

Isentropic Processes

For an isentropic process $s_1 = s_2$

$$s_2 - s_1 = s_2^o - s_1^o - R \ln\left(\frac{P_2}{P_1}\right) = 0$$

$$\frac{s_2^o - s_1^o}{R} = \ln\left(\frac{P_2}{P_1}\right)$$

$$\frac{P_2}{P_1} = \exp\left(\frac{s_2^o - s_1^o}{R}\right) = \frac{\exp(s_2^o / R)}{\exp(s_1^o / R)}$$

Define relative pressure $P_r \equiv \exp(s^o / R)$

$$\boxed{\left(\frac{P_2}{P_1}\right)_{s=const} = \frac{P_{r2}}{P_{r1}}}$$

For ideal gas $\frac{v_2}{v_1} = \frac{T_2}{T_1} \left(\frac{P_1}{P_2}\right) = \frac{T_2}{T_1} \left(\frac{P_{r1}}{P_{r2}}\right) = \frac{(T_2 / P_{r2})}{(T_1 / P_{r1})}$

Define $v_r \equiv T / P_r$, so $\boxed{\left(\frac{v_2}{v_1}\right)_{s=const} = \frac{v_{r2}}{v_{r1}}}$

Values of P_r and T_r as a function of temperature for air are tabulated in Table A-22

Isentropic Process for ideal gas with constant c_v and c_p

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{v_2}{v_1}\right) = 0$$

note $c_v = R/(k-1)$

$$0 = R \left[\frac{1}{k-1} \ln\left(\frac{T_2}{T_1}\right) + \ln\left(\frac{v_2}{v_1}\right) \right]$$

$$0 = \ln\left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}} + \ln\left(\frac{v_2}{v_1}\right)$$

$$0 = \ln\left[\left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}} \cdot \left(\frac{v_2}{v_1}\right) \right]$$

Take exponential of both sides

$$\exp(0) = 1 = \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}} \cdot \left(\frac{v_2}{v_1}\right)$$

$$\boxed{\left(\frac{T_2}{T_1}\right)_{\substack{s=\text{const} \\ c_p=\text{const}}} = \left(\frac{v_1}{v_2}\right)^{k-1}}$$

but $\frac{T_2}{T_1} = \frac{P_2 v_2}{P_1 v_1}$ substituting $\frac{P_2 v_2}{P_1 v_1} = \left(\frac{v_1}{v_2}\right)^{k-1}$

this yields $\boxed{\left(\frac{P_2}{P_1}\right)_{\substack{s=const \\ c_p=const}} = \left(\frac{v_1}{v_2}\right)^k}$ or $P_1 v_1^k = P_2 v_2^k$

Recall, for a polytropic compression or expansion process $Pv^n = \text{const}$, for the special case of an isentropic process (adiabatic and reversible) $\rightarrow n = k$

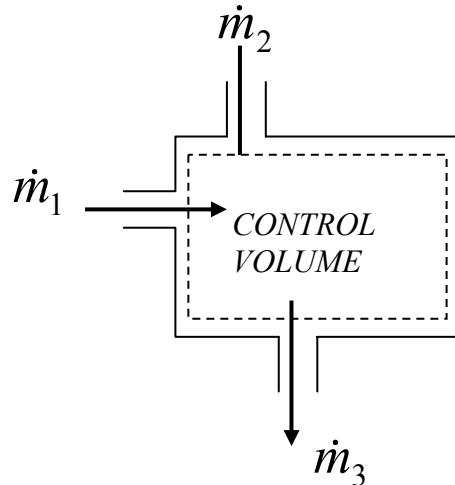
Combining the two equations yields

$$\left(\frac{v_1}{v_2}\right) = \left(\frac{P_2}{P_1}\right)^{\frac{1}{k}} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}}$$

$$\boxed{\left(\frac{T_2}{T_1}\right)_{\substack{s=const \\ c_p=const}} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}}}$$

Control volume entropy rate balance

Similar approach to that used to derive conservation of energy



$$\underbrace{\frac{dS_{CV}}{dt}}_{\text{Rate of Entropy change}} = \underbrace{\sum_j \frac{\dot{Q}_j}{T_j} + \sum_i \dot{m}_i s_i - \sum_e \dot{m}_e s_e}_{\text{Rate of entropy transfer}} + \underbrace{\dot{S}_{gen}}_{\text{Rate of Entropy production}}$$

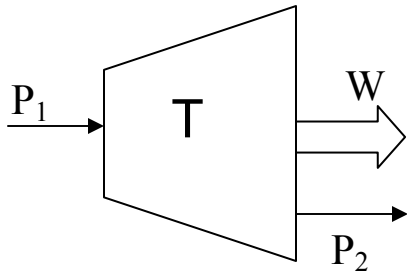
If temperature in CV is not uniform T_j corresponds to the temperature at different points on the control surface where heat is transferred

For steady-state, one inlet and one outlet, isothermal CV

$$0 = \frac{1}{\dot{m}} \left(\frac{\dot{Q}_{CV}}{T} \right) + s_{in} - s_{out} + \frac{\dot{S}_{gen}}{\dot{m}}$$

Isentropic efficiencies of Turbines and Compressors

Recall, for a turbine First Law (steady-state, neglecting KE and PE effects and heat losses) yields

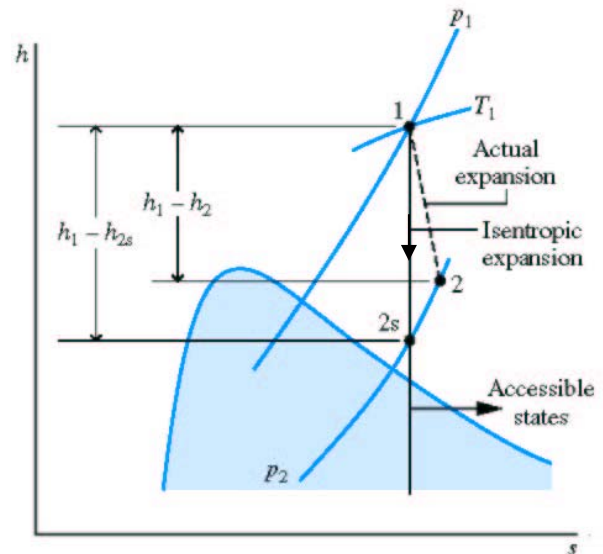
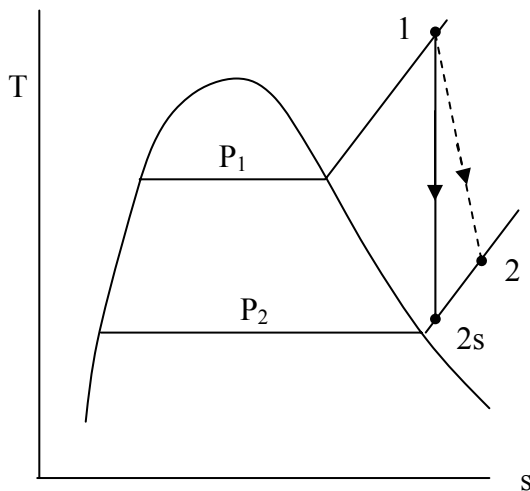


Expansion ($P_2 < P_1$)

$$\frac{\dot{W}_{CV}}{\dot{m}} = h_1 - h_2 > 0 \quad (W_{out})$$

An entropy balance yields $s_2 - s_1 = \frac{\dot{S}_{gen}}{\dot{m}} \geq 0$

For an actual turbine, irreversibilities are present, so accessible states are such that $s_2 > s_1$



The state labeled 2s on the T-s and h-s diagrams would be attained only in the limit of no irreversibilities, i.e., internally reversible expansion ($\dot{S}_{gen} = 0$) and thus $s_2 = s_1$

The maximum theoretical amount of turbine work output is obtained for an isentropic expansion

$$\frac{\dot{W}_{CV}}{\dot{m}} = h_1 - h_{2s}$$

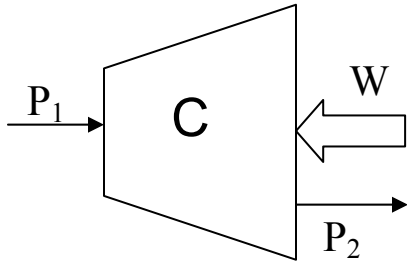
Since $h_1 - h_2 < h_1 - h_{2s}$ the actual work produced is less than the ideal isentropic turbine produces

The difference is gauged by the **isentropic turbine efficiency** defined by

$$\eta_t = \frac{\dot{W}_{CV} / \dot{m}}{(\dot{W}_{CV} / \dot{m})_s} = \frac{h_1 - h_2}{h_1 - h_{2s}}$$

Note, $\eta_t < 1$

Recall, for a compressor First Law (steady-state, neglecting KE and PE effects and heat losses) yields



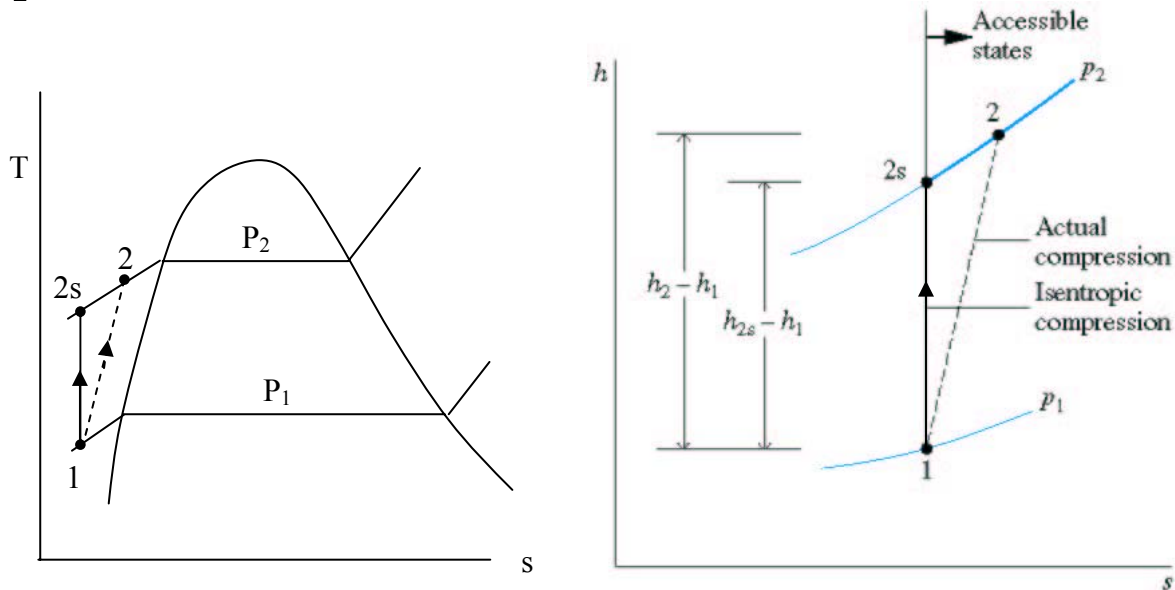
Compression ($P_2 > P_1$)

$$\frac{\dot{W}_{CV}}{\dot{m}} = h_1 - h_2 < 0 \quad (W_{in})$$

An entropy balance yields

$$s_2 - s_1 = \frac{\dot{S}_{gen}}{\dot{m}} \geq 0$$

For an actual compressor irreversibilities are always present so $s_2 > s_1$



The state labeled 2s on the T-s and h-s diagrams would be attained only in the limit of no irreversibilities, i.e., internally reversible compression where $\dot{S}_{gen} = 0$ and thus $s_2 = s_1$

The minimum theoretical amount of compressor work required corresponds to isentropic compression

$$\frac{\dot{W}_{CV}}{\dot{m}} = -(h_{2s} - h_1)$$

Since $h_2 - h_1 > h_{2s} - h_1$ the actual work input is more than the ideal isentropic compressor requires

The difference is gauged by the **isentropic compressor efficiency** defined by

$$\eta_c = \frac{(\dot{W}_{CV} / \dot{m})_s}{\dot{W}_{CV} / \dot{m}} = \frac{h_1 - h_{2s}}{h_1 - h_2}$$

Note, $\eta_c < 1$

Internally Reversible Steady-State Flow Work

For a single inlet and exit (1-inlet, 2-exit) CV at steady-state neglecting KE and PE effects conservation of energy

$$\frac{\dot{W}_{CV}}{\dot{m}} = \frac{\dot{Q}_{CV}}{\dot{m}} + (h_1 - h_2) + \left(\frac{V_1^2 - V_2^2}{2} \right) + g(z_1 - z_2)$$

For an internally reversible process $\dot{Q} / \dot{m} = \int T ds$

$$\frac{\dot{W}_{CV}}{\dot{m}} = \int_1^2 T ds + (h_1 - h_2)$$

Recall: $T ds = dh - v dp \rightarrow \int_1^2 T ds = (h_2 - h_1) - \int_1^2 v dp$

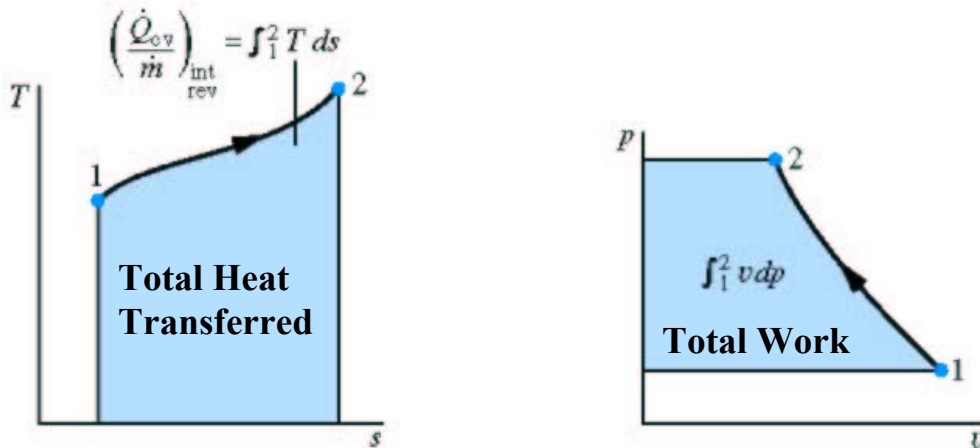
$$\frac{\dot{W}_{CV}}{\dot{m}} = (h_2 - h_1) - \int_1^2 v dP + (h_1 - h_2)$$

For pumps, turbines, compressors when $\Delta KE = \Delta PE = 0$

$$\left(\frac{\dot{W}_{CV}}{\dot{m}} \right)_{\text{int rev}} = -\int_1^2 v dP$$

Pumps and compressors $dP > 0 \rightarrow$ work done on system

Turbines $dP < 0 \rightarrow$ work done by system



Liquids – liquids are incompressible, so $v_1 = v_2 = v$

$$\left(\frac{\dot{W}_{CV}}{\dot{m}} \right)_{int rev} = -\int_1^2 v dP = -v(P_2 - P_1)$$

Gases - when each unit of gas through the CV undergoes a polytropic process $Pv^n = const$

$$\left(\frac{\dot{W}_{CV}}{\dot{m}} \right)_{int rev} = -\int_1^2 v dP = -(const)^{\frac{1}{n}} \int_1^2 \frac{dP}{P^{\frac{1}{n}}}$$

For the special case of an ideal gas where $Pv = RT$

$$\left(\frac{\dot{W}_{CV}}{\dot{m}} \right)_{int rev} = -\frac{nRT_1}{n-1} \left(\frac{T_2}{T_1} - 1 \right) \quad n \neq 1,$$

recall, for polytropic process $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}}$ so

$$\left(\frac{\dot{W}_{CV}}{\dot{m}}\right)_{\text{int rev}} = -\frac{nRT_1}{n-1} \left(\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1 \right) \quad n \neq 1$$

Recall: If the process is internally reversible and adiabatic (isentropic) for constant c_p and $c_v \rightarrow Pv^k = \text{const}$

Substitute $n = k$ in above equations to get work per unit mass for isentropic process (implies $k = \text{const} \neq f(T)$)

For the case of $n=1$: $P_1v_1 = P_2v_2 \rightarrow T_1 = T_2$ (isothermal)
 $\int v dP$ gives:

$$\left(\frac{\dot{W}_{CV}}{\dot{m}}\right)_{\text{int rev}} = -RT \ln(P_2/P_1) \quad n=1$$