**Solutions to quiz 1**
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# 1

(Bjarnadóttir, 2003, (Outline Kang, 2001))
$X_1, X_2$ are uniformly distributed between 0 and a. Let $G(a) \equiv E[max(x_1, x_2)^3]$ and consider $G(a+\varepsilon)$ when $X_1, X_2$ are uniformly distributed between 0 and $a+\varepsilon$, where $\varepsilon$ is very small.

Suppose $a < X_2 \leq a+\varepsilon$ and $0 \leq X_1 \leq a$. Then we know that $max(x_1, x_2) = x_2$. Therefore $E[max(x_1, x_2)^3] \equiv E[x_2^3]$. Since $X_1$ and $X_2$ are independent, $G(a+\varepsilon)$ for this case can be computed as follows:

$$G(a+\varepsilon) = E[max(x_1, x_2)^3] = E[x_2^3] = \int_a^{a+\varepsilon} (x_2)^3 f_{X_2}(x_2) dx_2 \,,$$

where $f_{X_2}(x_2)$ is the probability density function of $X_2$. Because $X_2$ is uniformly distributed over $(a, a+\varepsilon]$, $f_{X_2}(x_2) = \frac{1}{a}$. Thus,

$$\begin{aligned}
G(a+\varepsilon) &= \frac{1}{\varepsilon} \int_a^{a+\varepsilon} (x_2)^3 \, dx_2 \\
&= \frac{1}{\varepsilon} \left[ \frac{1}{4} x_2^4 \right]_a^{a+\varepsilon} \\
&= \frac{1}{\varepsilon} \cdot \frac{1}{4} \left( (a+\varepsilon)^4 - a^4 \right) \\
&= \frac{1}{\varepsilon} \cdot \frac{1}{4} \left( 4a^3\varepsilon + 6a^2\varepsilon^2 + 4a\varepsilon^3 + \varepsilon^4 \right) \\
&= \frac{1}{\varepsilon} \cdot \frac{1}{4} \left( (4a^3\varepsilon + o(\varepsilon) \right) ,
\end{aligned}$$

where $o(\varepsilon)$ represents higher order terms of $\varepsilon$ satisfying $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$ ("pathetic terms"). Therefore, $G(a+\varepsilon) \approx a^3$ as $\varepsilon \to 0$.

By symmetry we have $G(a+\varepsilon) \approx a^3$ as $\varepsilon \to 0$ when $0 \leq X_2 \leq a$ and $a < X_1 \leq a+\varepsilon$.

Finally, we do not have to compute $G(a+\varepsilon)$ for the case where $a < X_1 \leq a+\varepsilon$ and $a < X_2 \leq a+\varepsilon$ because the associated probability is negligible.

The following table summarizes $G(a+\varepsilon)$'s.

| Case | Probability of a case | $G(a+\varepsilon)$ given a case |
|---|---|---|
| hline $0 \le X_1 \le a,\ \ 0 \le X_2 \le a$ | $\frac{a}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = (\frac{a}{a+\varepsilon})^2$ | $G(a)$ |
| $a < X_1 \le a+\varepsilon,\ \ 0 \le X_2 \le a$ | $\frac{\varepsilon}{a+\varepsilon} \cdot \frac{a}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$ | $a^3$ |
| $0 \le X_1 \le a,\ \ a < X_2 \le a+\varepsilon$ | $\frac{a}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = \frac{\varepsilon a}{(a+\varepsilon)^2}$ | $a^3$ |
| $a < X_1 \le a+\varepsilon,\ \ a < X_2 \le a+\varepsilon$ | $\frac{\varepsilon}{a+\varepsilon} \cdot \frac{\varepsilon}{a+\varepsilon} = (\frac{\varepsilon}{a+\varepsilon})^2$ | We do not care. |

Using the total expectation theorem, we obtain

$$\begin{aligned} G(a+\varepsilon) &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + a^3\frac{\varepsilon a}{(a+\varepsilon)^2} + a^3\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2) \\ &= G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + 2a^3\frac{\varepsilon a}{(a+\varepsilon)^2} + o(\varepsilon^2) \\ &\approx G(a)\left(\frac{a}{a+\varepsilon}\right)^2 + 2a^3\frac{\varepsilon a}{(a+\varepsilon)^2}\,. \end{aligned}$$

From the formula of the sum of an infinite geometric series, we know

$$\frac{a}{a+\varepsilon} = \frac{1}{1+\frac{\varepsilon}{a}} = 1 - \frac{\varepsilon}{a} + \left(\frac{\varepsilon}{a}\right)^2 - \left(\frac{\varepsilon}{a}\right)^3 + \cdots\,.$$

Ignoring higher order terms of $\varepsilon$, we get

$$\frac{a}{a+\varepsilon} \approx 1 - \frac{\varepsilon}{a}\,.$$

This gives the following approximations:

$$\begin{aligned} \left(\frac{a}{a+\varepsilon}\right)^2 &\approx \left(1-\frac{\varepsilon}{a}\right)^2 = 1 - \frac{2\varepsilon}{a} + \frac{\varepsilon^2}{a^2} \approx 1 - \frac{2\varepsilon}{a}\,, \\ \frac{\varepsilon a}{(a+\varepsilon)^2} &= \frac{\varepsilon}{a}\left(\frac{a}{a+\varepsilon}\right)^2 \approx \frac{\varepsilon}{a}\left(1-\frac{2\varepsilon}{a}\right) = \frac{\varepsilon}{a} - \frac{2\varepsilon^2}{a^2} \approx \frac{\varepsilon}{a}\,. \end{aligned}$$

Therefore, we can rewrite $G(a+\varepsilon)$ as

$$G(a+\varepsilon) \approx G(a)\left(1-\frac{2\varepsilon}{a}\right) + 2a^3\cdot\frac{\varepsilon}{a} = G(a)\left(1-\frac{2\varepsilon}{a}\right) + 2a^2\varepsilon\,.$$

Rearranging terms, we have

$$\frac{G(a+\varepsilon)-G(a)}{\varepsilon} = -\frac{2G(a)}{a} + 2a^2\,.$$

If $\varepsilon \to 0$, we have the following differential equation:

$$G'(a) = -\frac{2G(a)}{a} + 2a^2 \,.$$

Seeing the $2a^2$ term, a "judicious" guess for the form of $G(a)$ is $Ba^3$ (keeping in mind that G(0)=0 and therefore there is no constant term in $G(a)$). Assuming $G(a) = Ba^3$ we have $G'(a) = 3Ba^2$. Plugging these values into our differential equation gives us:

$$\begin{gathered} 3Ba^2 = -2Ba^2 + 2a^2 \\ \Leftrightarrow 5B = 2 \\ \Leftrightarrow B = \tfrac{2}{5} \end{gathered}$$

This gives us the following solution:

$$G(a) \equiv E[max(x_1, x_2)^3] = \frac{2a^3}{5} \,.$$

## 2

(Bjarnadóttir, 2003)
Let assume $v_4$ is at some distance $k$ from the given point, with out loss of generality, we can assume $k = 1$ (then we do not have to carry $k$ through our calculations). Then we know that there are three other vehicles inside a circle of radius 1, which are uniformly distributed over the area of the circle.

Let A be the event that $v_4 > 4v_1$ and let B be the event that $v_4 > 2v_2$. We want to find the joint probability of these events, that is $P(A \cap B) = P(A) * P(B|A)$.

P(A) is the probability that at least one vechicle is within a circle of radius $\frac{1}{4}$. The compliment of A is the event that no vehicle is within radius $\frac{1}{4}$. For any one vehicle the probability of being outside a circle of radius $\frac{1}{4}$ is $\frac{(\pi*1^2-\pi*(1/4)^2}{1^2*\pi} = \frac{15}{16}$. Therefore $P(A) = 1 - P(A^c) = 1 - (\frac{15}{16})^3 = \frac{721}{4096}$

For event B ( $v_4 > 2v_2$ ) we need to have two vehicles within a circle of radius $\frac{1}{2}$. $P(B|A)$ is the event that the second vehicle is inside of a circle of radius $\frac{1}{2}$ given that the first vehicle is inside a circle of radius $\frac{1}{4}$. The compliment, $P(B^c|A)$ is then the event that the second nearest vehicle is outside of circle of radius $\frac{1}{2}$, given that the first one is within a circle of radius $\frac{1}{4}$ and $P(B|A) = 1 - P(B^c|A)$.

Now $P(B^c|A) = \frac{P(B^c \cap A)}{P(A)}$, where $P(B^c \cap A)$ is the event that two vehicles are outside of $\frac{1}{2}$ AND one vehicle inside of $\frac{1}{4}$. Therefore

$$P(B^c|A) = \frac{P(B^c \cap A)}{P(A)} = \frac{3 \cdot \frac{1}{16} \cdot (\frac{3}{4})^2}{\frac{721}{4096}} = \frac{432}{721}$$

Now

$$P(B|A) = 1 - P(B^c|A) = 1 - \frac{432}{721} = \frac{289}{721}$$

We then can put it all together:

$$P(A \cap B) = P(A) * P(B|A) = \frac{721}{4096} * \frac{289}{721} = \frac{289}{4096} \approx 0.071$$

## 3

(Bjarnadóttir, 2003)
(i) When considering the different probabilities for Mendel of entering in intervals of different lengths, we need to take into account random incidence: Mendel has $\frac{4}{4+5+6} = \frac{4}{15}$ chance of entering in an interval of length 4, $\frac{5}{15}$ of entering in an interval of length 5 and $\frac{6}{15}$ of entering in an interval of length 6. Given the Mendel enters in an interval of a certain length, his arrival is uniformly distributed over that interval. We can therefore compute the probability that he waits between 4 and 5 minutes for the next train as follows:

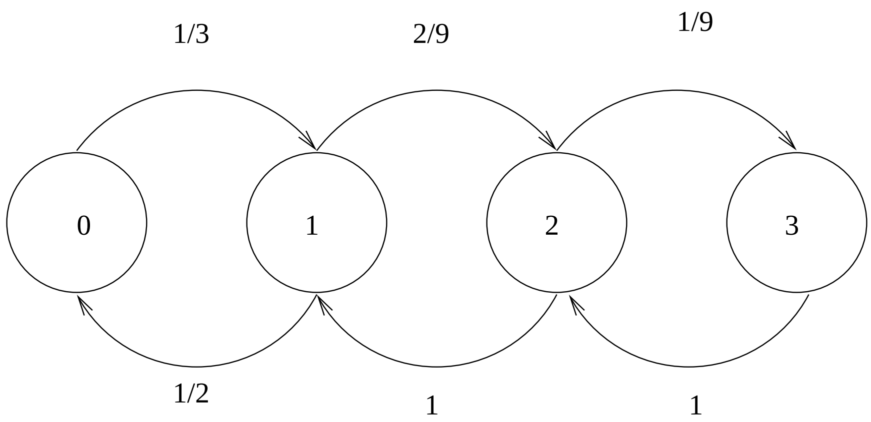
P(Mendel waiting between 4 and 5 minutes)= $\frac{4}{15} * 0 + \frac{5}{15} * \frac{1}{5} + \frac{6}{15} * \frac{1}{6} = \frac{2}{15}$

(ii) If the Lemon Line became less variable and all intervals between trains were exactly 5 minutes, the probability would go from $\frac{2}{15}$ to $\frac{1}{5}$, since Mendel would always arrive in an interval of length 5 and therefore the chance to wait between 4 and 5 minutes is always 1/5.

Intuitively, why does the answer move in that direction? (Barnett, 2003)
We see in the first part of the problem that the chance of waiting between 4 and 5 minutes is higher (20%) given an interval of length 5 than either one of length 4 (0%) or of length 6 (16.7%). Thus, if intervals of lengths 4 and 6 disappear in favor of 5's, the chance of waiting between 4 and 5 minutes must go up. (The average wait goes down under the change, because the possibility of waiting more than 5 minutes evaporates.)

## 4

(Odoni, 2003)
The small factory has 3 machines, therefore the total population is three. Our Birth-and-death chain has therefore only a 4 states, that is all machines can be running, one can be broken down, two can be broken down or all can be broken down. The following picture shows our queueing system.




We can now write our steady state equations:

$$\frac{1}{3}P_0 = \frac{1}{2}P_1$$
$$\frac{2}{9}P_1 = P_2$$
$$\frac{1}{9}P_2 = P_3$$
$$P_0 + P_1 + P_2 + P_3 = 1$$

Which gives us: $P_0 = \frac{243}{445}$, $P_1 = \frac{162}{445}$, $P_2 = \frac{36}{445}$ and $P_3 = \frac{4}{445}$. We can now find the expected number of machines that are operating, which three (the total population) minus the expected number in the system: $3 - L = 3 - (0 * P_0 + 1 * P_1 = 2 * P_2 + 3 * P_3) \approx 2.45$ operating machines.