

Application Example 3
(Bayes' Theorem)

**EXTRA-TERRESTRIAL LIFE AND THE DESIGN OF
EXPERIMENTS**

There is much controversy and uncertainty on whether there ever was life on planet Fedra. To reduce uncertainty, an unmanned experiment is planned. The experiment consists of sending an automatic lab to the surface of Fedra. The lab would conduct chemical and biological tests to reveal whether or not life ever existed on that planet. It is recognized, however, that the experiment may not provide definitive information on the existence of life, due to possible malfunctioning of the testing equipment or the communication system, or the fact that evidence of life may not be present at the locations and depth below the surface accessible by the lab.

Let A and B be the events

$$\begin{aligned} A &= \text{"there was life on Fedra"} \\ B &= \text{"lab says 'there was life on Fedra'"} \end{aligned} \tag{1}$$

and denote by A^c and B^c their complements. For a certain design of the mission (equipment, type of tests, communication system, etc.), NASA experts evaluate the performance characteristics of the experiment to be as follows.

- If there ever was life on Fedra, the probability that the lab will detect it and correctly report the finding is $P[B|A] = 0.5$. Hence, $P[B^c|A] = 0.5$;

- If there never was life on Fedra, the probability that the lab will erroneously indicate the presence of life is $P[B|A^c] = 0.1$. Hence, $P[B^c|A^c] = 0.9$.

Notice that the probabilities $P[B^c|A]$ and $P[B|A^c]$ are nonzero due to the possibility of errors. In particular, the large value of $P[B|A^c]$ reflects the fact that, in the judgment of NASA, many things can go wrong in the detection of early life on Fedra. In fact, it is believed that only a manned expedition and the retrieval of rock samples for analysis on Earth could significantly reduce the probability of this type of errors.

Suppose that, using currently available information, a fair assessment of the probability that life was ever present on Fedra is $P[A] = 0.1$. Therefore, $P[A^c] = 0.9$. These “prior” probabilities apply before the experiment is performed. The question is, how will these probabilities change after the experiment is performed and the lab reports either B (“there was life on Fedra”) or B^c (“there never was life on Fedra”)? Should the “posterior” probabilities $P[A|B]$ and $P[A|B^c]$ be close to the prior probability $P[A]$, little would be gained by performing the experiment.

To update the probability of A based on (hypothetical) information provided by the lab, we use Bayes’ theorem. According to this theorem,

$$P[A|B] = P[A] \frac{P[B|A]}{P[B]} \quad (2)$$

Using the total probability theorem, the probability $P[B]$ in the denominator can be calculated as $P[B] = P[A] P[B|A] + P[A^c] P[B|A^c]$. In the present case, $P[B] = (0.1)(0.5) + (0.9)(0.1) = 0.14$ and $P[B^c] = 1 - P[B] = 0.86$.

Then, from Eq. 2,

$$P[A|B] = (0.1)(0.5)/(0.14) = 0.357$$

$$P[A^c|B] = 1 - P[A|B] = 0.643 \tag{3}$$

$$P[A|B^c] = (0.1)(0.5)/(0.86) = 0.058$$

$$P[A^c|B^c] = 1 - P[A|B^c] = 0.942$$

Interestingly, the posterior probability that there ever was life on Fedra does not exceed 0.5. This is true also in the case when the lab gives a positive indication of life. The reason is that the outcome of the experiment is recognized to be quite likely erroneous, due to the high values of the probabilities $P[B^c|A]$ and $P[B|A^c]$.

Problem 3.1 Calculate and plot the posterior probabilities $P[A|B]$ and $P[A^c|B^c]$ against $P[B|A]$, for $P[B|A]$ between 0 and 1 and $P[B^c|A^c] = 0.5, 0.8, 0.99$. Comment on the results.

Given the high cost of the experiment, one might question whether the information gained from it is worthy of the expense. A conclusion might be that it is preferable to invest more resources in the mission and obtain more definitive information about life on Fedra. A redesigned experiment might consist of multiple probes (if the likelihood of identical errors is small). Another possibility might be to increase the number and diversity of experiments conducted by the lab to detect past life.

The important point here is that, by assessing through Bayes' Theorem the probability of A and A^c under various experimental outcomes, one can compare the informativeness of alternative experiments, prior to conducting the experiments.

For example, suppose that a modified experiment for life on Fedra consists of making two separate tests for organic material, each with possible outcomes B (indicating life) and B^c (no sign of life). Suppose that the probabilities of various combined outcomes of the two experiments given A or A^c are as follows:

- If there ever was life on Fedra, then

$$\begin{aligned}
 P[(B \cap B)|A] &= 0.4, \\
 P[(B^c \cap B^c)|A] &= 0.1 \\
 P[(B \cap B^c)|A] &= P[(B^c \cap B)|A] = 0.25
 \end{aligned}
 \tag{4a}$$

- If there never was life on Fedra, then

$$\begin{aligned}
 P[(B \cap B)|A^c] &= 0.05, \\
 P[(B^c \cap B^c)|A^c] &= 0.75 \\
 P[(B \cap B^c)|A^c] &= P[(B^c \cap B)|A^c] = 0.1
 \end{aligned}
 \tag{4b}$$

Problem 3.2

(a) Using Bayes' theorem, calculate the posterior probabilities $P[A|(B \cap B)]$, $P[A|(B \cap B^c)]$, $P[A|(B^c \cap B)]$ and $P[A|(B^c \cap B^c)]$ using $P[A] = 0.1$ as prior and the probabilities in Eqs. 4a and 4b. Comment on the informativeness of this experiment relative to the experiment in Eq. 3.

(b) Describe, in one paragraph each, two similar applications of Bayes' theorem to soil sampling, material/structural testing, environmental quality monitoring, or any other area you are interested in.