2.003J/1.053J Dynamics and Control I, Spring 2007
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2/28/2007

Lecture 7

2-D Motion of Rigid Bodies - Kinematics

Kinematics of Rigid Bodies

Williams 3-3 (No method of instant centers)

"Kinematics" - Description and analysis of the motions of objects without consideration of the forces and torques causing them.

Angular Velocity

Define Angular Velocity

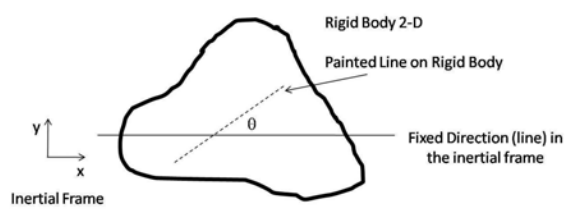

Figure 1: Rigid body in inertial frame. Figure by MIT OCW.

The angular velocity of the rigid body:

$$\underline{\omega} = \frac{d\theta}{dt}\hat{e}_z$$

$\underline{\omega}$ is a property of the body.

$\underline{\omega}$ is independent of the choice of 'painted line' or the reference fixed direction (Bedford & Fowler 6-2, 6-3)

Calculation of Velocity of a Point P on a Rotating Rigid Body

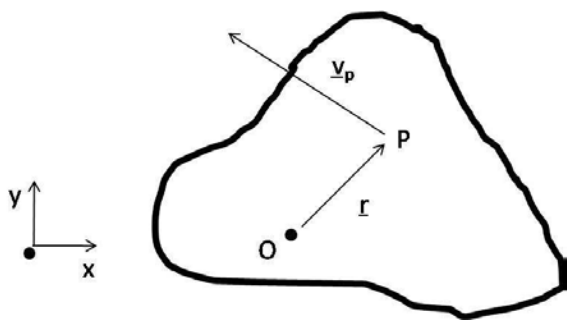

Figure 2: Rotating Rigid Body. O is fixed in the frame. Rotation axis passes through O. Figure by MIT OCW.

For any point on the body P:

$$\underline{v}_p = \underline{\omega} \times \underline{r}$$

More generally, for any vector $\underline{R}$ on the body:

$$\underline{R} = \underline{r}_2 - \underline{r}_1$$

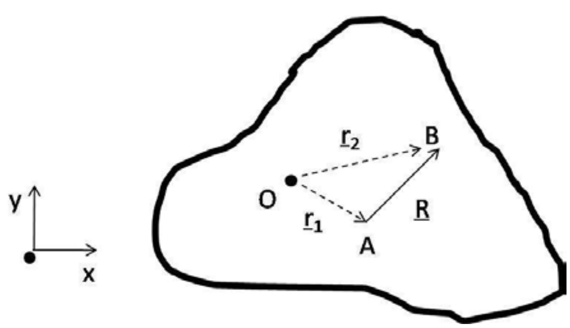

Figure 3: Rigid Body Rotating. Figure by MIT OCW.

Therefore:

$$\begin{aligned}\frac{d\underline{R}}{dt} &= \frac{d}{dt}(\underline{r}_2 - \underline{r}_1) \\ &= \underline{\omega} \times (\underline{r}_2 - \underline{r}_1) \\ &= \underline{\omega} \times \underline{R}\end{aligned}$$

B is moving on a circular path relative to $A \rightarrow$ although neither A or B is the axis of rotation.

Velocity of a Point P on Rotating and Translating Rigid Body

Often we have a combination of rotation and translation:

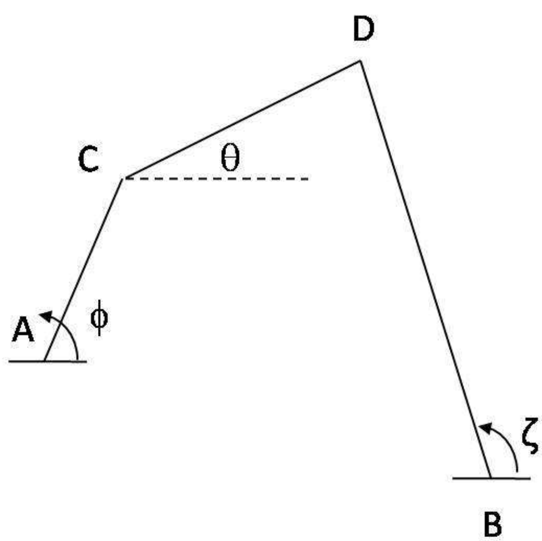

Figure 4: Three Bar Linkage. Figure by MIT OCW.

Bar AC: Fixed axis rotation about A:

$$\underline{\omega}_{AC} = \frac{d\phi}{dt}\hat{e}_z$$

Bar BD: Fixed axis rotation about B:

$$\underline{\omega}_{BD} = \frac{d\zeta}{dt}\hat{e}_z$$

Bar <u>CD</u>: Motion is a combination of rotation and translation.
At any point in time, we may locate CD by locating the point C. (Covers the translation) and identify the angle θ the CD makes with the horizontal.

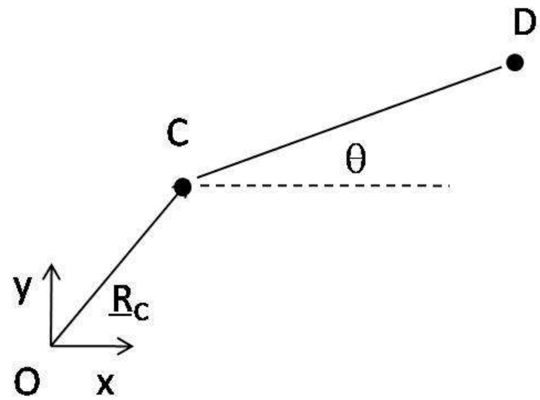

Figure 5: Free Body Diagram of Rod CD. Figure by MIT OCW.

Angular velocity is $\underline{\omega}_{CD} = \frac{d\theta}{dt}\hat{e}_z$

$\underline{\omega}_{CD}$ is independent of choice of point C. Bar has intrinsic rotation.

The motion of a rigid body is expressed as a combination of translation of a point fixed on the body and rotation about an axis passing through this point $\rightarrow$ need (x, y, θ).

(In a rigid body, particles are constrained to be the same distance apart.)

Aside:

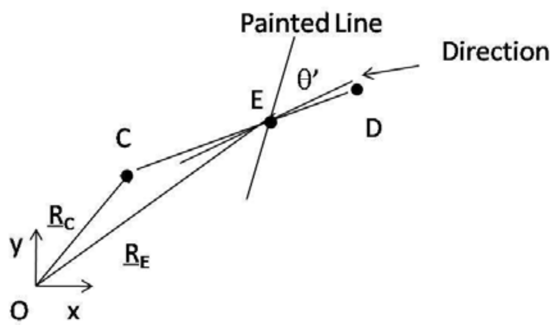

Figure 6: Angular velocity is not affected by location of direction line and painted line. Although the direction line and painted line are different than those in Figure 5, the angular velocities are the same. Suppose you had a different point E, with a direction line and painted line as shown. The relationship is $\underline{\omega} = \frac{d\theta}{dt}\hat{e}_z = \frac{d\theta}{dt}\hat{e}_z$. This independence of the angular velocity from the choice of direction line and painted line is explained in Williams, James H., Jr. *Fundamentals of Applied Dynamics*. New York, NY: John Wiley, 2006. ISBN: 9780470133859. Figure by MIT OCW.

Formalize:

Compute the velocity of any point P on a rigid body.

Now:

$$\underline{R}_p = \underline{R}_G + \underline{r}$$

$$\underline{\omega} = \frac{d\theta}{dt}\hat{e}_z$$

$$\underline{v}_p = \frac{d}{dt}\underline{R}_p = \frac{d\underline{R}_G}{dt} + \frac{d\underline{r}}{dt}$$

$$\underline{v}_p = \underline{v}_G + \underline{\omega} \times \underline{r}$$

Use $\underline{v}_p = \underline{v}_G + \underline{\omega} \times \underline{r}$. This relationship will be used often in finding the velocity of the body needed for the angular momentum principle.

$$\boxed{\underline{v}_P = \underline{v}_G + \underline{\omega} \times \underline{r}}$$

We can express the motion of any point on a rigid body in terms of translation of another point on the body and a rotation about that point.

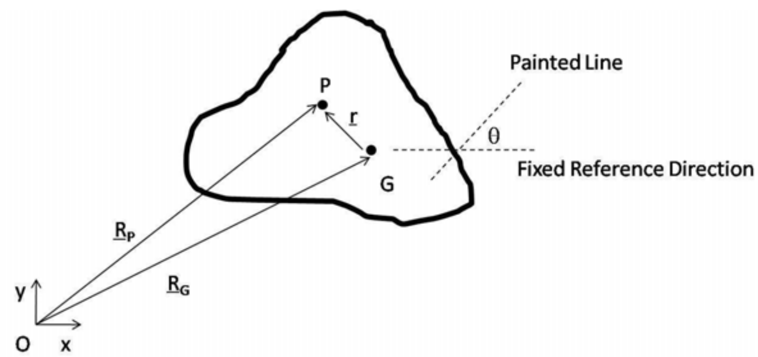

Figure 7: A rotating rigid body with two selected points P and G. The velocity at point P can be expressed in terms of the velocity at point G plus a term to represent the rotation of the point P around the point G. The term is the cross product of the angular velocity with the vector $\underline{r}$, which is position vector pointing from G to P. The angular velocity is set by the fixed reference direction, the painted line, and the rate at which the angle between those two lines changes. Figure by MIT OCW.

Example: Car With Swinging Bar

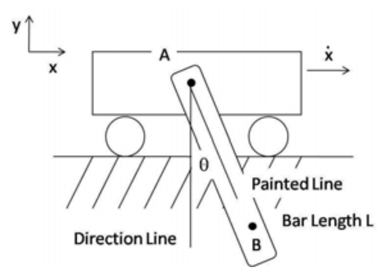

Figure 8: Car with swinging bar. Figure by MIT OCW.

Find the velocity of point B.

(Motion of Rod AB) = (Translation of A)+(Rotation about axis passing through A)

Angular velocity of the bar is $\underline{\omega}_{AB} = \dot{\theta}\hat{e}_z$.

$$\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{AB}$$
$$\underline{v}_A = \dot{x}\hat{e}_x$$

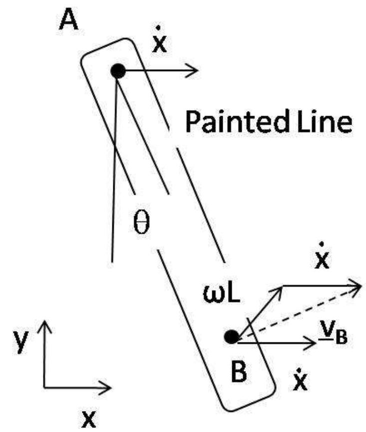

Figure 9: Kinematic Diagram of Rod AB. Figure by MIT OCW.

$$\underline{\omega} = \dot{\theta}\hat{e}_z$$

$$\underline{r}_{AB} = (L\sin\theta)\hat{e}_x - (L\cos\theta)\hat{e}_y$$

$$\underline{v}_{AB} = \dot{x}\hat{e}_x + (\dot{\theta}\hat{e}_z) \times (L\sin\theta\hat{e}_x - L\cos\theta\hat{e}_y) = (\dot{x} + \dot{\theta}L\cos\theta)\hat{e}_x + (\dot{\theta}L\sin\theta)\hat{e}_y$$

Geometric Constraints

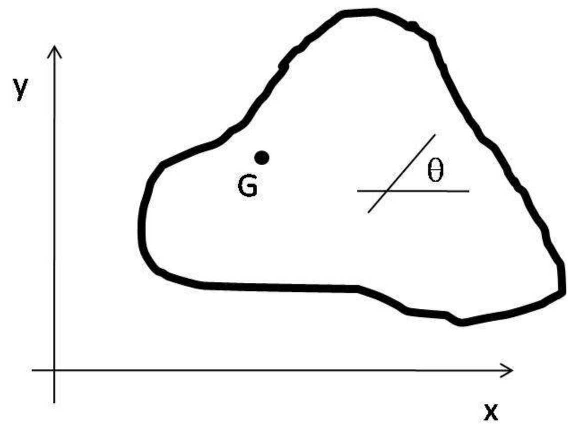

Figure 10: Rigid Body subject to rotation. Figure by MIT OCW.

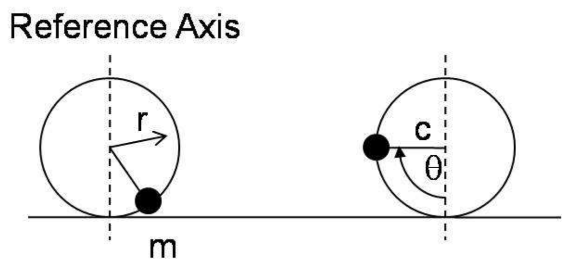

Figure 11: Ball rolling along x-direction. Figure by MIT OCW.

If you have (x_G, y_G, θ), then the state of your rigid body is uniquely defined. Complete set of coordinates.

A complete set of coordinates may not be independent; however, due to geometric constraints.

Example: Rolling Hoop

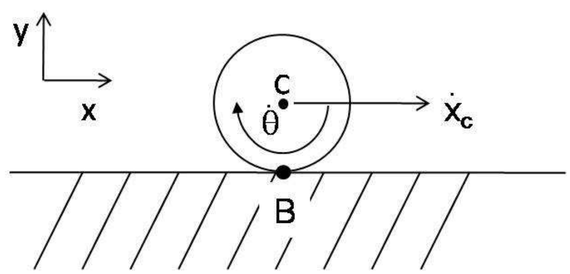

Figure 12: Rolling hoop. Hoop rolls without slipping. Mass m is attached to the hoop. Figure by MIT OCW.

Want to specify the position of the mass.

Pick c as the reference point.

$$x_m = x_c - r\sin\theta$$

$$y_m = y_c - r\cos\theta$$

x_c, y_c, θ form a complete coordinate system.
But there are 2 constraints on the surface.

1. Rolls on surface
2. Rolls withoug slipping (no sliding)

3 coordinates, 2 constrained. Therefore, only 1 generalized coordinate is required to describe m.

1. Rolls on surface

$$y_c = r$$

2. No Slip:

$$\underline{v}_B = 0 \text{ for no slip}$$

$$\underline{v}_B = \underline{v}_C + \underline{\omega} \times \underline{r}_{CB}$$

$$\underline{\omega} = -\dot{\theta}\hat{e}_z$$

$$\underline{r}_{CB} = -r\hat{e}_y$$

$$\underline{\omega} \times \underline{r}_{CB} = -r\dot{\theta}\hat{e}_x$$

$$\underline{v}_B = 0 = \dot{x}_c\hat{e}_x - r\dot{\theta}\hat{e}_x$$

$$\dot{x}_c = r\dot{\theta}$$

$$x_c = r\theta$$

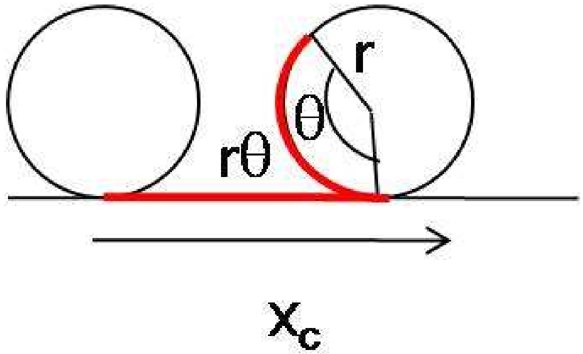

Figure 13: Hoop rolls along x-direction where distanced travelled equals $r\theta$. Figure by MIT OCW.

This statement means the displacement x_c is equal to the part of the hoop's circumference $r\theta$.

Thus, we can choose a single generalized coordinate to describe the state of the system. θ chosen!

$$x_M = r\theta - r\sin\theta$$

$$y_M = r - r\cos\theta$$

This completely defines the system.

Note: If slipping is allowed, need x_c and θ to describe state of the system. This is because the hoop could slide (translate without rotation).