Lecture 4: London’s Equations

Outline

1. **Drude Model of Conductivity**
2. **Superelectron model of perfect conductivity**
   - First London Equation
   - Perfect Conductor vs “Perfect Conducting Regime
3. **Superconductor: more than a perfect conductor**
4. **Second London Equation**
5. **Classical Model of a Superconductor**

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Drude Model of Conductivity

First microscopic explanation of Ohm’s Law (1900)

1. The conduction electrons are modeled as a gas of particles with no coulomb repulsion (screening)
2. Independent Electron Approximation
   - The response to applied fields is calculated for each electron separately.
   - The total response is the sum of the individual responses.
3. Electrons undergo collisions which randomize their velocities.
4. Electrons are in thermal equilibrium with the lattice.

Response of individual electrons

Consider an electron of mass *m* and velocity **v** in an applied electric **E** and magnetic **B.**

$$m\frac{d\mathbf{v}}{dt} = \mathbf{f}_{\text{em}} + \mathbf{f}_{\text{drag}}$$

$$\mathbf{f}_{\text{em}} = q\left(\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right)$$

Ohm's Law   Hall Effect

$$\mathbf{f}_{\text{em}} \approx q\mathbf{E}$$

$$\mathbf{f}_{\text{drag}} = -\frac{m}{\tau_{tr}}$$

Transport scattering time

$$\boxed{m\frac{d\mathbf{v}}{dt} + \frac{m}{\tau_{tr}}\mathbf{v} = q\mathbf{E}}$$

Response of a single electron

$$m\frac{d\mathbf{v}}{dt} + \frac{m}{\tau_{tr}}\mathbf{v} = q\mathbf{E}$$

Consider a sinusoidal drive and response of a single electron

$$\mathbf{E} = \text{Re}\left\{\hat{\mathbf{E}}\,e^{j\omega t}\right\} \qquad \text{and} \qquad \mathbf{v} = \text{Re}\left\{\hat{\mathbf{v}}\,e^{j\omega t}\right\}$$

Then, $$j\omega m\hat{\mathbf{v}} + \frac{m}{\tau_{tr}}\hat{\mathbf{v}} = q\hat{\mathbf{E}}$$

and

$$\hat{v} = \left(\frac{q\tau_{tr}}{m}\right)\frac{1}{1+j\omega\tau_{tr}}\hat{E}$$

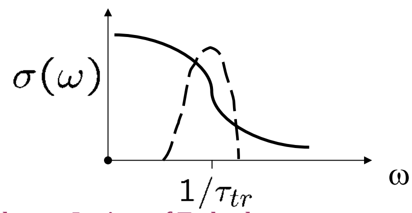
Total Response of conduction electrons

The density of conduction electrons, the number per unit volume, is *n*. The current density is

$$\hat{\mathbf{J}} = nq\hat{v} = \underbrace{\left(\frac{nq^2\tau_{tr}}{m}\right)\frac{1}{1+j\omega\tau_{tr}}}_{\displaystyle \sigma(\omega) = \sigma_o\frac{1}{1+j\omega\tau_{tr}}}\hat{E}$$

$$\Re(\sigma(\omega)) = \frac{\sigma_o}{1+(\omega\tau_{tr})^2}$$

$$\Im(\sigma(\omega)) = \frac{\sigma_o\,\omega\tau_{tr}}{1+(\omega\tau_{tr})^2}$$



Scattering time

To estimate the scattering time

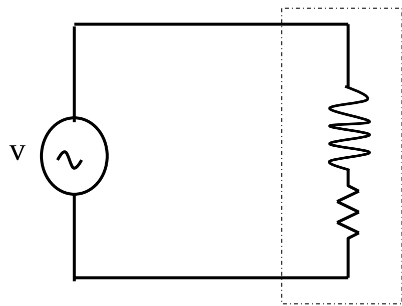
$$\tau_{tr} = \frac{m\sigma_o}{nq^2} = \frac{(9.1\times10^{-31}\text{kg})(5.8\times10^{7}\text{S})}{(8.5\times10^{28}\text{m}^{-3})(1.6\times10^{-19}\text{C})^2}$$

$$\tau_{tr} \approx 2.4\times10^{-14}$$

Hence for frequencies even as large at 1 THz,

$$\omega\tau_{tr} \ll 1 \qquad \text{and} \qquad \mathbf{J} = \sigma_o\mathbf{E}$$

Equivalent Circuit for a Metal



$$\tau_{RL} \Longleftrightarrow \tau_{tr}$$

$$L \Longleftrightarrow \frac{\tau_{tr}}{\sigma_o} = \frac{m}{nq^2}$$

$$R \Longleftrightarrow \frac{1}{\sigma_o} = \frac{m}{nq^2 \tau_{tr}}$$

$$Y(\omega) = \left(\frac{1}{R}\right) \frac{1}{1 + j\omega\tau_{RL}} \quad \Longleftrightarrow \quad \sigma(\omega) = \sigma_o \frac{1}{1 + j\omega\tau_{tr}}$$

Perfect Conductor vs. Perfectly Conducting Regime

$$\sigma(\omega) = \sigma_o \frac{1}{1 + j\omega\tau_{tr}}$$

Perfect conductor: $\omega\tau_{tr} >> 1$

$$\sigma(\omega) = \frac{\sigma_o}{1 + j\omega\tau_{tr}} = \frac{1}{j\omega(m/nq^2)}$$

A perfect inductor
Purely reactive
Lossless

Perfectly conducting regime:

$$\sigma(\omega) = \sigma_o \to \infty$$

A perfect resistor
Purely resistive
Lossy

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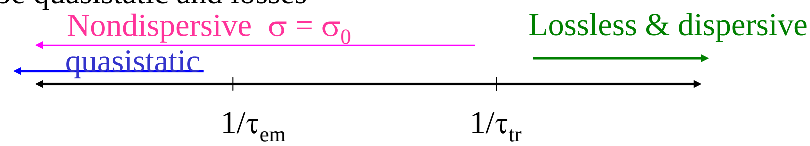
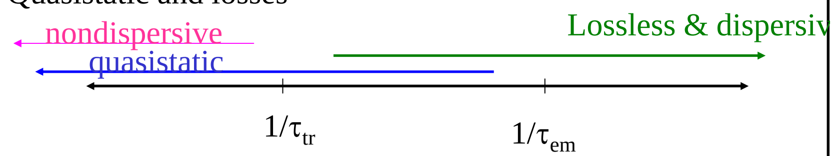
Ordering of time constants

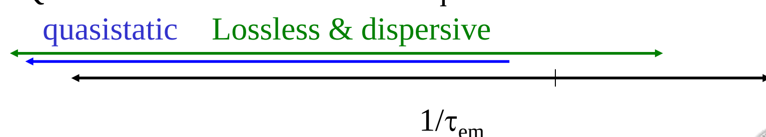
Cannot be quasistatic and losses



Can be Quasistatic and losses

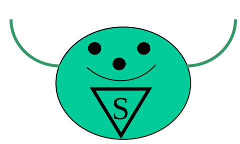


Can be Quasistatic and losses for all frequencies: Perfect conductor



First London Equation

Superelectron
Or Cooper Pair:



$$m^\star = 2m_e$$
$$q^\star = 2q_e$$
$$n^\star = \frac{n}{2}$$

$$\sigma(\omega) = \frac{1}{j\omega(m^\star/n^\star q^{\star\,2})} = \frac{1}{j\omega\Lambda}$$ and $$\Lambda \equiv \frac{m^\star}{n^\star(q^\star)^2}$$

Therefore,

$$\mathbf{J}(\omega) = \frac{1}{j\omega\Lambda}\mathbf{E}(\omega)$$

And we have the First London Equation $$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda\mathbf{J})$$

MQS and First London

$\nabla \times \mathbf{H} = \mathbf{J}$ and $\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J})$ describe the perfect conductor.

So that $\nabla \times \nabla \times \mathbf{H} = \nabla \times \mathbf{J}$

$$\boxed{\frac{\partial}{\partial t}} \left\{ \nabla \cdot (\overset{0}{\cancel{\nabla \cdot \mathbf{H}}}) - \nabla^2 \mathbf{H} = \nabla \times \mathbf{J} \right\}$$

$$-\nabla^2 \frac{\partial \mathbf{H}}{\partial t} = \nabla \times \frac{\mathbf{E}}{\Lambda}$$ using the first London EQN

Therefore,

$$\left(\frac{\mu_o}{\Lambda} - \nabla^2 \right) \frac{\partial}{\partial t} \mathbf{H} = 0$$ governs a perfect conductor.

The Penetration Depth

$$\left(\frac{\mu_o}{\Lambda} - \nabla^2 \right) \frac{\partial}{\partial t} \mathbf{H} = 0$$

So that

$$\left(\frac{1}{\lambda^2} - \nabla^2 \right) \frac{\partial}{\partial t} \mathbf{H} = 0$$

The penetration depth $\lambda \equiv \sqrt{\frac{\Lambda}{\mu_o}}$ is independent of frequency.

And is of the order of about 0.1 microns for Nb

Perfectly Conducting Infinite Slab

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Please see: Figure 2.13, page 43, from Orlando, T., and K. Delin *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Let $\mathbf{H}(\mathbf{r},t) = \mathsf{Re}\left\{\widehat{H}(y)\, e^{j\omega t}\right\}\mathbf{i}_z$

Therefore,

$$j\omega\left(\frac{1}{\lambda^2} - \frac{d^2}{dy^2}\right)\widehat{H}(y) = 0$$

and

$$\widehat{H}(y) = C\cosh(y/\lambda)$$

$$\mathbf{H}_{\mathsf{app}} = \mathsf{Re}\left\{\widehat{H}_o\, e^{j\omega t}\right\}\mathbf{i}_z$$

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\frac{\partial}{\partial t}\mathbf{H} = 0$$

Boundary Conditions demand

$$H_z(a) = H_z(-a) = C\cosh(a/\lambda) = \widehat{H}_o$$

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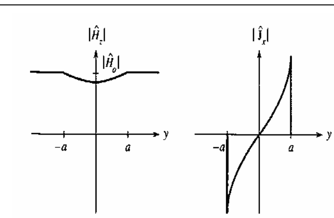
Fields and Currents for |y|< a

$$\mathbf{H} = \mathsf{Re}\left\{\widehat{H}_o\,\frac{\cosh(y/a)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_z \quad \mathbf{J} = \mathsf{Re}\left\{\frac{\widehat{H}_o}{\lambda}\,\frac{\sinh(y/\lambda)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_x$$

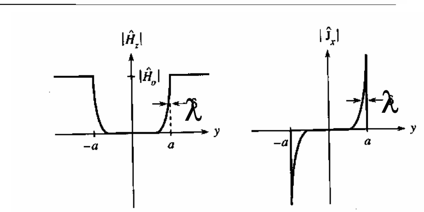
Thin film limit

$$a \ll \lambda$$



Bulk limit

$$a \gg \lambda$$



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Ohmic vs. perfect conductor

Ohmic conductor

$$\mathbf{H} = \mathrm{Re}\left\{\hat{H}_o \frac{\cosh ky}{\cosh ka}\, e^{j\omega t}\right\}\mathbf{i}_z \qquad \mathbf{J} = \mathrm{Re}\left\{\hat{H}_o k \frac{\sinh ky}{\cosh ka}\, e^{j\omega t}\right\}\mathbf{i}_x$$

$$k = \frac{(1+j)}{\delta}$$

Complex *k* mean a damped wave: lossy

Perfect conductor

$$\mathbf{H} = \mathrm{Re}\left\{\hat{H}_o \frac{\cosh(y/a)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_z \quad \mathbf{J} = \mathrm{Re}\left\{\frac{\hat{H}_o}{\lambda} \frac{\sinh(y/\lambda)}{\cosh(a/\lambda)}\, e^{j\omega t}\right\}\mathbf{i}_x$$

Real λ means an evanescent wave: Lossless

Modeling a perfect conductor

Thus, to model perfect conductivity, we cannot self-consistently neglect the frequency dependent part of σ, *which depends on* τ_{tr}, *and simultaneously let the frequency independent part grow to large values.* To be consistent, we must consider the entire expression for $\sigma(\omega)$ in the limit where $\tau_{tr} \to \infty$, to be the definition of a perfect conductor.

Perfectly Conducting Infinite Slab: General Solution

$$\mathbf{H}_{\text{app}} = \text{Re}\left\{\widehat{H}_o(\mathbf{r},t)\right\}\,\mathbf{i}_z$$

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\frac{\partial}{\partial t}\mathbf{H} = 0$$

Let $\mathbf{H}(\mathbf{r},t) = \text{Re}\left\{\widehat{H}(y,t)\right\}\mathbf{i}_z$

Therefore,

$$\frac{\partial}{\partial t}H(y,t) = C(t)\cosh(y/\lambda)$$

Boundary Conditions demand

$$\frac{\partial}{\partial t}H(y,t) = \frac{\partial}{\partial t}H(a,t)\,\frac{\cosh(y/\lambda)}{\cosh(a/\lambda)}$$

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Integrating over time gives:

$$H(y,t) = [H(a,t) - H(a,0)]\,\frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} + H(y,0)$$

Perfectly Conducting Infinite Slab: General Solution

$$H(y,t) = [H(a,t) - H(a,0)]\,\frac{\cosh(y/\lambda)}{\cosh(a/\lambda)} + H(y,0)$$

For a thin film, H(y,t) = H(a,t) for all time.

Bulk limit near surface y = a

$$H(y,t) = [H(a,t) - H(a,0)]\;e^{-(a-y)/\lambda} + H(y,0)$$

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Please see: Figure 2.13, page 43, from Orlando, T., and K. Delin *Foundations of Applied Superconductivity*. Reading, MA: Addison-Wesley, 1991. ISBN: 0201183234.

Deep in the perfect conductor

$$H(y,t) = H(y,0)$$

The perfect conductor preserves the original flux distribution, in the bulk limit.

A perfect conductor is a flux conserving medium; a superconductor is a flux expelling medium.

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Towards a superconductor

Perfectly conducting regime	Perfect conductor	Superconductor
$\mathbf{J} = \sigma_o \mathbf{E} \quad \sigma_o > \frac{1}{\ell}\sqrt{\frac{\epsilon}{\mu}}$	$\mathbf{J}(\omega) = \sigma(\omega)\,\mathbf{E}(\omega)$	$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J})$
	$\mathbf{J}(\omega) = \frac{1}{j\omega\Lambda}\mathbf{E}(\omega)$	$\mathbf{J}(\omega) = \frac{1}{j\omega\Lambda}\mathbf{E}(\omega)$
$\tau_m > \tau_c$	$\Lambda \equiv \frac{m^\star}{n^\star (q^\star)^2}$	$\Lambda \equiv \frac{m^\star}{n^\star (q^\star)^2}$
$\left(\mu\sigma_o \frac{\partial}{\partial t} - \nabla^2\right)\mathbf{H} = 0$	$\left(\frac{1}{\lambda^2} - \nabla^2\right)\frac{\partial}{\partial t}\mathbf{H} = 0$	$\left(\frac{1}{\lambda^2} - \nabla^2\right)\mathbf{H} = 0$
$\delta \equiv \sqrt{\frac{2}{\omega\mu\sigma_o}}$	$\lambda \equiv \sqrt{\frac{\Lambda}{\mu_o}}$	$\lambda \equiv \sqrt{\frac{\Lambda}{\mu_o}}$
Flux “expulsion” in the bulk limit, not for $\omega = 0$.	Flux conserving in the bulk limit	Flux expulsion in the bulk limit, even for $\omega = 0$.

Second London Equation

For a superconductor we want to have

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\mathbf{H} = 0$$

Working backwards

$$\nabla \cdot (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\frac{\mu_0}{\Lambda}\mathbf{H}$$

$$\nabla \times (\nabla \times \mathbf{H}) = -\frac{\mu_0}{\Lambda}\mathbf{H}$$

Therefore, the second London Equation

$$\nabla \times (\Lambda \mathbf{J}) = -\mathbf{B}$$

Superconductor: Classical Model

$$\mathbf{E} = \frac{\partial}{\partial t}(\Lambda \mathbf{J})$$ first London Equation

$$\nabla \times (\Lambda \mathbf{J}) = -\mathbf{B}$$ second London Equation

$$\Lambda \equiv \frac{m^\star}{n^\star (q^\star)^2} \qquad \lambda \equiv \sqrt{\frac{\Lambda}{\mu_o}}$$ penetration depth

When combined with Maxwell's equation in the MQS limit

$$\left(\frac{1}{\lambda^2} - \nabla^2\right)\mathbf{H} = 0$$