol) resulting in 0.3 total ISI
- As a result middle symbol is detected in error

Bandlimited communication systems

- Block detector vs. symbol-by-symbol

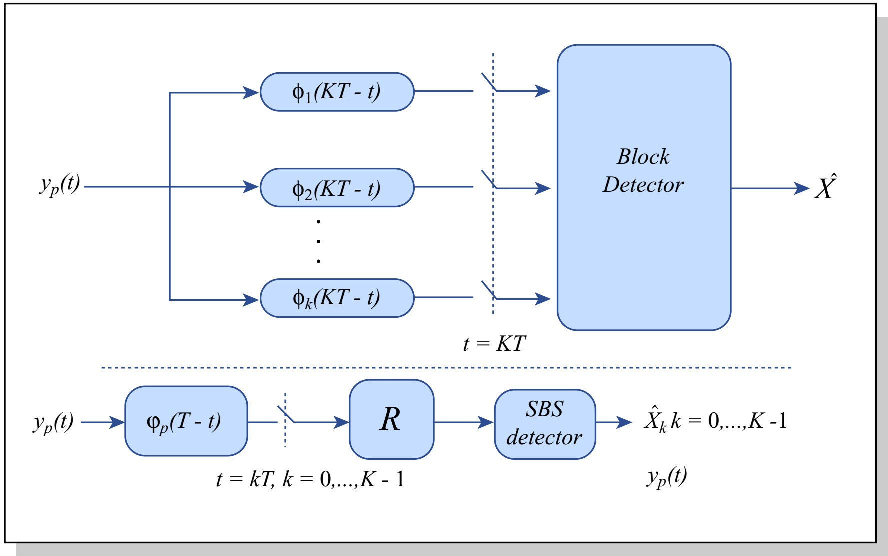
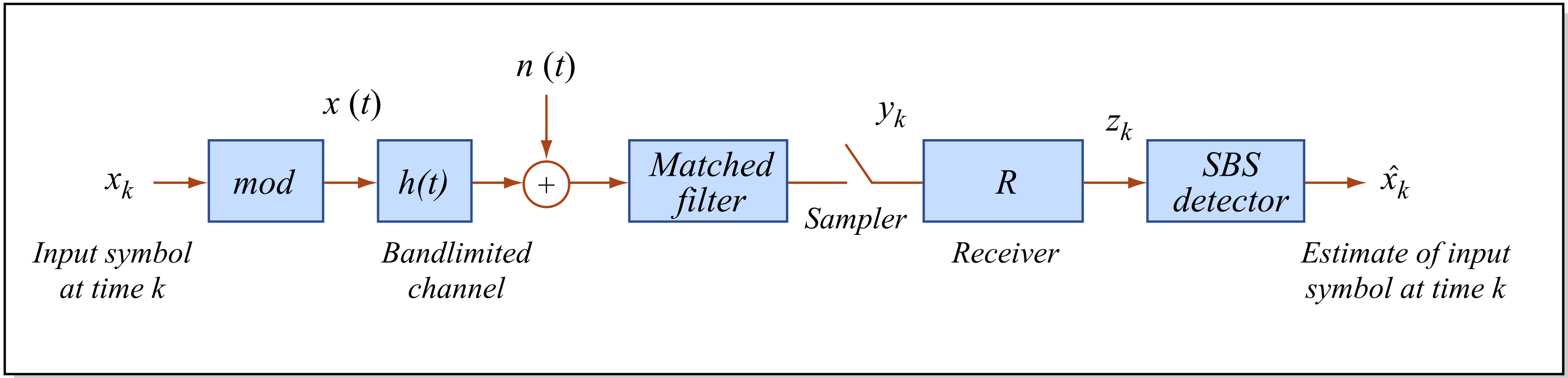


Figure by MIT OpenCourseWare.

- Block of K symbols – M^K messages
  - MAP/ML detector complexity grows exponentially
    - M^K basis functions (branches in the matched filter)
      - Sequence detection can bound that growth
  - Simpler detector is "Symbol-By-Symbol"
    - Optimal for AWGN channel

Symbol-by-symbol detection



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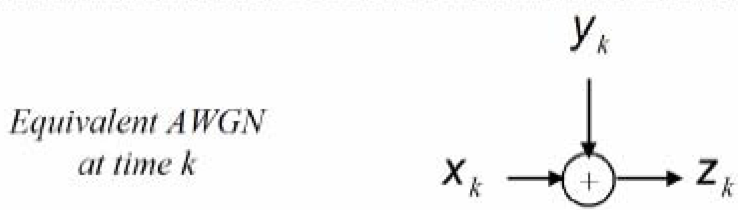


$$\text{SNR} = \frac{\mathcal{E}_x}{E\{|e_k|^2\}}$$

$$x(t) = \sum_{k=0}^{K-1} x_k(t - kT)$$

$$x(t) = \sum_{k=0}^{K-1} \sum_{n=1}^{N} x_{kn}\varphi_n(t - kT)$$

- Suffers significantly from Intersymbol-interference (channel memory), so need to remove ISI to get almost AWGN channel
- Need to adapt basis functions to the particular channel, to avoid ISI
- Alternatively, use equalization to remove ISI

Vector channel - revisited

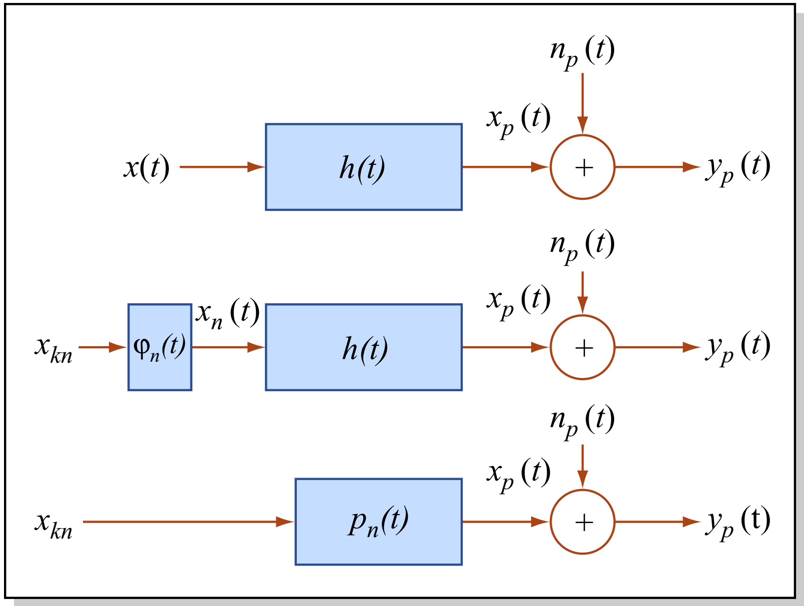


Figure by MIT OpenCourseWare.

$$x_p(t) = \sum_{k=0}^{K-1} \sum_{n=1}^{N} x_{kn}\varphi_n(t-kT) * h(t)$$
$$= \sum_{k=0}^{K-1} \sum_{n=1}^{N} x_{kn}p_n(t-kT)$$

$$x_p(t) = \sum_{k=0}^{K-1} x_k\varphi(t-kT) * h(t)$$
$$= \sum_{k=0}^{K-1} x_k p(t-kT) \quad .$$

$$x_{p,k} \triangleq x_k \|p\|$$

$$p(t) = \varphi(t) * h(t)$$
$$\varphi_p(t) \triangleq \frac{p(t)}{\|p\|}$$

$$x_p(t) = \sum_{k=0}^{K-1} x_{p,k}\varphi_p(t-kT)$$

ISI impact

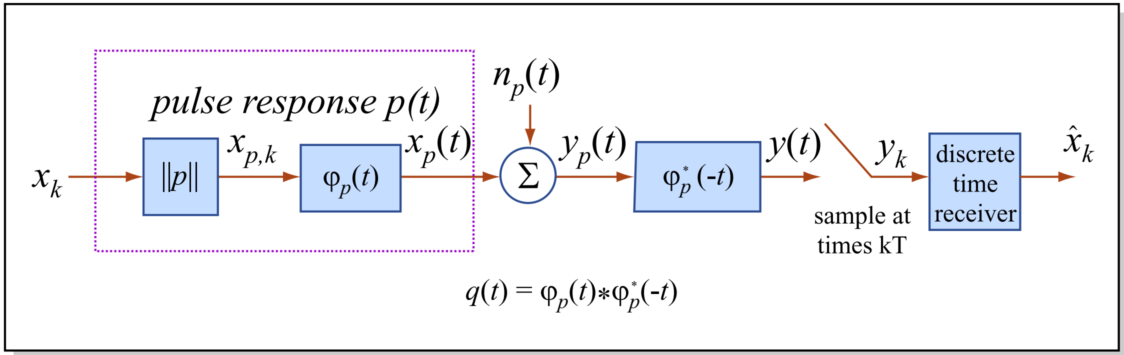


Figure by MIT OpenCourseWare.

$$y_k = \underbrace{\|p\| \cdot x_k}_{\text{scaled input (desired)}} + \underbrace{n_k}_{\text{noise}} + \underbrace{\|p\| \sum_{m \neq k} x_m q_{k-m}}_{\text{ISI}}$$

- Mean-distortion
  - Treat ISI as noise

$$\mathcal{D}_{ms} \triangleq E\left\{ \left| \sum_{m \neq k} x_{p,m} q_{k-m} \right|^2 \right\} = \mathcal{E}_x \cdot \|p\|^2 \cdot \sum_{m \neq 0} |q_m|^2 \quad ,$$

$$P_e \approx N_e Q\left[\frac{\|p\| d_{\min}}{2\sqrt{\sigma^2 + \bar{\mathcal{D}}_{ms}}} \right]$$

- Peak-distortion
  - Treat worst-case ISI as constellation offset

$$\mathcal{D}_p \triangleq |x|_{max} \cdot \|p\| \cdot \sum_{m \neq 0} |q_m|$$

$$P_e \leq N_e Q\left[\frac{\|p\| \frac{d_{\min}}{2} - \mathcal{D}_p}{\sigma} \right]$$

Matched Filter Bound

- You can't do better with successive transmissions than with one-shot
- Matched filter collects the pulse energy $||p||^2$
- Then calculate performance as on AWGN

$$SNR_{MFB} = \frac{\bar{\mathcal{E}}_x ||p||^2}{\frac{\mathcal{N}_0}{2}}$$

- Example – binary transmission

$$x_p(t) = \sum_k x_k p(t - kT) \quad x_k = \pm\sqrt{\mathcal{E}_x}. \quad P_e \geq Q(\sqrt{\mathrm{SNR}_{MFB}})$$

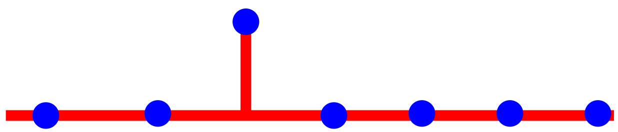
- Will use MFB to compare different ISI compensation techniques

Nyquist criterion – 6.011 revisited

- A channel specified by pulse response p(t) is ISI free if

$$Q(e^{-\jmath\omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Q(\omega + \frac{2\pi n}{T}) = 1$$

$$q(t) = \frac{p(t) * p^*(-t)}{\|p\|^2},$$



$$\begin{aligned} q(kT) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Q(\omega) e^{\jmath\omega kT} d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\frac{(2n-1)\pi}{T}}^{\frac{(2n+1)\pi}{T}} Q(\omega) e^{\jmath\omega kT} d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} Q(\omega + \frac{2\pi n}{T}) e^{\jmath(\omega + \frac{2\pi n}{T})kT} d\omega \\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} Q_{eq}(\omega) e^{\jmath\omega kT} d\omega \quad , \end{aligned}$$

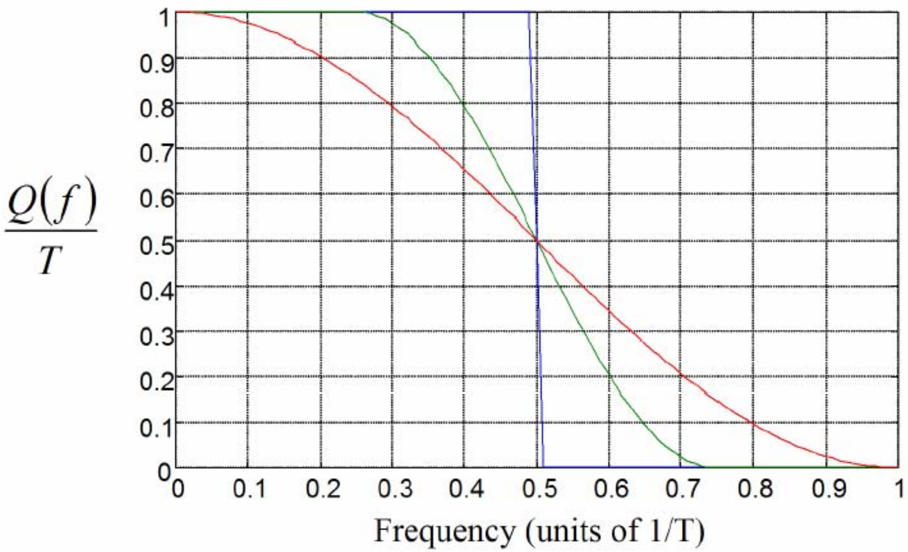
$$Q_{eq}(\omega) \triangleq \sum_{n=-\infty}^{\infty} Q(\omega + \frac{2\pi n}{T})$$

$$\frac{1}{T} Q_{eq}(\omega) = Q(e^{-\jmath\omega T}) \triangleq \sum_{k=-\infty}^{\infty} q_k e^{-\jmath\omega kT}$$

- Nyquist frequency: w=pi/T or f=1/2T

Raised-cosine pulses

- Can have “excess” bandwidth as long as there is symmetry that “fills” the aliased spectrum flat

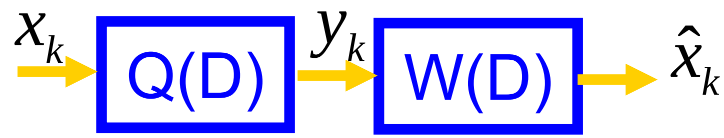


$$q(t) = sinc\left(\frac{t}{T}\right) \cdot \left[\frac{\cos\left(\frac{\alpha\pi t}{T}\right)}{1-\left(\frac{2\alpha t}{T}\right)^2}\right]$$

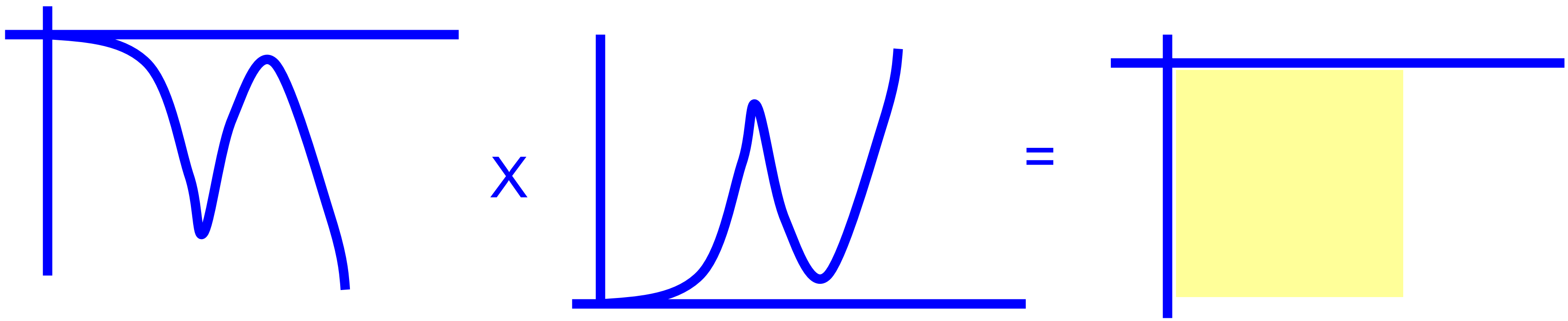
$$Q(\omega) = \begin{cases} T & |\omega| \le \frac{\pi}{T}(1-\alpha) \\ \frac{T}{2}\left[1-\sin\left(\frac{T}{2\alpha}\left(|\omega|-\frac{\pi}{T}\right)\right)\right] & \frac{\pi}{T}(1-\alpha) \le |\omega| \le \frac{\pi}{T}(1+\alpha) \\ 0 & \frac{\pi}{T}(1+\alpha) \le |\omega| \end{cases}$$

Basic equalization concepts

- Zero-forcing equalization

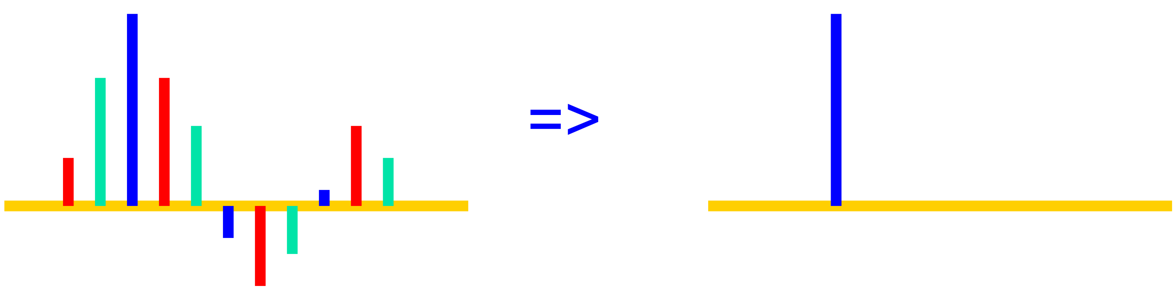


  - Flattens equalized channel transfer function



Channel Q(w) Equalizer W(w) Equalized

- H(D)=Q(D)*W(D) Wzfe(D)=1/(Q(D)||p||)



Linear equalization

- Zero-forcing not good on channels with nulls
  - Equalizer enhances noise
- Remember, P_e depends on both noise and ISI
- Balance noise and ISI in the mean-square sense

$$e_k = x_k - w_k * y_k = x_k - z_k \qquad E(D) = X(D) - W(D)Y(D)$$

- Minimizing MMSE wrt. W_k $\quad \sigma^2_{MMSE-LE} \triangleq \min_{w_k} E\left[|x_k - z_k|^2\right]$
  - Same as using the orthogonality principle

$$E\left[E(D)Y^*(D^{-*})\right] = 0$$

$$R_{xx}(D) \triangleq E\left[X(D)X^*(D^{-*})\right] \quad \leftarrow \sum_k x_k x^*_{k-j}$$

$$\bar{R}_{xy}(D) = E\left[X(D)Y^*(D^{-*})\right]/N = \|p\|Q(D)\bar{\mathcal{E}}_x$$

$$\bar{R}_{yy}(D) = E\left[Y(D)Y^*(D^{-*})\right]/N = \|p\|^2Q^2(D)\bar{\mathcal{E}}_x + \frac{\mathcal{N}_0}{2}Q(D) = Q(D)\left(\|p\|^2Q(D)\bar{\mathcal{E}}_x + \frac{\mathcal{N}_0}{2}\right)$$

$$W(D) = \frac{\bar{R}_{xy}(D)}{\bar{R}_{yy}(D)} = \frac{1}{\|p\|\left(Q(D) + 1/\mathrm{SNR}_{MFB}\right)}$$

ZFE vs. MMSE - LE

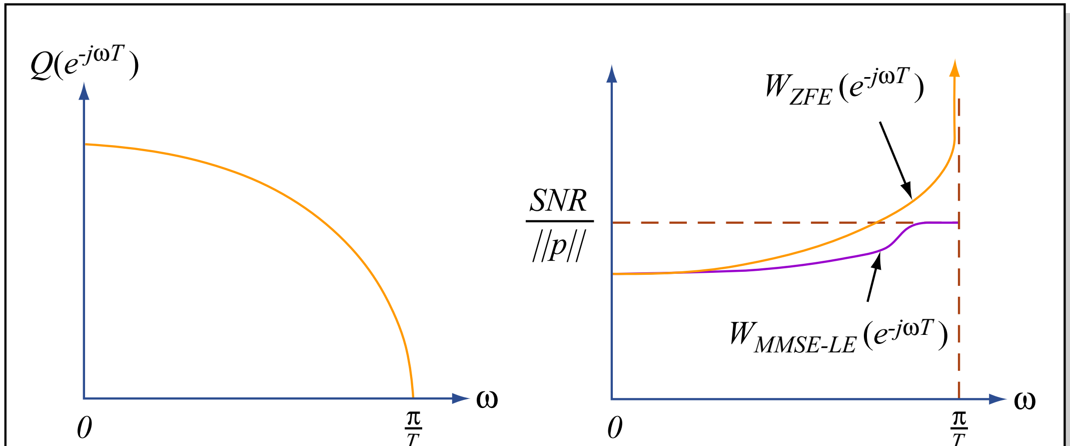


Figure by MIT OpenCourseWare.

$$\sigma^2_{ZFE} = \frac{\mathcal{N}_0}{2}\frac{w_0}{\|p\|}$$

$$\sigma^2_{MMSE-LE} = w_0 \frac{\frac{\mathcal{N}_0}{2}}{\|p\|}$$

$$\sigma^2_{ZFE} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \bar{R}_{ZFE}(e^{-\jmath\omega T})d\omega = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\frac{\mathcal{N}_0}{2}}{\|p\|^2 Q(e^{-\jmath\omega T})}d\omega = \frac{\frac{\mathcal{N}_0}{2}}{\|p\|^2}\gamma_{ZFE} = \frac{\frac{\mathcal{N}_0}{2}}{\|p\|}\cdot w_0$$

$$\gamma_{ZFE} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{1}{Q(e^{-\jmath\omega T})}d\omega = w_0 \cdot \|p\|$$

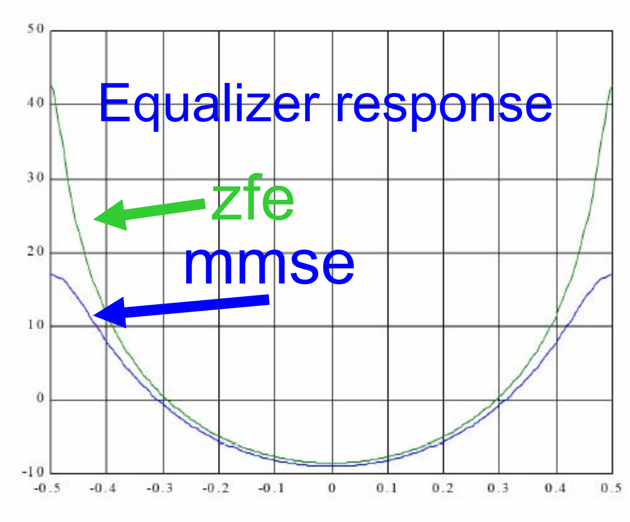
$$\text{SNR}_{ZFE} = \frac{\bar{\mathcal{E}}_x}{\sigma^2_{ZFE}} = \text{SNR}_{MFB}\cdot\frac{1}{\gamma_{ZFE}}$$

$$\sigma^2_{MMSE-LE} \le \sigma^2_{ZFE}$$

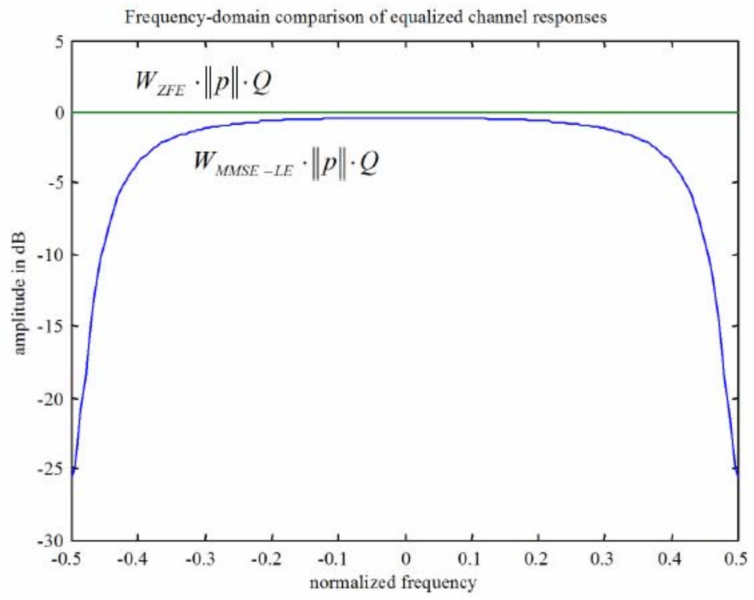
$$\sigma^2_{MMSE-LE} = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \bar{R}_{ee}(e^{-\jmath\omega T})d\omega = \frac{T}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{\frac{\mathcal{N}_0}{2}d\omega}{\|p\|^2\left(Q(e^{-\jmath\omega T}) + 1/\text{SNR}_{MFB}\right)}$$

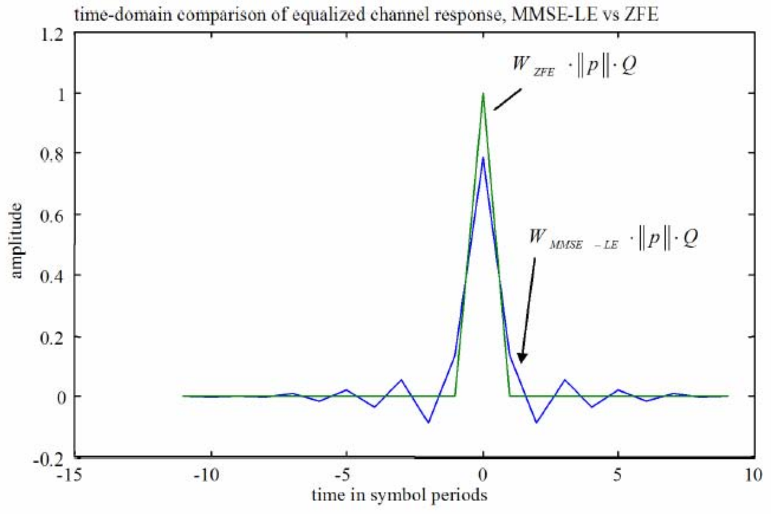
Example: ZFE vs. MMSE LE

- 1+0.9D channel



$$P(\omega) = \begin{cases} \sqrt{T}\left(1 + .9e^{\jmath\omega T}\right) & |\omega| \leq \frac{\pi}{T} \\ 0 & |\omega| > \frac{\pi}{T} \end{cases}$$





Fractional equalizers

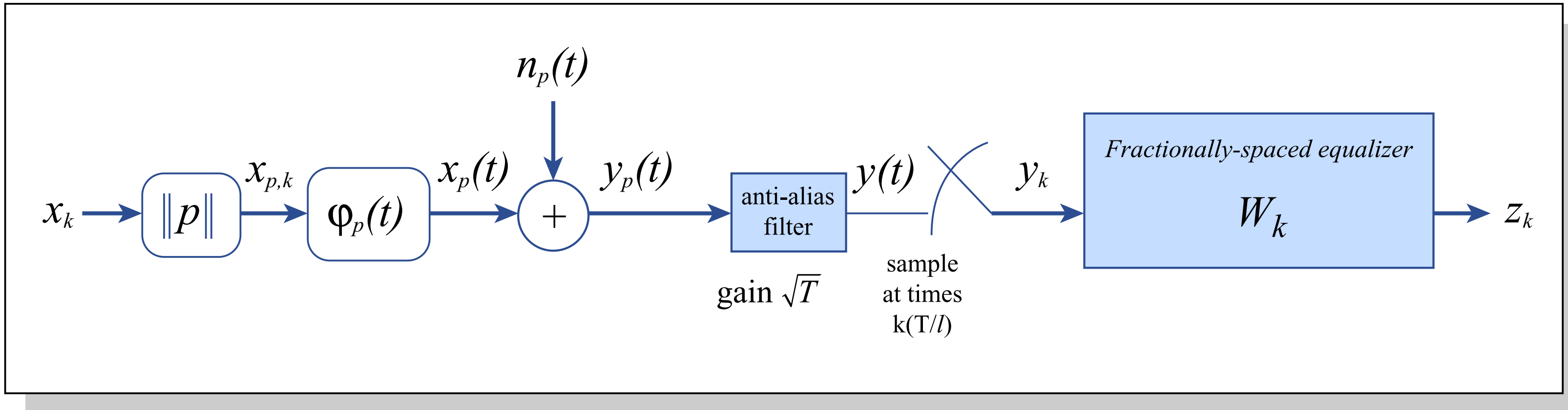
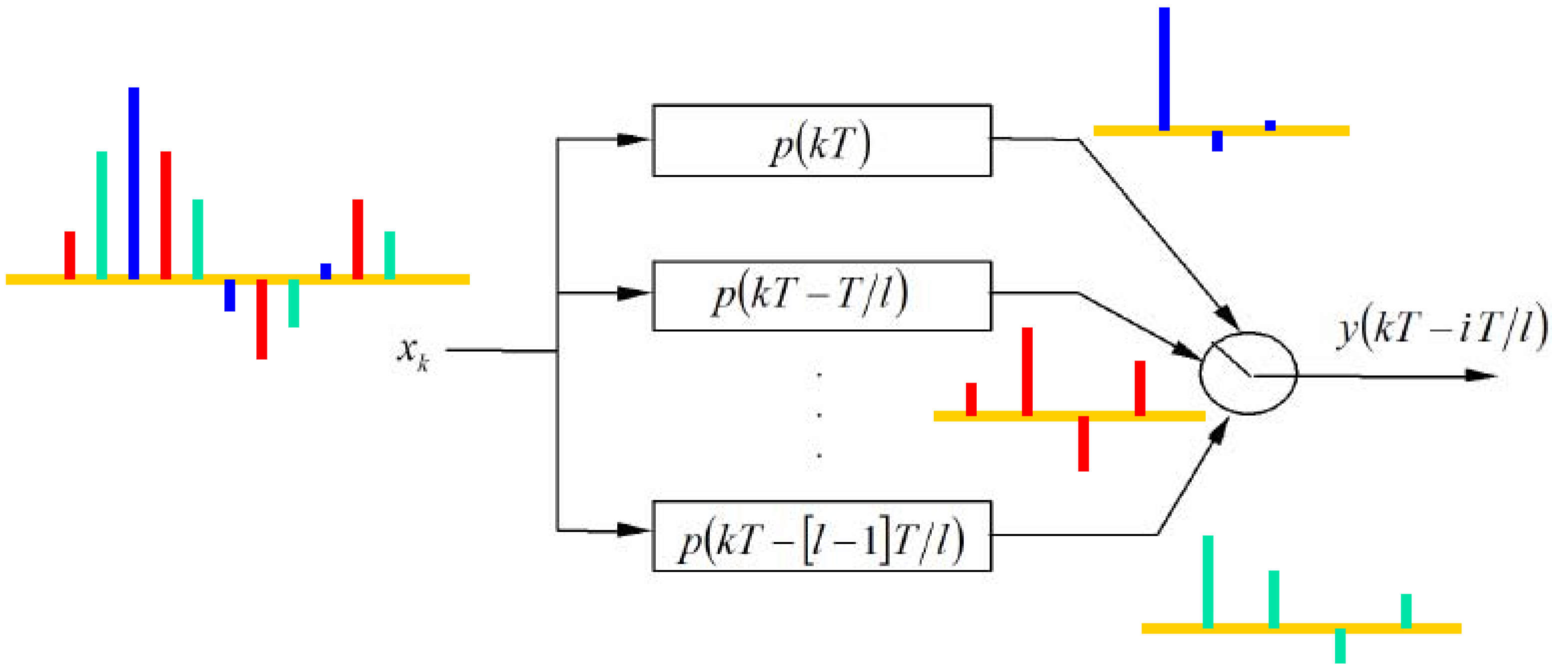


Figure by MIT OpenCourseWare.

- Oversampling in the receiver
  - Can merge matched filter and equalizer
    - Can reconstruct original signal from oversampled signal (as long as original is band-limited)

ISI channel model

- Oversampled channel representation (3x e.g)

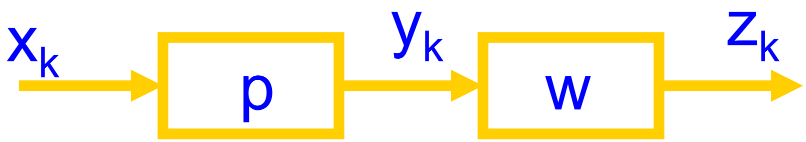


$$y(t) = \sum_m x_m \cdot \tilde{p}(t - mT) + n(t)$$

$$\mathbf{y}_k = \sum_{m=-\infty}^{\infty} x_m \mathbf{p}_{k-m} + \mathbf{n}_k = \sum_{m=-\infty}^{\infty} x_{k-m} \mathbf{p}_m + \mathbf{n}_k$$

$$\mathbf{y}_k = \begin{bmatrix} y(kT) \\ y(kT - \frac{T}{l}) \\ \vdots \\ y(kT - \frac{l-1}{l}T) \end{bmatrix} \quad \mathbf{p}_k = \begin{bmatrix} \tilde{p}(kT) \\ \tilde{p}(kT - \frac{T}{l}) \\ \vdots \\ \tilde{p}(kT - \frac{l-1}{l}T) \end{bmatrix} \quad \mathbf{n}_k = \begin{bmatrix} n(kT) \\ n(kT - \frac{T}{l}) \\ \vdots \\ n(kT - \frac{l-1}{l}T) \end{bmatrix}$$

Finite length equalizer formulation



- Write convolution as multiply with Toeplitz matrix

$$\mathbf{Y}_k \triangleq \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-N_f+1} \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & \dots & p_\nu & 0 & 0 & \dots & 0 \\ 0 & p_0 & p_1 & \dots & \dots & p_\nu & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \dots & 0 & 0 & p_0 & p_1 & \dots & p_\nu \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ \vdots \\ x_{k-N_f-\nu+1} \end{bmatrix} + \begin{bmatrix} n_k \\ n_{k-1} \\ \vdots \\ n_{k-N_f+1} \end{bmatrix}$$

$$\mathbf{Y}_k = \mathbf{P}\mathbf{X}_k + \mathbf{N}_k \qquad z_k = \boldsymbol{w}\mathbf{Y}_k \qquad e_k = x_{k-\Delta} - z_k \qquad \Delta \approx \frac{\nu + N_f}{2}$$

$$z_k = wPx_k$$

ZFE and MMSE solution

- Zero forcing equalizer (ZFE)

$$z_k = x_{k-\Delta} = wPx_k \;\Rightarrow\; 1_\Delta = w_{zfe}P \;\Rightarrow\; w_{zfe} = 1_\Delta^T P^T \left(PP^T\right)^{-1}, \qquad 1_\Delta = [0\,0 \ldots 1 \ldots 0\,0]^T$$

- Minimum-mean square error (MMSE) equalizer

$$\sigma^2_{MMSE-LE} = E\left\{|e_k|^2\right\} = E\left\{e_k e_k^*\right\} = E\left\{\left(x_{k-\Delta} - z_k\right)\left(x_{k-\Delta} - z_k\right)^*\right\}$$

$$E\left\{e_k \boldsymbol{Y}_k^*\right\} = 0 \qquad E\left\{x_{k-\Delta}\boldsymbol{Y}_k^*\right\} - \boldsymbol{w}E\left\{\boldsymbol{Y}_k\boldsymbol{Y}_k^*\right\} = 0$$

$$\boldsymbol{w} = \boldsymbol{R}^*_{\boldsymbol{Y}x}\boldsymbol{R}^{-1}_{\boldsymbol{YY}} = \boldsymbol{R}_{x\boldsymbol{Y}}\boldsymbol{R}^{-1}_{\boldsymbol{YY}} \quad => \quad \boldsymbol{w} = \boldsymbol{1}^*_\Delta \boldsymbol{P}^*\left[\boldsymbol{PP}^* + \frac{1}{\text{SNR}}\boldsymbol{R_{nn}}\right]^{-1}$$

$$\boldsymbol{R}_{\boldsymbol{YY}} \triangleq E\left\{\boldsymbol{Y}_k\boldsymbol{Y}_k^*\right\}/N$$

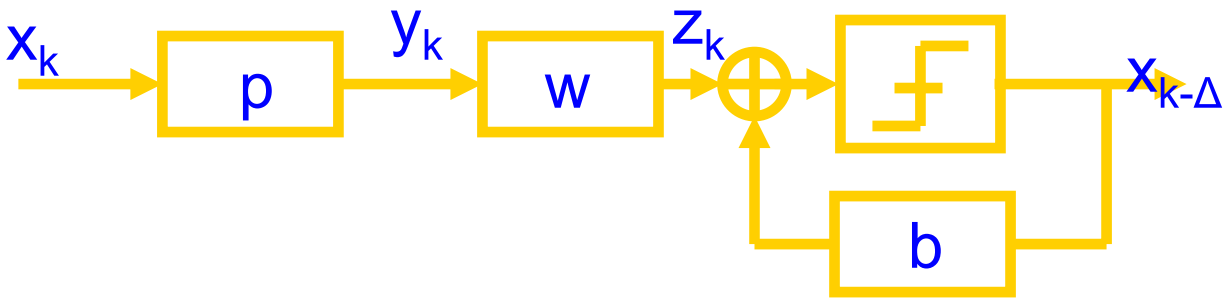
$$\boldsymbol{R}_{\boldsymbol{Y}x} \triangleq E\left\{\boldsymbol{Y}_k x^*_{k-\Delta}\right\}/N$$

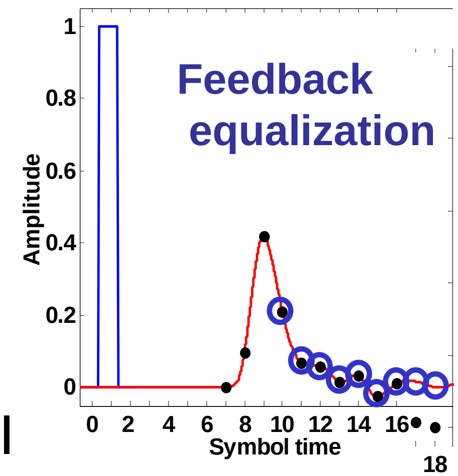
$$\begin{aligned}\boldsymbol{R}_{x\boldsymbol{Y}} &= \overline{E\left\{x_{k-\Delta}\boldsymbol{Y}_k^*\right\}} = \overline{E\left\{x_{k-\Delta}\boldsymbol{X}_k^*\right\}}\boldsymbol{P}^* + \overline{E\left\{x_{k-\Delta}\boldsymbol{N}_k^*\right\}} \\ &= \left[0 \ldots 0\; \bar{\mathcal{E}}_x\; 0 \ldots 0\right]\boldsymbol{P}^* + 0 \\ &= \bar{\mathcal{E}}_x\left[0 \ldots 0\; \boldsymbol{p}_\nu^* \ldots \boldsymbol{p}_0^*\; 0 \ldots 0\right]\end{aligned}$$

$$\begin{aligned}\boldsymbol{R}_{\boldsymbol{YY}} &= \overline{E\left\{\boldsymbol{Y}_k\boldsymbol{Y}_k^*\right\}} = \boldsymbol{P}\overline{E\left\{\boldsymbol{X}_k\boldsymbol{X}_k^*\right\}}\boldsymbol{P}^* + \overline{E\left\{\boldsymbol{N}_k\boldsymbol{N}_k^*\right\}} \\ &= \bar{\mathcal{E}}_x\boldsymbol{PP}^* + l\cdot\frac{\mathcal{N}_0}{2}\cdot\boldsymbol{R_{nn}} \;,\end{aligned}$$

$$[0 \ldots 0\; \boldsymbol{p}_\nu^* \ldots \boldsymbol{p}_0^*\; 0 \ldots 0] = \boldsymbol{1}^*_\Delta \boldsymbol{P}^*$$

Decision feedback equalizer





- Feed-forward equalizer
  - Matched + whitening filter + remove pre-cursor ISI
- Feed-back equalizer
  - Removes trailing ISI
  - To get w, first puncture the channel matrix to emulate the effect of feedback on the equalized pulse response wP
  - Then, get b from the causal taps of equalized pulse response wP

$$Y_k \triangleq \begin{bmatrix} y_k \\ y_{k-1} \\ \vdots \\ y_{k-N_f+1} \end{bmatrix} = \begin{bmatrix} p_0 & p_1 & \cdots & p_\nu & 0 & 0 & \cdots & 0 \\ 0 & p_0 & p_1 & \cdots & \cdots & p_\nu & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \\ 0 & \cdots & 0 & 0 & p_0 & p_1 & \cdots & p_\nu \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ \vdots \\ x_{k-N_f-\nu+1} \end{bmatrix} + \begin{bmatrix} n_k \\ n_{k-1} \\ \vdots \\ n_{k-N_f+1} \end{bmatrix}$$

MMSE DFE

$$MSE = E\left\{|x_{k-\Delta} - \boldsymbol{w}\boldsymbol{Y}_k - \boldsymbol{b}x_{k-\Delta-1}|^2\right\}$$

$$\boldsymbol{b} \triangleq [\, b_1 \; b_2 \; \ldots \; b_{N_b}]$$

$$\tilde{\boldsymbol{w}} \triangleq \left[\boldsymbol{w} \; \vdots \; -\boldsymbol{b}\right]$$

$$\tilde{\boldsymbol{Y}}_k \triangleq \left[\begin{array}{c} \boldsymbol{Y}_k \\ x_{k-\Delta-1} \end{array}\right]$$

$$MSE = E\left\{|x_{k-\Delta} - \tilde{\boldsymbol{w}}\tilde{\boldsymbol{Y}}_k|^2\right\}$$

Selects the feedback taps

$$\boldsymbol{b} = \boldsymbol{w}\boldsymbol{P}\boldsymbol{J}_\Delta \quad ,$$

$$\boldsymbol{w}\left(\boldsymbol{P}\boldsymbol{P}^* - \boldsymbol{P}\boldsymbol{J}_\Delta\boldsymbol{J}_\Delta^*\boldsymbol{P}^* + \frac{l}{\mathrm{SNR}}\boldsymbol{R}_{nn}\right) = \boldsymbol{1}_\Delta^*\boldsymbol{P}^*$$

$$\boldsymbol{w} = \boldsymbol{1}_\Delta^*\boldsymbol{P}^*\left(\boldsymbol{P}\boldsymbol{P}^* - \boldsymbol{P}\boldsymbol{J}_\Delta\boldsymbol{J}_\Delta^*\boldsymbol{P}^* + \frac{l}{\mathrm{SNR}}\boldsymbol{R}_{nn}\right)^{-1}$$

$$\left[\boldsymbol{w} \vdots -\boldsymbol{b}\right] \cdot \bar{\mathcal{E}}_{\boldsymbol{x}} \cdot \left[\begin{array}{cc} \boldsymbol{P}\boldsymbol{P}^* + \frac{l}{\mathrm{SNR}}\boldsymbol{R}_{nn} & \boldsymbol{P}\boldsymbol{J}_\Delta \\ \boldsymbol{J}_\Delta^*\boldsymbol{P}^* & \boldsymbol{I}_{N_b} \end{array}\right] = \left[\bar{\mathcal{E}}_{\boldsymbol{x}} \cdot \boldsymbol{1}_\Delta^*\boldsymbol{P}^* \vdots 0\right]$$

Basic multitone modulation

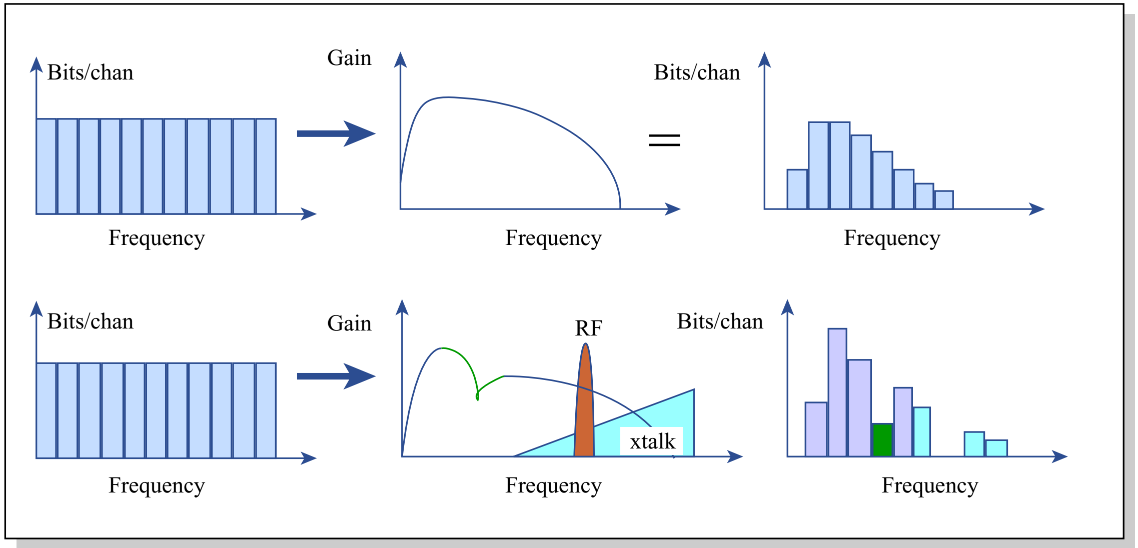


Figure by MIT OpenCourseWare.

- Best performance if basis functions are tailored to the channel
  - Use each tone as a basis function
  - Each tone transmits narrow QAM signal and satisfies Nyquist criterion – i.e. no ISI per tone
  - Put less energy where channel is bad or where there is more noise

A bit of history

- 1948 Shannon constructs capacity bounds
  - AWGN channel with linear ISI – effectively uses multi-tone modulation
- Analog multi-tone
  - 1958 Collins Kineplex modem (first voiceband modem) – analog multitone
  - 1964 Holsinger's MIT thesis – modem that approximates Shannon's "water-filling"
  - 1967 Saltzberg, 1973 Bell Labs, 1980 IBM …
- Digital multi-tone ~ 1990s
  - DMT for DSL - Major push by prof. Cioffi's group at Stanford
  - Use DSP power to improve the robustness and algorithms for discrete multi-tone modulation
  - We will mostly focus on this type of modulation

Basic multitone transmission

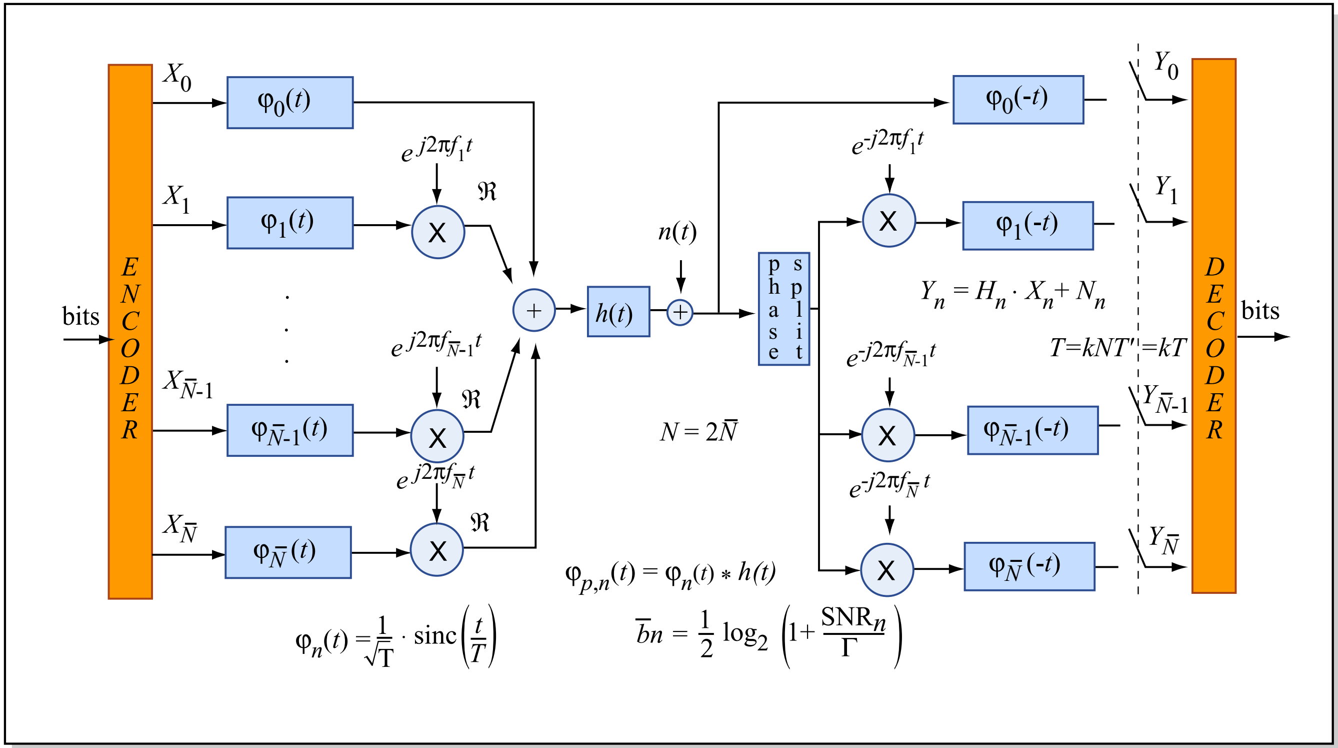


Figure by MIT OpenCourseWare.

$$\varphi_n(t) = \frac{1}{\sqrt{T}} \cdot \mathrm{sinc}\left(\frac{t}{T}\right) \qquad \bar{b}_n = \frac{1}{2}\log_2\left(1 + \frac{\mathrm{SNR}_n}{\Gamma}\right)$$

- Each tone sees AWGN channel (no ISI)
  - N QAM-like symbols (complex)
  - 1 PAM symbol

The effect of the channel

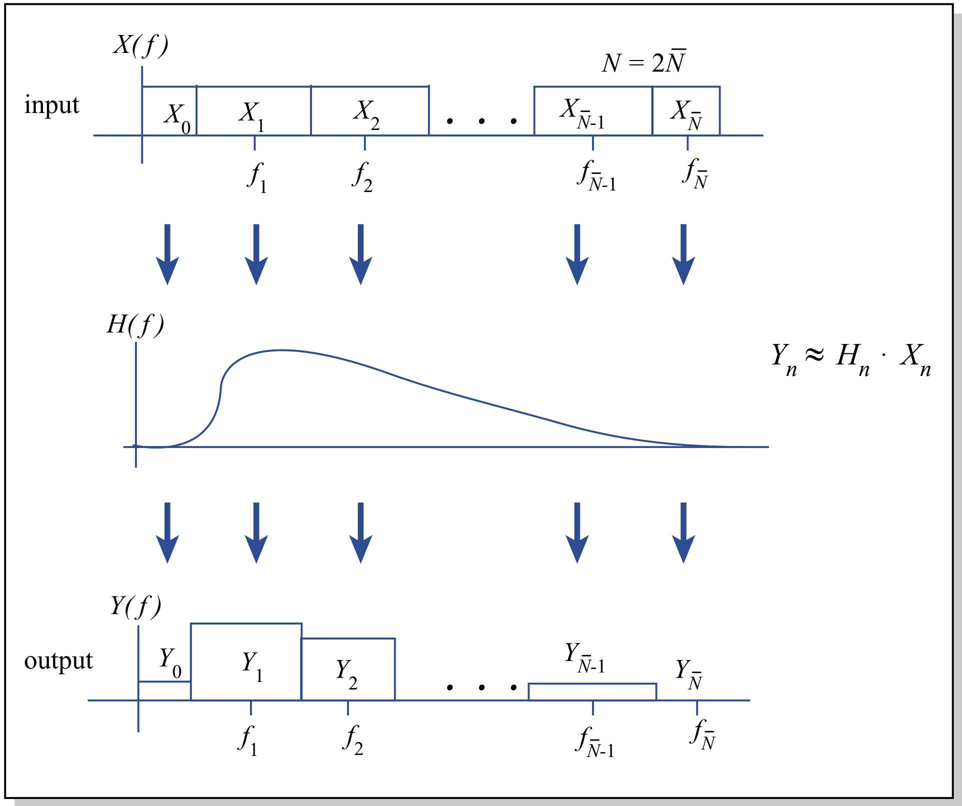


Figure by MIT OpenCourseWare.

- Each channel can be treated as AWGN
  - With only one basis function – hence simple symbol-by-symbol detector is optimal

Gap review

$$\bar{c}_n = \frac{1}{2}\log_2(1 + \mathrm{SNR}_n)$$

$$\bar{b}_n = \frac{1}{2}\log_2\left(1 + \frac{\mathrm{SNR}_n}{\Gamma}\right)$$

$$\Gamma \triangleq \frac{2^{2\bar{c}} - 1}{2^{2\bar{b}} - 1} = \frac{\mathrm{SNR}}{2^{2\bar{b}} - 1}$$

$\bar{b}$	.5	1	2	3	4	5
SNR for $P_e = 10^{-6}$ (dB)	8.8	13.5	20.5	26.8	32.9	38.9
$2^{2\bar{b}} - 1$ (dB)	0	4.7	11.7	18.0	24.1	30.1
Γ (dB)	8.8	8.8	8.8	8.8	8.8	8.8

Table of SNR Gaps for $P_e = 10^{-6}$.

Figure by MIT OpenCourseWare.

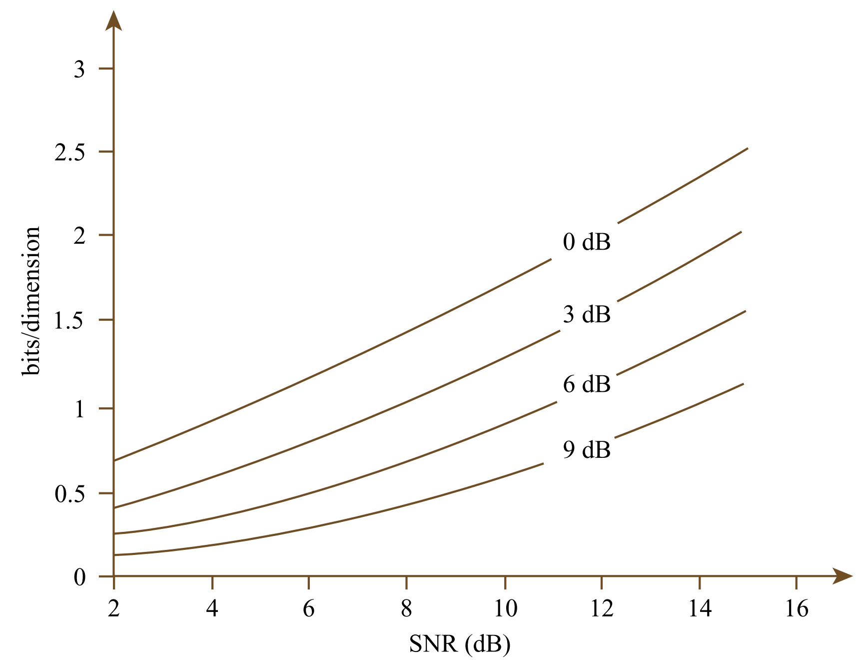


Illustration of bit rates versus SNR for various gaps.

Figure by MIT OpenCourseWare.

Example – simplified multitone

- 1+0.9D channel
  - With gap 4.4 dB

$$SNR_{m,u} \triangleq \left[\left(\prod_{n=1}^{N} \left[1 + \frac{SNR_n}{\Gamma} \right] \right)^{1/N} - 1 \right] \cdot \Gamma$$

n	$\mathcal{E}_n$	H_n	SNR_n	b_n	arg Q-func (dB)
0	8/7	1.9	22.8 (13.6 dB)	1.6	9.2
1	16/7	$1 + .9e^{j\pi/4} = 1.76\angle 21.3^o$	19.5 (12.9 dB)	2×1.5	9.2
2	16/7	$1 + .9e^{j\pi/2} = 1.35\angle 42.0^o$	11.4 (10.6 dB)	2×1.2	9.1
3	16/7	$1 + .9e^{j3\pi/4} = .733\angle 60.25^o$	3.4 (5.3 dB)	$2 \times .5$	10

- Put unit energy per dimension (simply guessed)
  - Same as baseband DFE
- Data rate 1bit/dimension
  - Re-calculate the necessary SNR – margin
- SNRmfb=10dB
  - SNRmultitone=8.8dB (with more tones to better approx no-ISI case)
  - SNRdfe=7.1dB
  - Can do even better with multitone, if allocated energy properly

Water-filling derivation

- Find optimum energy allocation that maximizes *b* for given total energy constraint
  - *b* is a convex function in energy/dimension

$$b = \frac{1}{2}\sum_{n=1}^{N}\log_2\left(1+\frac{\mathcal{E}_n \cdot g_n}{\Gamma}\right) \qquad g_n = |H_n|^2/(\sigma_n^2) \qquad \sum_{n=1}^{N}\mathcal{E}_n = N\bar{\mathcal{E}}_x$$

  - Use Lagrange multipliers to solve for ε_n

$$\max_{\mathcal{E}_n} b = \sum_{n=1}^{N}\frac{1}{2}\log_2\left(1+\frac{\mathcal{E}_n \cdot g_n}{\Gamma}\right)$$

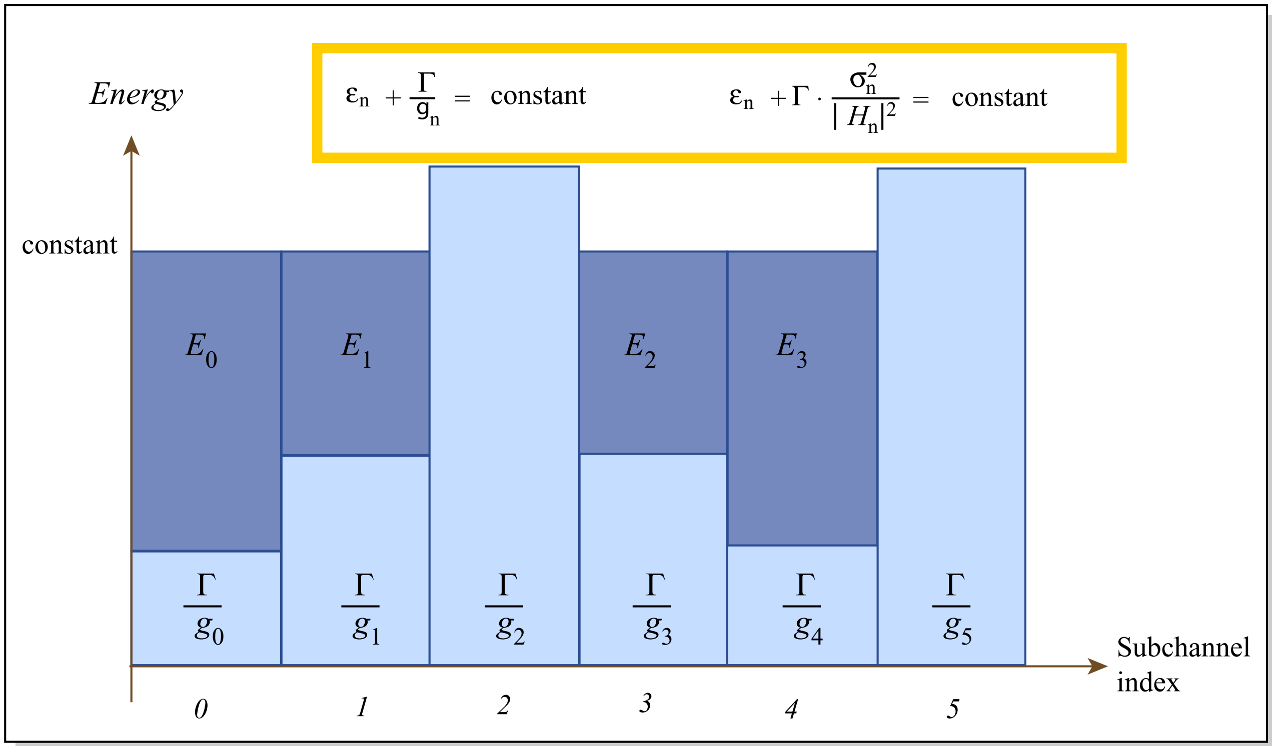
$$\textit{subject to: } N\bar{\mathcal{E}}_x = \sum_{n=1}^{N}\mathcal{E}_n$$

$$\frac{1}{2\ln(2)}\sum_n \ln\left(1+\frac{\mathcal{E}_n \cdot g_n}{\Gamma}\right)+\lambda\left(\sum_n \mathcal{E}_n - N\bar{\mathcal{E}}_x\right) \quad \xRightarrow{\frac{d}{d\varepsilon_n}} \quad \frac{1}{2\ln(2)}\frac{1}{1+\frac{\mathcal{E}_n \cdot g_n}{\Gamma}} = -\lambda$$

$$\mathcal{E}_n + \frac{\Gamma}{g_n} = \text{constant} \qquad \mathcal{E}_n + \Gamma \cdot \frac{\sigma_n^2}{|H_n|^2} = \text{constant}$$

Water-filling spectrum

- Flip the channel and pour in energy like water



Channel

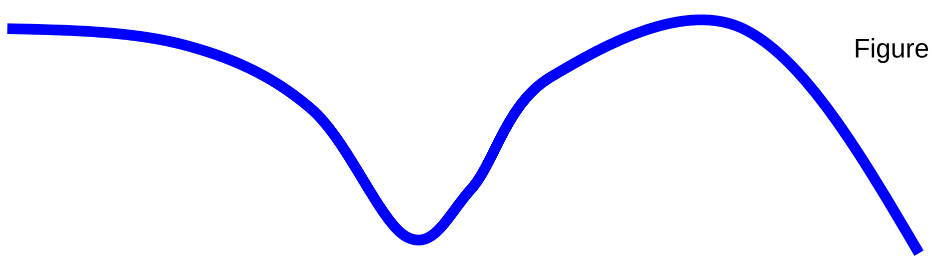


Figure by MIT OpenCourseWare.

Water-fill loading algorithms

- Rate maximization

$$\max_{\mathcal{E}_n} b = \sum_{n=1}^{N} \frac{1}{2} \log_2 \left(1 + \frac{\mathcal{E}_n \cdot g_n}{\Gamma}\right)$$

$$\textit{subject to: } N\bar{\mathcal{E}}_{\boldsymbol{x}} = \sum_{n=1}^{N} \mathcal{E}_n$$

- Margin maximization

$$\min_{\mathcal{E}_n} \mathcal{E}_{\boldsymbol{x}} = \sum_{n=1}^{N} \mathcal{E}_n$$

$$\textit{subject to: } b = \sum_{n=1}^{N} \frac{1}{2} \log_2 \left(1 + \frac{\mathcal{E}_n \cdot g_n}{\Gamma}\right)$$

Rate-adaptive loading

$$\begin{aligned} \mathcal{E}_1 + \Gamma/g_1 &= K \\ \mathcal{E}_2 + \Gamma/g_2 &= K \\ \vdots &= \vdots \\ \mathcal{E}_N + \Gamma/g_N &= K \\ \mathcal{E}_1 + \ldots + \mathcal{E}_N &= N\bar{\mathcal{E}}_{\boldsymbol{x}} \end{aligned}$$

Solve through matrix inversion

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & -1 \\ 0 & 1 & 0 & 0 & \ldots & -1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \vdots \\ \mathcal{E}_N \\ K \end{bmatrix} = \begin{bmatrix} -\Gamma/g_1 \\ -\Gamma/g_2 \\ \vdots \\ -\Gamma/g_N \\ N\bar{\mathcal{E}}_{\boldsymbol{x}} \end{bmatrix}$$

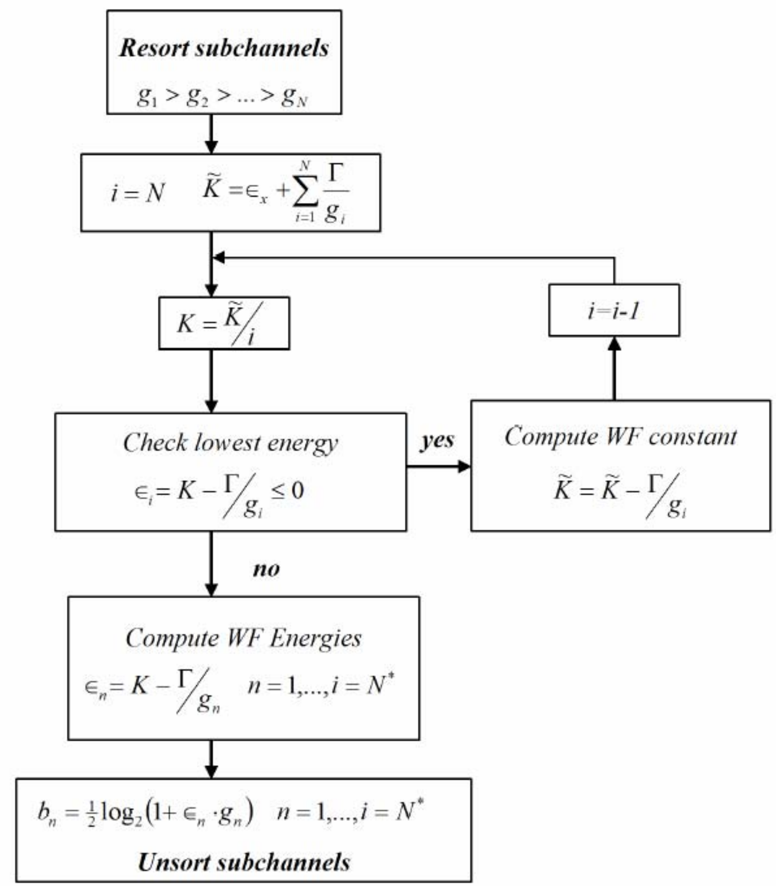
Solve iteratively

$$K = \frac{1}{N}\left[N\bar{\mathcal{E}}_{\boldsymbol{x}} + \Gamma \cdot \sum_{n=1}^{N} \frac{1}{g_n}\right]$$

$$\mathcal{E}_n = K - \Gamma/g_n \quad \forall\, n = 1, \ldots, N$$

$$K = \frac{1}{N-i}\left[\mathcal{E}_{\boldsymbol{x}} + \Gamma \cdot \sum_{n=1}^{N-i} \frac{1}{g_n}\right]$$

$$\mathcal{E}_n = K - \Gamma/g_n \quad \forall\, n = 1, \ldots, N^* = N - i$$



Water-filling example (rate-adaptive)

- 1+0.9D again (Gap=1, so calculating capacity)

$$g_0 = \frac{1.9^2}{.181} = 19.94$$
$$g_1 = \frac{1.7558^2}{.181} = 17.03$$
$$g_2 = \frac{1.3454^2}{.181} = 10.00$$
$$g_3 = \frac{.7329^2}{.181} = 2.968$$
$$g_4 = \frac{.1^2}{.181} = .0552$$

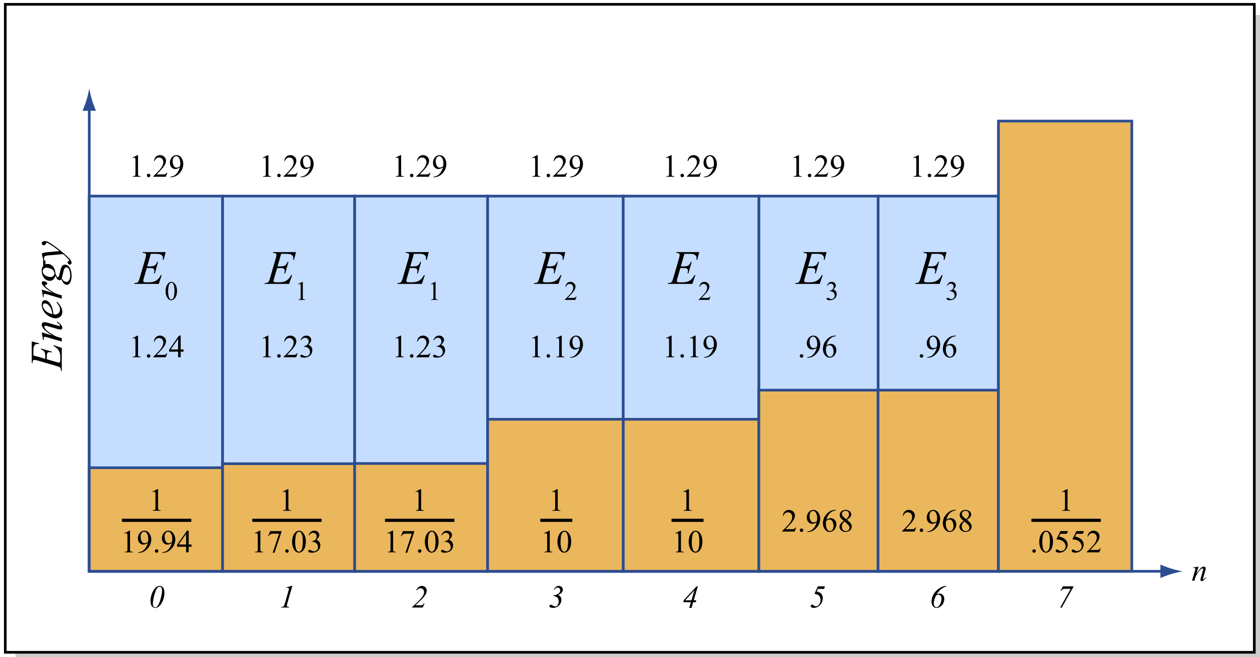


Figure by MIT OpenCourseWare.

Try 8 dim first

$$K = \frac{1}{8}\left[8 + 1\cdot(\frac{1}{19.94} + \frac{2}{17.03} + \frac{2}{10.0} + \frac{2}{2.968} + \frac{1}{.0552})\right] = 3.3947$$

=> $\mathcal{E}_4 = 3.3947 - 1/.05525 = -14.7 < 0$

Try 7 dim next

$$K = \frac{1}{7}\left[8 + 1\cdot(\frac{1}{20} + \frac{2}{17} + \frac{2}{9.8} + \frac{2}{3})\right] = 1.292$$

Capacity=1.55bits/dim

Summary

- Bandlimited communication
  - Block vs. symbol-by-symbol detector
- Try to make bandlimited channel look AWGN
  - Use complex block detectors to orthogonalize basis functions (MAP)
  - Simplify with equalization+sbs detector
  - Generate basis functions that don't loose orthogonality when passing through frequency selective channle (multitone modulation)
- Equalization
  - ZFE removes ISI but enhances noise
  - Trade-off by designing MMSE equalizer
  - DFE removes trailing ISI without noise enhancement
- Multitone
  - Optimal transmission with proper allocation of energy/dimension (waterfilling)
- Next – practical loading algorithms and DMT/OFDM, Vector coding