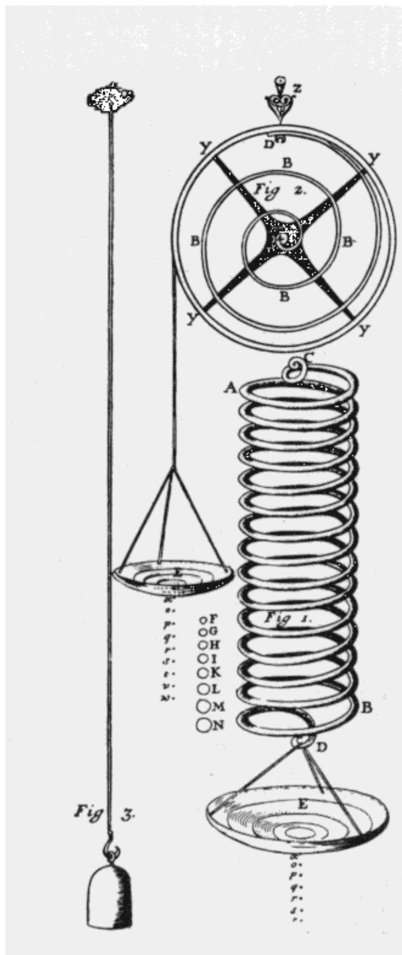
DRAFT

Lecture Notes

Introduction to

MECHANICS of MATERIALS

Fundamentals of Inelastic Analysis



©VICTOR E. SAOUMA

Dept. of Civil Environmental and Architectural Engineering
University of Colorado, Boulder, CO 80309-0428

PREFACE

One of the most fundamental question that an Engineer has to ask him/herself is what is how does it deform, and when does it break. Ultimately, it its the answer to those two questions which would provide us with not only a proper safety assesment of a structure, but also how to properly design it. Ironically, botht he ACI and the AISC codes are based on limit state design, yet practically all design analyses are linear and elastic. On the other hand, the Engineer is often confronted with the task of determining the ultimate load carying capacity of a structure or to assess its progressive degradation (in the ontect of a forensic study, or the rehabilitation, or life extension of an existing structure). In those particular situations, the Engineer should be capable of going beyond the simple linear elastic analysis investigation.

Whereas the Finite Element Method has proved to be a very powerful investigative tool, its proper (and correct) usage in the context of non-linear analysis requires a solid and thorough understanding of the fundamentals of Mechanics. Unfortunately, this is often forgotten as students rush into ever more advanced FEM classes without a proper solid background in Mechanics.

In the humble opinion of the author, this understanding is best achieved in two stages. First, the student should be exposed to the basic principles of Continuum Mechanics. Detailed coverage of (3D) Stress, Strain, General Principles, and Constitutive Relations is essential. In here we shall go from the general to the specific.

Then material models should be studied. Plasticity will provide a framework from where to determine the ultimate strength, Fracture Mechanics a framework to check both strength and stability of flawed structures, and finally Damage Mechanics will provide a framework to assess stiffness degradation under increased load.

The course was originally offered to second year undergraduate Materials Science students at the Swiss Institute of Technology during the author's sabbatical leave in French. The notes were developed with the following objectives in mind. First they must be complete and rigorous. At any time, a student should be able to trace back the development of an equation. Furthermore, by going through all the derivations, the student would understand the limitations and assumptions behind every model. Finally, the rigor adopted in the coverage of the subject should serve as an example to the students of the rigor expected from them in solving other scientific or engineering problems. This last aspect is often forgotten.

The notes are broken down into a very hierarchical format. Each concept is broken down into a small section (a byte). This should not only facilitate comprehension, but also dialogue among the students or with the instructor.

Whenever necessary, Mathematical preliminaries are introduced to make sure that the student is equipped with the appropriate tools. Illustrative problems are introduced whenever possible, and last but not least problem set using *Mathematica* is given in the Appendix.

The author has no illusion as to the completeness or exactness of all these set of notes. They were entirely developed during a single academic year, and hence could greatly benefit from a thorough review. As such, corrections, criticisms and comments are welcome.

Victor E. Saouma
Boulder, January 2002

Contents

List of Figures

Part I

CONTINUUM MECHANICS

Chapter 1

MATHEMATICAL PRELIMINARIES; Part I Vectors and Tensors

1 Physical laws should be independent of the position and orientation of the observer. For this reason, physical laws are **vector equations** or **tensor equations**, since both vectors and tensors transform from one coordinate system to another in such a way that if the law holds in one coordinate system, it holds in any other coordinate system.

1.1 Indicial Notation

2 Whereas the Engineering notation may be the simplest and most intuitive one, it often leads to long and repetitive equations. Alternatively, the tensor form will lead to shorter and more compact forms.

3 While working on general relativity, Einstein got tired of writing the summation symbol with its range of summation below and above (such as $\sum_{i=1}^{n=3} a_{ij} b_i$) and noted that most of the time the upper range (n) was equal to the dimension of space (3 for us, 4 for him), and that when the summation involved a product of two terms, the summation was over a repeated index (i in our example). Hence, he decided that there is no need to include the summation sign $\sum$ if there was repeated indices (i), and thus any repeated index is a **dummy index** and is summed over the range 1 to 3. An index that is not repeated is called **free index** and assumed to take a value from 1 to 3.

4 Hence, this so called **indicial notation** is also referred to **Einstein's notation**.

5 The following rules define indicial notation:

1. If there is one letter index, that index goes from i to n (range of the tensor). For instance:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor = \left\{ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \right\} \qquad i = 1, 3 \tag{1.1}$$

   assuming that $n = 3$.

2. A repeated index will take on all the values of its range, and the resulting tensors summed. For instance:

$$a_{1i} x_i = a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \tag{1.2}$$

3. Tensor's order:

- First order tensor (such as force) has only one free index:

$$a_i = a^i = \lfloor\ a_1 \quad a_2 \quad a_3\ \rfloor \tag{1.3}$$

  other first order tensors $a_{ij}b_j$, F_{ikk}, $\varepsilon_{ijk}u_jv_k$

- Second order tensor (such as stress or strain) will have two free indeces.

$$D_{ij} = \begin{bmatrix} D_{11} & D_{22} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{1.4}$$

  other examples A_{ijip}, $\delta_{ij}u_kv_k$.

- A fourth order tensor (such as Elastic constants) will have four free indeces.

4. Derivatives of tensor with respect to x_i is written as $,i$. For example:

$$\frac{\partial \Phi}{\partial x_i} = \Phi_{,i} \quad \frac{\partial v_i}{\partial x_i} = v_{i,i} \quad \frac{\partial v_i}{\partial x_j} = v_{i,j} \quad \frac{\partial T_{i,j}}{\partial x_k} = T_{i,j,k} \tag{1.5}$$

6 Usefulness of the indicial notation is in presenting systems of equations in compact form. For instance:

$$x_i = c_{ij}z_j \tag{1.6}$$

this simple compacted equation, when expanded would yield:

$$\begin{aligned} x_1 &= c_{11}z_1 + c_{12}z_2 + c_{13}z_3 \\ x_2 &= c_{21}z_1 + c_{22}z_2 + c_{23}z_3 \\ x_3 &= c_{31}z_1 + c_{32}z_2 + c_{33}z_3 \end{aligned} \tag{1.7}$$

Similarly:

$$A_{ij} = B_{ip}C_{jq}D_{pq} \tag{1.8}$$

$$\begin{aligned} A_{11} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{12} &= B_{11}C_{11}D_{11} + B_{11}C_{12}D_{12} + B_{12}C_{11}D_{21} + B_{12}C_{12}D_{22} \\ A_{21} &= B_{21}C_{11}D_{11} + B_{21}C_{12}D_{12} + B_{22}C_{11}D_{21} + B_{22}C_{12}D_{22} \\ A_{22} &= B_{21}C_{21}D_{11} + B_{21}C_{22}D_{12} + B_{22}C_{21}D_{21} + B_{22}C_{22}D_{22} \end{aligned} \tag{1.9}$$

7 Using indicial notation, we may rewrite the definition of the dot product

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a_ib_i} \tag{1.10}$$

and of the cross product

$$\boxed{\mathbf{a} \times \mathbf{b} = \varepsilon_{pqr}a_qb_r\mathbf{e}_p} \tag{1.11}$$

we note that in the second equation, there is one free index p thus there are three equations, there are two repeated (dummy) indices q and r, thus each equation has nine terms.

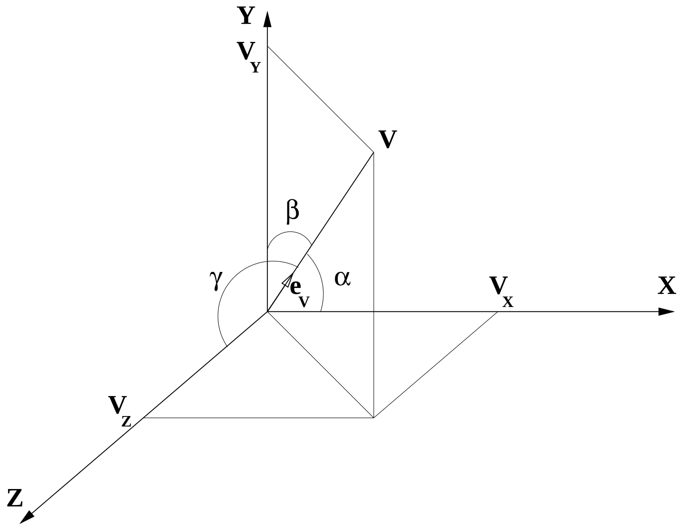


Figure 1.1: Direction Cosines

1.2 Vectors

8 A vector is a directed line segment which can denote a variety of quantities, such as position of point with respect to another (**position vector**), a force, or a traction.

9 A vector may be defined with respect to a particular coordinate system by specifying the **components** of the vector in that system. The choice of the coordinate system is arbitrary, but some are more suitable than others (axes corresponding to the major direction of the object being analyzed).

10 The **rectangular Cartesian coordinate system** is the most often used one (others are the cylindrical, spherical or curvilinear systems). The rectangular system is often represented by three mutually perpendicular axes $Oxyz$, with corresponding **unit vector triad** $\mathbf{i}, \mathbf{j}, \mathbf{k}$ (or $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) such that:

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}; \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}; \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}; \tag{1.12-a}$$

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \tag{1.12-b}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \tag{1.12-c}$$

Such a set of base vectors constitutes an **orthonormal basis**.

11 An arbitrary vector $\mathbf{v}$ may be expressed by

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \tag{1.13}$$

where

$$v_x = \mathbf{v}\cdot\mathbf{i} = v\cos\alpha \tag{1.14-a}$$

$$v_y = \mathbf{v}\cdot\mathbf{j} = v\cos\beta \tag{1.14-b}$$

$$v_z = \mathbf{v}\cdot\mathbf{k} = v\cos\gamma \tag{1.14-c}$$

are the projections of $\mathbf{v}$ onto the coordinate axes, Fig. 1.1.

12 The unit vector in the direction of $\mathbf{v}$ is given by

$$\mathbf{e}_v = \frac{\mathbf{v}}{v} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k} \tag{1.15}$$

Since $\mathbf{v}$ is arbitrary, it follows that any unit vector will have **direction cosines** of that vector as its **Cartesian components**.

13 The length or more precisely the magnitude of the vector is denoted by $\| \mathbf{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

14 †We will denote the **contravariant components** of a vector by superscripts v^k, and its **covariant components** by subscripts v_k (the significance of those terms will be clarified in Sect. 1.2.2.1.

1.2.1 Operations

15 **Addition:** of two vectors $\mathbf{a} + \mathbf{b}$ is geometrically achieved by connecting the tail of the vector $\mathbf{b}$ with the head of $\mathbf{a}$, Fig. 1.2. Analytically the sum vector will have components $\lfloor\ a_1 + b_1 \quad a_2 + b_2 \quad a_3 + b_3\ \rfloor$.

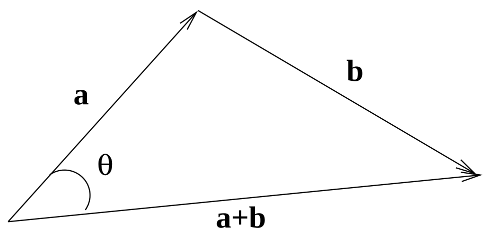


Figure 1.2: Vector Addition

16 **Scalar multiplication:** $\alpha \mathbf{a}$ will scale the vector into a new one with components $\lfloor\ \alpha a_1 \quad \alpha a_2 \quad \alpha a_3\ \rfloor$.

17 **Vector multiplications** of $\mathbf{a}$ and $\mathbf{b}$ comes in two major varieties:

Dot Product (or scalar product) is a scalar quantity which relates not only to the lengths of the vector, but also to the angle between them.

$$\boxed{\mathbf{a}\cdot\mathbf{b} \equiv \| \mathbf{a} \| \| \mathbf{b} \| \cos\theta(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{3} a_i b_i} \tag{1.16}$$

where $\cos\theta(\mathbf{a},\mathbf{b})$ is the cosine of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. The dot product measures the relative orientation between two vectors.

The dot product is both *commutative*

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \tag{1.17}$$

and *distributive*

$$\alpha\mathbf{a}\cdot(\beta\mathbf{b} + \gamma\mathbf{c}) = \alpha\beta(\mathbf{a}\cdot\mathbf{b}) + \alpha\gamma(\mathbf{a}\cdot\mathbf{c}) \tag{1.18}$$

The dot product of $\mathbf{a}$ with a unit vector $\mathbf{n}$ gives the projection of $\mathbf{a}$ in the direction of $\mathbf{n}$.

The dot product of base vectors gives rise to the definition of the **Kronecker delta** defined as

$$\boxed{\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}} \tag{1.19}$$

where

$$\boxed{\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}} \tag{1.20}$$

Cross Product (or vector product) $\mathbf{c}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the vector

$$\boxed{\mathbf{c} = \mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3} \tag{1.21}$$

which can be remembered from the determinant expansion of

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{1.22}$$

and is equal to the area of the parallelogram described by $\mathbf{a}$ and $\mathbf{b}$, Fig. 1.3.

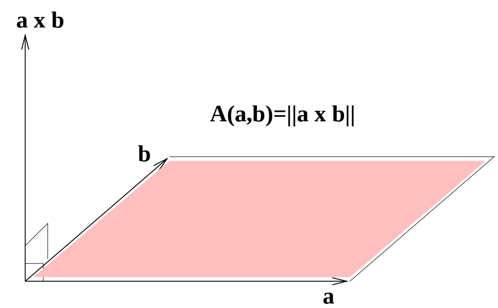


Figure 1.3: Cross Product of Two Vectors

$$\boxed{A(\mathbf{a},\mathbf{b}) = \| \mathbf{a}\times\mathbf{b} \|} \tag{1.23}$$

The cross product is not commutative, but satisfies the condition of **skew symmetry**

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{1.24}$$

The cross product is distributive

$$\alpha\mathbf{a}\times(\beta\mathbf{b} + \gamma\mathbf{c}) = \alpha\beta(\mathbf{a}\times\mathbf{b}) + \alpha\gamma(\mathbf{a}\times\mathbf{c}) \tag{1.25}$$

18 †Other forms of vector multiplication

†**Triple Scalar Product:** of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is designated by $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and it corresponds to the (scalar) volume defined by the three vectors, Fig. 1.4.

$$V(\mathbf{a},\mathbf{b},\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \tag{1.26}$$

$$= \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{1.27}$$

The triple scalar product of base vectors represents a fundamental operation

$$\boxed{(\mathbf{e}_i\times\mathbf{e}_j)\cdot\mathbf{e}_k = \varepsilon_{ijk} \equiv \begin{cases} 1 & \text{if } (i,j,k) \text{ are in cyclic order} \\ 0 & \text{if any of } (i,j,k) \text{ are equal} \\ -1 & \text{if } (i,j,k) \text{ are in acyclic order} \end{cases}} \tag{1.28}$$

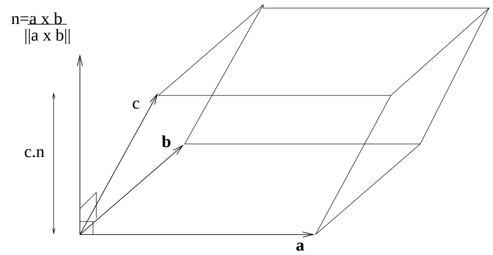


Figure 1.4: Cross Product of Two Vectors

The scalars ε_{ijk} is the **permutation tensor**. A cyclic permutation of 1,2,3 is $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$, an acyclic one would be $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$. Using this notation, we can rewrite

$$\boxed{\mathbf{c} = \mathbf{a}\times\mathbf{b} \Rightarrow c_i = \varepsilon_{ijk} a_j b_k} \tag{1.29}$$

Vector Triple Product is a cross product of two vectors, one of which is itself a cross product.

$$\boxed{\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = \mathbf{d}} \tag{1.30}$$

and the product vector **d** lies in the plane of **b** and **c**.

1.2.2 Coordinate Transformation

1.2.2.1 † General Tensors

19 Let us consider two arbitrary coordinate systems $\mathbf{b}(x_1, x_2, x_3)$ and $\overline{\mathbf{b}}(\overline{x}_1, \overline{x}_2 \overline{x}_3)$, Fig. 1.5, in a three-dimensional Euclidian space.

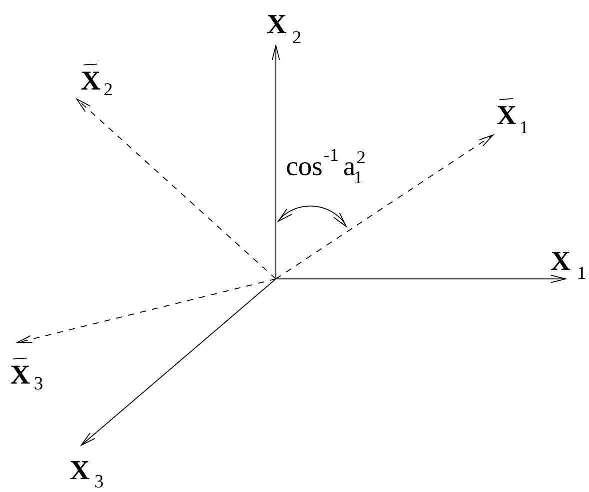


Figure 1.5: Coordinate Transformation

20 We define a set of **coordinate transformation equations** as

$$\overline{x}^i = \overline{x}^i(x^1, x^2, x^3) \tag{1.31}$$

which assigns to any point (x^1, x^2, x^3) in base bb a new set of coordinates $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ in the $\overline{\mathbf{b}}$ system.

21 The transformation relating the two sets of variables (coordinates in this case) are assumed to be single-valued, continuous, differential functions, and the must have the determinant[1] of its **Jacobian**

$$J = \begin{vmatrix} \frac{\partial \overline{x}^1}{\partial x^1} & \frac{\partial \overline{x}^1}{\partial x^2} & \frac{\partial \overline{x}^1}{\partial x^3} \\ \frac{\partial \overline{x}^2}{\partial x^1} & \frac{\partial \overline{x}^2}{\partial x^2} & \frac{\partial \overline{x}^2}{\partial x^3} \\ \frac{\partial \overline{x}^3}{\partial x^1} & \frac{\partial \overline{x}^3}{\partial x^2} & \frac{\partial \overline{x}^3}{\partial x^3} \end{vmatrix} \neq 0 \tag{1.32}$$

different from zero (the superscript is a label and not an exponent).

22 It is important to note that so far, the coordinate systems are completely general and may be Cartesian, curvilinear, spherical or cylindrical.

1.2.2.1.1 ‡Contravariant Transformation

23 Expanding on the definitions of the two bases $\mathbf{b}_j(x_1, x_2, x_3)$ and $\overline{\mathbf{b}}_j(\overline{x}_1, \overline{x}_2\overline{x}_3)$, Fig. 1.5. Each unit vector in one basis must be a linear combination of the vectors of the other basis

$$\overline{\mathbf{b}}_j = a_j^p \mathbf{b}_p \quad \text{and} \quad \mathbf{b}_k = b_q^k \overline{\mathbf{b}}_q \tag{1.33}$$

(summed on p and q respectively) where a_j^p (subscript new, superscript old) and b_q^k are the coefficients for the forward and backward changes respectively from $\overline{\mathbf{b}}$ to $\mathbf{b}$ respectively. Explicitly

$$\begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} = \begin{bmatrix} b_1^1 & b_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{e}}_1 \\ \overline{\mathbf{e}}_2 \\ \overline{\mathbf{e}}_3 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \overline{\mathbf{e}}_1 \\ \overline{\mathbf{e}}_2 \\ \overline{\mathbf{e}}_3 \end{Bmatrix} = \begin{bmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{bmatrix} \begin{Bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{Bmatrix} \tag{1.34}$$

24 But the vector representation in both systems must be the same

$$\mathbf{v} = \overline{v}^q \overline{\mathbf{b}}_q = v^k \mathbf{b}_k = v^k (b_k^q \overline{\mathbf{b}}_q) \Rightarrow (\overline{v}^q - v^k b_k^q)\overline{\mathbf{b}}_q = \mathbf{0} \tag{1.35}$$

since the base vectors $\overline{\mathbf{b}}_q$ are linearly independent, the coefficients of $\overline{\mathbf{b}}_q$ must all be zero hence

$$\boxed{\overline{v}^q = b_k^q v^k \quad \text{and inversely} \quad v^p = a_j^p \overline{v}^j} \tag{1.36}$$

showing that the forward change from components v^k to $\overline{v}^q$ used the coefficients b_k^q of the backward change from base $\overline{\mathbf{b}}_q$ to the original $\mathbf{b}_k$. This is why these components are called **contravariant**.

25 Generalizing, a **Contravariant Tensor of order one** (recognized by the use of the superscript) transforms a set of quantities r^k associated with point P in x^k through a coordinate transformation into a new set $\overline{r}^q$ associated with $\overline{x}^q$

$$\boxed{\overline{r}^q = \underbrace{\frac{\partial \overline{x}^q}{\partial x^k}}_{b_k^q} r^k} \tag{1.37}$$

26 By extension, the **Contravariant tensors of order two** requires the tensor components to obey the following transformation law

$$\boxed{\overline{r}^{ij} = \frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^j}{\partial x^s} r^{rs}} \tag{1.38}$$

[1]You may want to review your Calculus III, (Multivariable Calculus) notes.

1.2.2.1.2 Covariant Transformation

27 Similarly to Eq. 1.36, a **covariant component transformation** (recognized by subscript) will be defined as

$$\boxed{\overline{v}_j = a_j^p v_p \text{ and inversely } v_k = b_q^k \overline{v}_q} \tag{1.39}$$

We note that contrarily to the contravariant transformation, the covariant transformation uses the same transformation coefficients as the ones for the base vectors.

28 † Finally transformation of tensors of order one and two is accomplished through

$$\overline{r}_q = \frac{\partial x^k}{\partial \overline{x}^q} r_k \tag{1.40}$$

$$\overline{r}_{ij} = \frac{\partial x^r}{\partial \overline{x}^i} \frac{\partial x^s}{\partial \overline{x}^j} r_{rs} \tag{1.41}$$

1.2.2.2 Cartesian Coordinate System

29 If we consider two different sets of cartesian orthonormal coordinate systems $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \overline{\mathbf{e}}_3\}$, any vector $\mathbf{v}$ can be expressed in one system or the other

$$\mathbf{v} = v_j \mathbf{e}_j = \overline{v}_j \overline{\mathbf{e}}_j \tag{1.42}$$

30 To determine the relationship between the two sets of components, we consider the dot product of $\mathbf{v}$ with one (any) of the base vectors

$$\overline{\mathbf{e}}_i \cdot \mathbf{v} = \overline{v}_i = v_j (\overline{\mathbf{e}}_i \cdot \mathbf{e}_j) \tag{1.43}$$

31 We can thus define the nine scalar values

$$\boxed{a_i^j \equiv \overline{\mathbf{e}}_i \cdot \mathbf{e}_j = \cos(\overline{x}_i, x_j)} \tag{1.44}$$

which arise from the dot products of base vectors as the **direction cosines**. (Since we have an orthonormal system, those values are nothing else than the cosines of the angles between the nine pairing of base vectors.)

32 Thus, one set of vector components can be expressed in terms of the other through a **covariant transformation** similar to the one of Eq. 1.39.

$$\overline{v}_j = a_j^p v_p \tag{1.45}$$

$$v_k = b_q^k \overline{v}_q \tag{1.46}$$

we note that the free index in the first and second equations appear on the upper and lower index respectively.

33 †Because of the orthogonality of the unit vector we have $a_p^s a_q^s = \delta_{pq}$ and $a_r^m a_r^n = \delta_{mn}$.

34 As a further illustration of the above derivation, let us consider the transformation of a vector $\mathbf{V}$ from $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ coordinate system to $(\mathbf{x}, \mathbf{y}, \mathbf{z})$, Fig. 1.6:

35 Eq. 1.45 would then result in

$$V_j = a_j^K \overline{V}_K \quad \text{or} \quad \text{or} \quad \left\{ \begin{array}{lcl} V_1 & = & a_1^1 \overline{V}_1 + a_1^2 \overline{V}_2 + a_1^3 \overline{V}_3 \\ V_2 & = & a_2^1 \overline{V}_2 + a_1^2 \overline{V}_2 + a_2^3 \overline{V}_3 \\ V_3 & = & a_3^1 \overline{V}_3 + a_1^2 \overline{V}_2 + a_3^3 \overline{V}_3 \end{array} \right. \tag{1.47}$$

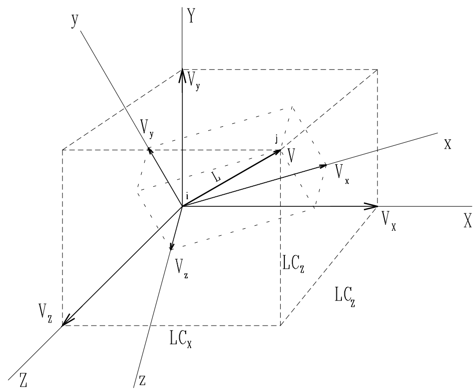


Figure 1.6: Arbitrary 3D Vector Transformation

or

$$\left\{ \begin{array}{c} V_x \\ V_y \\ V_z \end{array} \right\} = \left[\begin{array}{ccc} a_x^X & a_x^Y & a_x^Z \\ a_y^X & a_y^Y & a_y^Z \\ a_z^X & a_z^Y & a_z^Z \end{array} \right] \left\{ \begin{array}{c} V_X \\ V_Y \\ V_Z \end{array} \right\} \tag{1.48}$$

and a_i^j is the direction cosine of axis i with respect to axis j

- $a_x^j = (a_x^X, a_x^Y, a_x^Z)$ direction cosines of $\mathbf{x}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $a_y^j = (a_y^X, a_y^Y, a_y^Z)$ direction cosines of $\mathbf{y}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$
- $a_z^j = (a_z^X, a_z^Y, a_z^Z)$ direction cosines of $\mathbf{z}$ with respect to $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$

36 Finally, for the 2D case and from Fig. 1.7, the transformation matrix is written as

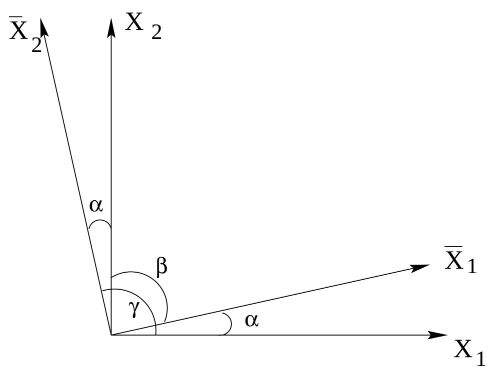


Figure 1.7: Rotation of Orthonormal Coordinate System

$$T = \left[\begin{array}{cc} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{array} \right] = \left[\begin{array}{cc} \cos\alpha & \cos\beta \\ \cos\gamma & \cos\alpha \end{array} \right] \tag{1.49}$$

but since $\gamma = \frac{\pi}{2} + \alpha$, and $\beta = \frac{\pi}{2} - \alpha$, then $\cos\gamma = -\sin\alpha$ and $\cos\beta = \sin\alpha$, thus the transformation matrix becomes

$$T = \left[\begin{array}{cc} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{array} \right] \tag{1.50}$$

1.3 Tensors

1.3.1 Definition

37 We now seek to generalize the concept of a vector by introducing the **tensor** (**T**), which essentially *exists to operate on vectors* **v** *to produce other vectors* (or on tensors to produce other tensors!). We designate this operation by **T·v** or simply **Tv**.

38 We hereby adopt the **dyadic** notation for tensors as **linear vector operators**

$$\mathbf{u} = \mathbf{T}\cdot\mathbf{v} \quad \text{or} \quad u_i = T_{ij}v_j \quad \text{or} \quad \left\{ \begin{array}{lcl} u_1 & = & T_{11}v_1 + T_{12}v_2 + T_{13}v_3 \\ u_2 & = & T_{21}v_1 + T_{22}v_2 + T_{23}v_3 \\ u_3 & = & T_{31}v_1 + T_{32}v_2 + T_{33}v_3 \end{array} \right. \quad \text{or} \quad \mathbf{u} = \mathbf{v}\cdot\mathbf{S} \quad \text{where} \quad \mathbf{S} = \mathbf{T}^T \tag{1.51}$$

39 † In general the vectors may be represented by either covariant or contravariant components v_j or v^j. Thus we can have different types of linear transformations

$$\begin{array}{lclclcl} u_i & = & T_{ij}v^j; & u^i & = & T^{ij}v_j \\ u_i & = & T_i^{\cdot j}v_j; & u^i & = & T^i_{\cdot j}v^j \end{array} \tag{1.52}$$

involving the **covariant components** T_{ij}, the **contravariant components** T^{ij} and the **mixed components** $T^i_{\cdot j}$ or $T_i^{\cdot j}$.

40 Whereas a tensor is essentially an operator on vectors (or other tensors), it is also a physical quantity, independent of any particular coordinate system yet specified most conveniently by referring to an appropriate system of coordinates.

41 Tensors frequently arise as physical entities whose components are the coefficients of a linear relationship between vectors.

42 A tensor is classified by the rank or order. A Tensor of order zero is specified in any coordinate system by one coordinate and is a scalar. A tensor of order one has three coordinate components in space, hence it is a vector. In general 3-D space the number of components of a tensor is 3^n where n is the order of the tensor.

43 A force and a stress are tensors of order 1 and 2 respectively.

1.3.2 Tensor Operations

44 **Sum:** The sum of two (second order) tensors is simply defined as:

$$\boxed{\mathbf{S}_{ij} = \mathbf{T}_{ij} + \mathbf{U}_{ij}} \tag{1.53}$$

45 **Multiplication by a Scalar:** The multiplication of a (second order) tensor by a scalar is defined by:

$$\boxed{\mathbf{S}_{ij} = \lambda\mathbf{T}_{ij}} \tag{1.54}$$

46 **Contraction:** In a contraction, we make two of the indeces equal (or in a mixed tensor, we make a subscript equal to the superscript), thus producing a tensor of order two less than that to which it is

applied. For example:

$$\begin{array}{rclrcl} T_{ij} & \rightarrow & T_{ii}; & 2 & \rightarrow & 0 \\ u_i v_j & \rightarrow & u_i v_i; & 2 & \rightarrow & 0 \\ A^{mr}_{..sn} & \rightarrow & A^{mr}_{..sm} = B^{r}_{.s}; & 4 & \rightarrow & 2 \\ E_{ij} a_k & \rightarrow & E_{ij} a_i = c_j; & 3 & \rightarrow & 1 \\ A^{mpr}_{qs} & \rightarrow & A^{mpr}_{qr} = B^{mp}_{q}; & 5 & \rightarrow & 3 \end{array} \quad (1.55)$$

47 **Outer Product:** The outer product of two tensors (not necessarily of the same type or order) is a set of tensor components obtained simply by writing the components of the two tensors beside each other with no repeated indices (that is by multiplying each component of one of the tensors by every component of the other). For example

$$a_i b_j = T_{ij} \quad (1.56\text{-a})$$
$$A^i B_j^{.k} = C^{i.k}.j \quad (1.56\text{-b})$$
$$v_i T_{jk} = S_{ijk} \quad (1.56\text{-c})$$

48 **Inner Product:** The inner product is obtained from an outer product by contraction involving one index from each tensor. For example

$$a_i b_j \rightarrow a_i b_i \quad (1.57\text{-a})$$
$$a_i E_{jk} \rightarrow a_i E_{ik} = f_k \quad (1.57\text{-b})$$
$$E_{ij} F_{km} \rightarrow E_{ij} F_{jm} = G_{im} \quad (1.57\text{-c})$$
$$A^i B_i^{.k} \rightarrow A^i B_i^{.k} = D^k \quad (1.57\text{-d})$$

49 **Scalar Product:** The scalar product of two tensors is defined as

$$\boxed{\mathbf{T} : \mathbf{U} = T_{ij} U_{ij}} \quad (1.58)$$

in any rectangular system.

50 The following **inner-product** axioms are satisfied:

$$\mathbf{T} : \mathbf{U} = \mathbf{U} : \mathbf{T} \quad (1.59\text{-a})$$
$$\mathbf{T} : (\mathbf{U} + \mathbf{V}) = \mathbf{T} : \mathbf{U} + \mathbf{T} : \mathbf{V} \quad (1.59\text{-b})$$
$$\alpha(\mathbf{T} : \mathbf{U}) = (\alpha\mathbf{T}) : \mathbf{U} = \mathbf{T} : (\alpha\mathbf{U}) \quad (1.59\text{-c})$$
$$\mathbf{T} : \mathbf{T} > 0 \text{ unless } \mathbf{T} = \mathbf{0} \quad (1.59\text{-d})$$

51 **Product of Two Second-Order Tensors:** The product of two tensors is defined as

$$\boxed{\mathbf{P} = \mathbf{T}{\cdot}\mathbf{U}; \qquad P_{ij} = T_{ik} U_{kj}} \quad (1.60)$$

in any rectangular system.

52 The following axioms hold

$$(\mathbf{T}{\cdot}\mathbf{U}){\cdot}\mathbf{R} = \mathbf{T}{\cdot}(\mathbf{U}{\cdot}\mathbf{R}) \quad (1.61\text{-a})$$

$$\begin{aligned} \mathbf{T}\cdot(\mathbf{R}+\mathbf{U}) &= \mathbf{T}\cdot\mathbf{R}+\mathbf{t}\cdot\mathbf{U} && (1.61\text{-b}) \\ (\mathbf{R}+\mathbf{U})\cdot\mathbf{T} &= \mathbf{R}\cdot\mathbf{T}+\mathbf{U}\cdot\mathbf{T} && (1.61\text{-c}) \\ \alpha(\mathbf{T}\cdot\mathbf{U}) &= (\alpha\mathbf{T})\cdot\mathbf{U} = \mathbf{T}\cdot(\alpha\mathbf{U}) && (1.61\text{-d}) \\ \mathbf{1T} &= \mathbf{T}\cdot\mathbf{1} = \mathbf{T} && (1.61\text{-e}) \end{aligned}$$

Note again that some authors omit the dot.
Finally, the operation is not commutative

53 **Trace:** The trace of a second-order tensor, denoted tr $\mathbf{T}$ is a scalar invariant function of the tensor and is defined as

$$\boxed{\text{tr } \mathbf{T} \equiv T_{ii}} \tag{1.62}$$

Thus it is equal to the sum of the diagonal elements in a matrix.

54 **Inverse Tensor:** An inverse tensor is simply defined as follows

$$\boxed{\mathbf{T}^{-1}(\mathbf{Tv}) = \mathbf{v} \quad \text{and} \quad \mathbf{T}(\mathbf{T}^{-1}\mathbf{v}) = \mathbf{v}} \tag{1.63}$$

alternatively $\mathbf{T}^{-1}\mathbf{T} = \mathbf{TT}^{-1} = \mathbf{I}$, or $T^{-1}_{ik}T_{kj} = \delta_{ij}$ and $T_{ik}T^{-1}_{kj} = \delta_{ij}$

1.3.3 Rotation of Axes

55 The rule for changing second order tensor components under rotation of axes goes as follow:

$$\begin{aligned} \overline{u}_i &= a_i^j u_j && \text{From Eq. 1.45} \\ &= a_i^j T_{jq} v_q && \text{From Eq. 1.51} \\ &= a_i^j T_{jq} a_p^q \overline{v}_p && \text{From Eq. 1.45} \end{aligned} \tag{1.64}$$

But we also have $\overline{u}_i = \overline{T}_{ip}\overline{v}_p$ (again from Eq. 1.51) in the barred system, equating these two expressions we obtain

$$\overline{T}_{ip} - (a_i^j a_p^q T_{jq})\overline{v}_p = 0 \tag{1.65}$$

hence

$$\boxed{\begin{aligned} \overline{T}_{ip} &= a_i^j a_p^q T_{jq} \quad \text{in Matrix Form} \quad [\overline{T}] = [A]^T[T][A] && (1.66) \\ T_{jq} &= a_i^j a_p^q \overline{T}_{ip} \quad \text{in Matrix Form} \quad [T] = [A][\overline{T}][A]^T && (1.67) \end{aligned}}$$

By extension, higher order tensors can be similarly transformed from one coordinate system to another.

56 If we consider the 2D case, From Eq. 1.50

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.68-a}$$

$$T = \begin{bmatrix} T_{xx} & T_{xy} & 0 \\ T_{xy} & T_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.68-b}$$

$$\overline{T} = A^T T A = \begin{bmatrix} \overline{T}_{xx} & \overline{T}_{xy} & 0 \\ \overline{T}_{xy} & \overline{T}_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.68-c}$$

$$= \begin{bmatrix} \cos^2\alpha T_{xx} + \sin^2\alpha T_{yy} + \sin 2\alpha T_{xy} & \frac{1}{2}(-\sin 2\alpha T_{xx} + \sin 2\alpha T_{yy} + 2\cos 2\alpha T_{xy} & 0 \\ \frac{1}{2}(-\sin 2\alpha T_{xx} + \sin 2\alpha T_{yy} + 2\cos 2\alpha T_{xy} & \sin^2\alpha T_{xx} + \cos\alpha(\cos\alpha T_{yy} - 2\sin\alpha T_{xy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.68-d}$$

alternatively, using $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, this last equation can be rewritten as

$$\boxed{\begin{Bmatrix} \overline{T}_{xx} \\ \overline{T}_{yy} \\ \overline{T}_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} T_{xx} \\ T_{yy} \\ T_{xy} \end{Bmatrix}} \tag{1.69}$$

57 Note the analogy with the transformation of a stiffness matrix from one coordinate system to another.

1.3.4 Principal Values and Directions of Symmetric Second Order Tensors

58 Since the two fundamental tensors in continuum mechanics are of the second order and symmetric (stress and strain), we examine some important properties of these tensors.

59 For every symmetric tensor T_{ij} defined at some point in space, there is associated with each direction (specified by unit normal n_j) at that point, a vector given by the inner product

$$v_i = T_{ij}n_j \tag{1.70}$$

If the direction is one for which v_i is **parallel** to n_i, the inner product may be expressed as

$$T_{ij}n_j = \lambda n_i \tag{1.71}$$

and the direction n_i is called **principal direction** of T_{ij}. Since $n_i = \delta_{ij}n_j$, this can be rewritten as

$$(T_{ij} - \lambda\delta_{ij})n_j = 0 \tag{1.72}$$

which represents a system of three equations for the four unknowns n_i and λ.

$$\begin{aligned} (T_{11} - \lambda)n_1 + T_{12}n_2 + T_{13}n_3 &= 0 \\ T_{21}n_1 + (T_{22} - \lambda)n_2 + T_{23}n_3 &= 0 \\ T_{31}n_1 + T_{32}n_2 + (T_{33} - \lambda)n_3 &= 0 \end{aligned} \tag{1.73}$$

To have a non-trivial slution ($n_i = 0$) the determinant of the coefficients must be zero,

$$\boxed{|T_{ij} - \lambda\delta_{ij}| = 0} \tag{1.74}$$

60 Expansion of this determinant leads to the following **characteristic equation**

$$\boxed{\lambda^3 - I_T\lambda^2 + II_T\lambda - III_T = 0} \tag{1.75}$$

the roots are called the **principal values** T_{ij} and

$$I_T = T_{ij} = \text{tr } T_{ij} \tag{1.76}$$

$$II_T = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ij}) \tag{1.77}$$

$$III_T = |T_{ij}| = \det T_{ij} \tag{1.78}$$

or first, second and third **invariants** respectively of the second order tensor T_{ij}.

61 It is customary to order those roots as $\lambda_1 > \lambda_2 > \lambda_3$

62 For a symmetric tensor with real components, the principal values are also real. If those values are distinct, the three principal directions are mutually orthogonal.

1.3.5 † Powers of Second Order Tensors; Hamilton-Cayley Equations

63 When expressed in term of the principal axes, the tensor array can be written in matrix form as

$$\mathcal{T} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 \\ 0 & \lambda_{(2)} & 0 \\ 0 & 0 & \lambda_{(3)} \end{bmatrix} \tag{1.79}$$

64 By direct matrix multiplication, the square of the tensor T_{ij} is given by the inner product $T_{ik}T_{kj}$, the cube as $T_{ik}T_{km}T_{mn}$. Therefore the nth power of T_{ij} can be written as

$$\mathcal{T}^n = \begin{bmatrix} \lambda^n_{(1)} & 0 & 0 \\ 0 & \lambda^n_{(2)} & 0 \\ 0 & 0 & \lambda^n_{(3)} \end{bmatrix} \tag{1.80}$$

Since each of the principal values satisfies Eq. 1.75 and because the diagonal matrix form of $\mathcal{T}$ given above, then the tensor itself will satisfy Eq. 1.75.

$$\boxed{\mathcal{T}^3 - I_T\mathcal{T}^2 + II_T\mathcal{T} - III_T\mathcal{I} = 0} \tag{1.81}$$

where $\mathcal{I}$ is the identity matrix. This equation is called the **Hamilton-Cayley equation**.

Chapter 2

KINETICS

Or How Forces are Transmitted

2.1 Force, Traction and Stress Vectors

1 There are two kinds of **forces** in continuum mechanics

Body forces: act on the elements of volume or mass inside the body, e.g. gravity, electromagnetic fields. $d\mathbf{F} = \rho\mathbf{b}d\text{Vol}$.

Surface forces: are contact forces acting on the free body at its bounding surface. Those will be defined in terms of **force per unit area.**

2 The surface force per unit area acting on an element dS is called **traction** or more accurately **stress vector**[1].

$$\int_S \mathbf{t}dS = \mathbf{i}\int_S t_x dS + \mathbf{j}\int_S t_y dS + \mathbf{k}\int_S t_z dS \tag{2.1}$$

3 The traction vectors on planes perpendicular to the coordinate axes are particularly useful. When the vectors acting at a point on three such mutually perpendicular planes is given, the **stress vector** at that point on any other arbitrarily inclined plane can be expressed in terms of the first set of tractions.

4 A **stress**, Fig 2.1 is a second order cartesian tensor, σ_{ij} where the 1st subscript (i) refers to the direction of outward facing normal, and the second one (j) to the direction of component force.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \left\{ \begin{array}{c} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{array} \right\} \tag{2.2}$$

5 In fact the nine rectangular components σ_{ij} of $\boldsymbol{\sigma}$ turn out to be the three sets of three vector components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, $(\sigma_{21}, \sigma_{22}, \sigma_{23})$, $(\sigma_{31}, \sigma_{32}, \sigma_{33})$ which correspond to the three tractions $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$ which are acting on the x_1, x_2 and x_3 faces (It should be noted that those tractions are not necesarily normal to the faces, and they can be decomposed into a normal and shear traction if need be). In other words, stresses are nothing else than the components of tractions (stress vector), Fig. 2.2.

6 The state of stress at a point cannot be specified entirely by a single vector with three components; it requires the second-order tensor with all nine components.

[1]Most authors limit the term traction to an actual bounding surface of a body, and use the term **stress vector** for an imaginary interior surface (even though the state of stress is a tensor and not a vector).

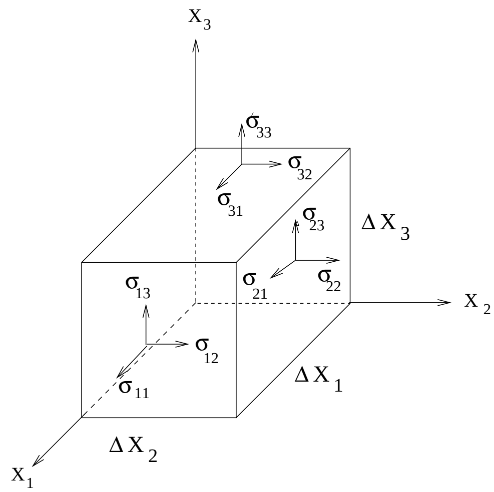


Figure 2.1: Stress Components on an Infinitesimal Element

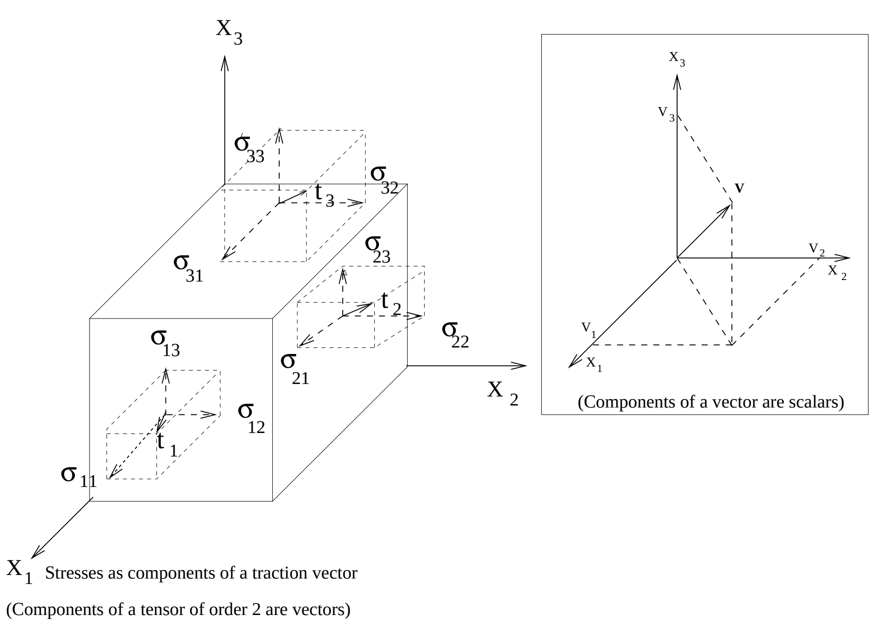


Figure 2.2: Stresses as Tensor Components

2.2 Traction on an Arbitrary Plane; Cauchy's Stress Tensor

7 Let us now consider the problem of determining the traction acting on the surface of an oblique plane (characterized by its normal $\mathbf{n}$) in terms of the known tractions normal to the three principal axis, $\mathbf{t}_1, \mathbf{t}_2$ and $\mathbf{t}_3$. This will be done through the so-called Cauchy's tetrahedron shown in Fig. 2.3.

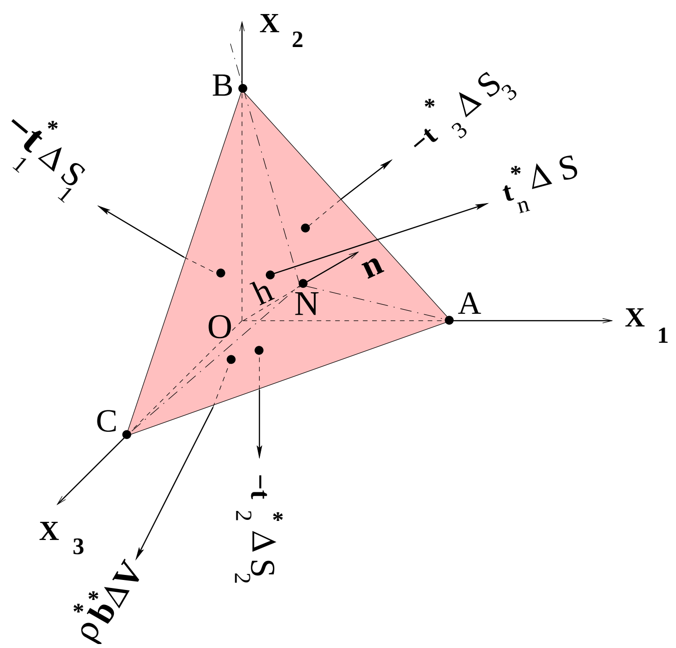


Figure 2.3: Cauchy's Tetrahedron

8 The components of the unit vector $\mathbf{n}$ are the direction cosines of its direction:

$$n_1 = \cos(\angle AON); \quad n_2 = \cos(\angle BON); \quad n_3 = \cos(\angle CON); \tag{2.3}$$

The altitude ON, of length h is a leg of the three right triangles ANO, BNO and CNO with hypothenuses OA, OB and OC. Hence

$$h = OAn_1 = OBn_2 = OCn_3 \tag{2.4}$$

9 The volume of the tetrahedron is one third the base times the altitude

$$\Delta V = \frac{1}{3}h\Delta S = \frac{1}{3}OA\Delta S_1 = \frac{1}{3}OB\Delta S_2 = \frac{1}{3}OC\Delta S_3 \tag{2.5}$$

which when combined with the preceding equation yields

$$\Delta S_1 = \Delta S n_1; \quad \Delta S_2 = \Delta S n_2; \quad \Delta S_3 = \Delta S n_3; \tag{2.6}$$

or $\Delta S_i = \Delta S n_i$.

10 In Fig. 2.3 are also shown the *average* values of the body force and of the surface tractions (thus the asterix). The negative sign appears because $\mathbf{t}_i^*$ denotes the average traction on a surface whose outward normal points in the negative x_i direction. We seek to determine $\mathbf{t}_n^*$.

11 We invoke the **momentum principle of a collection of particles** (more about it later on) which is postulated to apply to our idealized continuous medium. This principle states that

The vector sum of all external forces acting on the free body is equal to the rate of change of the total momentum[2].

12 The total momentum is $\int_{\Delta m} \mathbf{v} dm$. By the mean-value theorem of the integral calculus, this is equal to $\mathbf{v}^* \Delta m$ where $\mathbf{v}^*$ is average value of the velocity. Since we are considering the momentum of a given collection of particles, Δm does not change with time and $\Delta m \frac{d\mathbf{v}^*}{dt} = \rho^* \Delta V \frac{d\mathbf{v}^*}{dt}$ where ρ^* is the average density. Hence, the momentum principle yields

$$\mathbf{t}_n^* \Delta S + \rho^* \mathbf{b}^* \Delta V - \mathbf{t}_1^* \Delta S_1 - \mathbf{t}_2^* \Delta S_2 - \mathbf{t}_3^* \Delta S_3 = \rho^* \Delta V \frac{d\mathbf{v}^*}{dt} \tag{2.7}$$

Substituting for ΔV, ΔS_i from above, dividing throughout by ΔS and rearanging we obtain

$$\mathbf{t}_n^* + \frac{1}{3} h \rho^* \mathbf{b}^* = \mathbf{t}_1^* n_1 + \mathbf{t}_2^* n_2 + \mathbf{t}_3^* n_3 + \frac{1}{3} h \rho^* \frac{d\mathbf{v}}{dt} \tag{2.8}$$

and now we let $h \to 0$ and obtain

$$\boxed{\mathbf{t}_n = \mathbf{t}_1 n_1 + \mathbf{t}_2 n_2 + \mathbf{t}_3 n_3 = \mathbf{t}_i n_i} \tag{2.9}$$

13 We observe that we dropped the asterix as the length of the vectors approached zero.

14 Hence, this equation enables us to *determine the traction* $\mathbf{t}_n$ *at a point acting on an arbitrary plane through the point, when we know the tractions on three mutually perpendicular planes through the point.*

15 It is important to note that this result was obtained without any assumption of equilibrium and that it applies as well in fluid dynamics as in solid mechanics.

16 This equation is a vector equation. Substituting $\mathbf{t}_n$ from Eq. 2.2 we obtain

$$\boxed{\begin{array}{lrcl} & t_{n_1} & = & \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ & t_{n_2} & = & \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ & t_{n_3} & = & \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \\ \text{Indicial notation} & t_{n_i} & = & \sigma_{ji} n_j \\ \text{dyadic notation} & \mathbf{t}_n & = & \mathbf{n} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \cdot \mathbf{n} \end{array}} \tag{2.10}$$

17 We have thus established that the nine components σ_{ij} are components of the second order tensor, **Cauchy's stress tensor**. This tensor associates with a unit normal vector $\mathbf{n}$ a traction vector $\mathbf{t}_n$ acting at that point on a surface whose normal is $\mathbf{n}$.

18 Note that this stress tensor is really defined in the deformed space (Eulerian), and this issue will be revisited in Sect. 4.6.

■ Example 2-1: Stress Vectors

if the stress tensor at point P is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{Bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{Bmatrix} \tag{2.11}$$

We seek to determine the traction (or stress vector) $\mathbf{t}$ passing through P and parallel to the plane ABC where $A(4,0,0)$, $B(0,2,0)$ and $C(0,0,6)$.

[2] This is really **Newton's second law** $\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$

Solution:
The vector normal to the plane can be found by taking the cross products of vectors AB and AC:

$$\mathbf{N} = AB \times AC = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -4 & 2 & 0 \\ -4 & 0 & 6 \end{vmatrix} \quad (2.12\text{-a})$$

$$= 12\mathbf{e}_1 + 24\mathbf{e}_2 + 8\mathbf{e}_3 \quad (2.12\text{-b})$$

The unit normal of N is given by

$$\mathbf{n} = \frac{3}{7}\mathbf{e}_1 + \frac{6}{7}\mathbf{e}_2 + \frac{2}{7}\mathbf{e}_3 \quad (2.13)$$

Hence the stress vector (traction) will be

$$\lfloor \tfrac{3}{7} \quad \tfrac{6}{7} \quad \tfrac{2}{7} \rfloor \begin{bmatrix} 7 & -5 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \lfloor -\tfrac{9}{7} \quad \tfrac{5}{7} \quad \tfrac{10}{7} \rfloor \quad (2.14)$$

and thus $\mathbf{t} = -\frac{9}{7}\mathbf{e}_1 + \frac{5}{7}\mathbf{e}_2 + \frac{10}{7}\mathbf{e}_3$ ■

2.3 Principal Stresses

19 Regardless of the state of stress (as long as the stress tensor is symmetric), at a given point, it is always possible to choose a special set of axis through the point so that the shear stress components vanish when the stress components are referred to this system of axis. these special axes are called **principal axes** of the **principal stresses**.

20 To determine the principal directions at any point, we consider $\mathbf{n}$ to be a unit vector in one of the unknown directions. It has components n_i. Let λ represent the principal-stress component on the plane whose normal is $\mathbf{n}$ (note both $\mathbf{n}$ and λ are yet unknown). Since we know that there is no shear stress component on the plane perpendicular to $\mathbf{n}$,

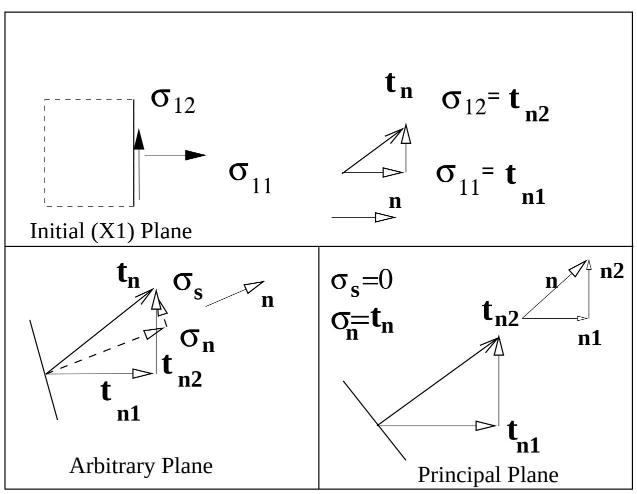


Figure 2.4: Principal Stresses

the stress vector on this plane must be parallel to $\mathbf{n}$ and

$$\mathbf{t}_n = \lambda\mathbf{n} \quad (2.15)$$

21 From Eq. 2.10 and denoting the stress tensor by $\boldsymbol{\sigma}$ we get

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \lambda\mathbf{n} \tag{2.16}$$

in indicial notation this can be rewritten as

$$n_r\sigma_{rs} = \lambda n_s \tag{2.17}$$

or

$$(\sigma_{rs} - \lambda\delta_{rs})n_r = 0 \tag{2.18}$$

in matrix notation this corresponds to

$$n\left([\boldsymbol{\sigma}] - \lambda[\mathbf{I}]\right) = 0 \tag{2.19}$$

where I corresponds to the identity matrix. We really have here a set of three homogeneous algebraic equations for the direction cosines n_i.

22 Since the direction cosines must also satisfy

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{2.20}$$

they can not all be zero. hence Eq.2.19 has solutions which are not zero if and only if the determinant of the coefficients is equal to zero, i.e

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0 \tag{2.21}$$

$$|\sigma_{rs} - \lambda\delta_{rs}| = 0 \tag{2.22}$$

$$|\boldsymbol{\sigma} - \lambda\mathbf{I}| = 0 \tag{2.23}$$

23 For a given set of the nine stress components, the preceding equation constitutes a cubic equation for the three unknown magnitudes of λ.

24 Cauchy was first to show that since the matrix is symmetric and has real elements, the roots are all real numbers.

25 The three lambdas correspond to the three principal stresses $\sigma_{(1)} > \sigma_{(2)} > \sigma_{(3)}$. When any one of them is substituted for λ in the three equations in Eq. 2.19 those equations reduce to only two independent linear equations, which must be solved together with the quadratic Eq. 2.20 to determine the direction cosines n_r^i of the normal $\mathbf{n}^i$ to the plane on which σ_i acts.

26 The three directions form a right-handed system and

$$\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 \tag{2.24}$$

27 In 2D, it can be shown that the principal stresses are given by:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{2.25}$$

2.3.1 Invariants

28 The principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore **invariants** of the stress state.

29 When the determinant in the characteristic Eq. 19.21-c is expanded, the cubic equation takes the form

$$\lambda^3 - I_\sigma \lambda^2 - II_\sigma \lambda - III_\sigma = 0 \quad (2.26)$$

where the symbols I_σ, II_σ and III_σ denote the following scalar expressions in the stress components:

$$\begin{aligned}
I_\sigma &= \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} = \text{tr}\,\boldsymbol{\sigma} && (2.27)\\
II_\sigma &= -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 && (2.28)\\
&= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}I_\sigma^2 && (2.29)\\
&= \frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_\sigma^2) && (2.30)\\
III_\sigma &= \det\boldsymbol{\sigma} = \frac{1}{6}e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} && (2.31)
\end{aligned}$$

30 In terms of the principal stresses, those invariants can be simplified into

$$\begin{aligned}
I_\sigma &= \sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} && (2.32)\\
II_\sigma &= -(\sigma_{(1)}\sigma_{(2)} + \sigma_{(2)}\sigma_{(3)} + \sigma_{(3)}\sigma_{(1)}) && (2.33)\\
III_\sigma &= \sigma_{(1)}\sigma_{(2)}\sigma_{(3)} && (2.34)
\end{aligned}$$

2.3.2 Spherical and Deviatoric Stress Tensors

31 If we let σ denote the mean normal stress p

$$\sigma = -p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\text{tr}\,\boldsymbol{\sigma} \quad (2.35)$$

then the stress tensor can be written as the sum of two tensors:

Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape in an isotropic medium.

$$\sigma_{hyd} = -p\mathbf{I} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \quad (2.36)$$

Deviatoric Stress: which causes the change in shape.

$$\sigma_{dev} = \begin{bmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{bmatrix} \quad (2.37)$$

2.4 Stress Transformation

32 From Eq. 1.66 and 1.67, the stress transformation for the second order stress tensor is given by

$$\begin{aligned}
\overline{\sigma}_{ip} &= a_i^j a_p^q \sigma_{jq} \quad \text{in Matrix Form } [\overline{\sigma}] = [A]^T[\sigma][A] && (2.38)\\
\sigma_{jq} &= a_i^j a_p^q \overline{\sigma}_{ip} \quad \text{in Matrix Form } [\sigma] = [A][\overline{\sigma}][A]^T && (2.39)
\end{aligned}$$

33 For the 2D plane stress case we rewrite Eq. 1.69

$$\left\{ \begin{array}{c} \overline{\sigma}_{xx} \\ \overline{\sigma}_{yy} \\ \overline{\sigma}_{xy} \end{array} \right\} = \left[\begin{array}{ccc} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos\alpha\sin\alpha & \cos^2\alpha - \sin^2\alpha \end{array} \right] \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} \tag{2.40}$$

Example 2-2: Principal Stresses

The stress tensor is given at a point by

$$\boldsymbol{\sigma} = \left[\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{array} \right] \tag{2.41}$$

determine the principal stress values and the corresponding directions.

Solution:

From Eq.19.21-c we have

$$\left| \begin{array}{ccc} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{array} \right| = 0 \tag{2.42}$$

Or upon expansion (and simplification) $(\lambda+2)(\lambda-4)(\lambda-1) = 0$, thus the roots are $\sigma_{(1)} = 4$, $\sigma_{(2)} = 1$ and $\sigma_{(3)} = -2$. We also note that those are the three **eigenvalues** of the stress tensor.

If we let $\overline{x}_1$ axis be the one corresponding to the direction of $\sigma_{(3)}$ and n_i^3 be the direction cosines of this axis, then from Eq. 2.19 we have

$$\left\{ \begin{array}{rcl} (3+2)n_1^3 + n_2^3 + n_3^3 & = & 0 \\ n_1^3 + 2n_2^3 + 2n_3^3 & = & 0 \\ n_1^3 + 2n_2^3 + 2n_3^3 & = & 0 \end{array} \right. \Rightarrow n_1^3 = 0; \quad n_2^3 = \frac{1}{\sqrt{2}}; \quad n_3^3 = -\frac{1}{\sqrt{2}} \tag{2.43}$$

Similarly If we let $\overline{x}_2$ axis be the one corresponding to the direction of $\sigma_{(2)}$ and n_i^2 be the direction cosines of this axis,

$$\left\{ \begin{array}{rcl} 2n_1^2 + n_2^2 + n_3^2 & = & 0 \\ n_1^2 - n_2^2 + 2n_3^2 & = & 0 \\ n_1^2 + 2n_2^2 - n_3^2 & = & 0 \end{array} \right. \Rightarrow n_1^2 = \frac{1}{\sqrt{3}}; \quad n_2^2 = -\frac{1}{\sqrt{3}}; \quad n_3^2 = -\frac{1}{\sqrt{3}} \tag{2.44}$$

Finally, if we let $\overline{x}_3$ axis be the one corresponding to the direction of $\sigma_{(1)}$ and n_i^1 be the direction cosines of this axis,

$$\left\{ \begin{array}{rcl} -n_1^1 + n_2^1 + n_3^1 & = & 0 \\ n_1^1 - 4n_2^1 + 2n_3^1 & = & 0 \\ n_1^1 + 2n_2^1 - 4n_3^1 & = & 0 \end{array} \right. \Rightarrow n_1^1 = -\frac{2}{\sqrt{6}}; \quad n_2^1 = -\frac{1}{\sqrt{6}}; \quad n_3^1 = -\frac{1}{\sqrt{6}} \tag{2.45}$$

Finally, we can convince ourselves that the two stress tensors have the same invariants I_σ, II_σ and III_σ. ■

Example 2-3: Stress Transformation

Show that the transformation tensor of direction cosines previously determined transforms the original stress tensor into the diagonal principal axes stress tensor.

Solution:

From Eq. 2.38

$$\overline{\boldsymbol{\sigma}} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \quad (2.46\text{-a})$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (2.46\text{-b})$$

■

2.5 †Simplified Theories; Stress Resultants

34 For many applications of continuum mechanics the problem of determining the three-dimensional stress distribution is too difficult to solve. However, in many (civil/mechanical)applications, one or more dimensions is/are small compared to the others and possess certain symmetries of geometrical shape and load distribution.

35 In those cases, we may apply "**engineering theories**" for shells, plates or beams. In those problems, instead of solving for the stress components throughout the body, we solve for certain *stress resultants* (normal, shear forces, and Moments and torsions) resulting from an integration over the body. We consider separately two of those three cases.

36 Alternatively, if a continuum solution is desired, and engineering theories prove to be either too restrictive or inapplicable, we can use numerical techniques (such as **the Finite Element Method**) to solve the problem.

2.5.1 Shell

37 Fig. 2.5 illustrates the stresses acting on a differential element of a shell structure. The resulting forces in turn are shown in Fig. 2.6 and for simplification those acting per unit length of the middle surface are shown in Fig. 2.7. The net resultant forces are given by:

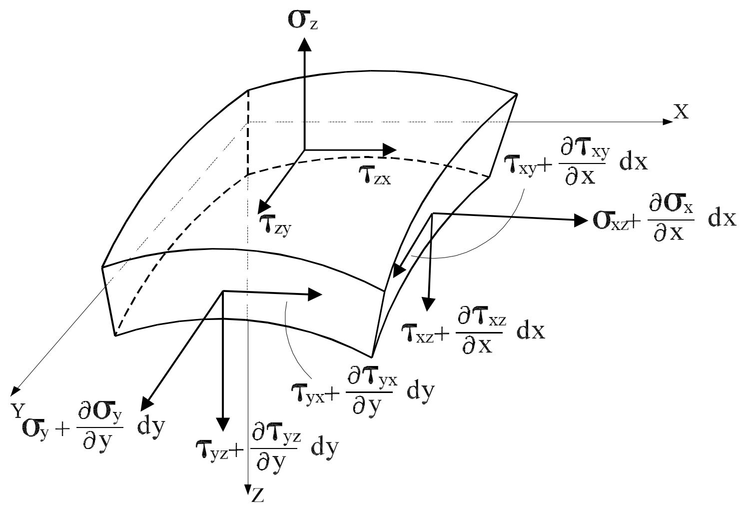


Figure 2.5: Differential Shell Element, Stresses

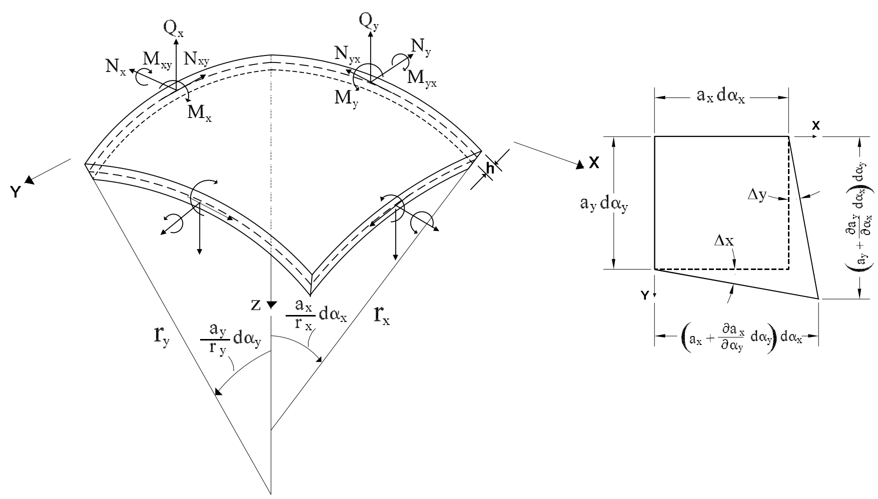


Figure 2.6: Differential Shell Element, Forces

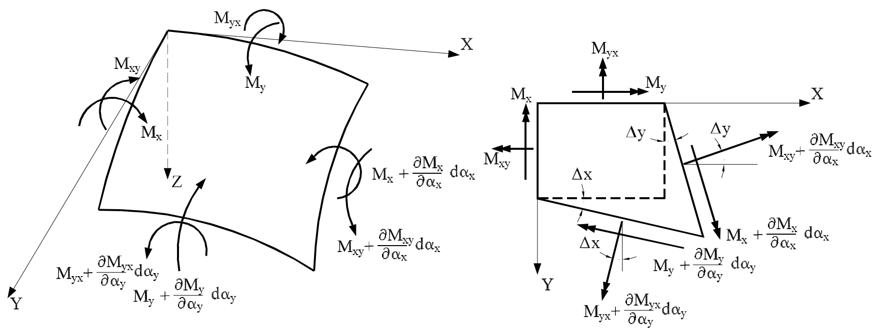


Figure 2.7: Differential Shell Element, Vectors of Stress Couples

Membrane Force

$$
\mathbf{N} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma}\left(1-\frac{z}{r}\right)dz \qquad
\begin{cases}
N_{xx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xx}\left(1-\frac{z}{r_y}\right)dz \\
N_{yy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yy}\left(1-\frac{z}{r_x}\right)dz \\
N_{xy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy}\left(1-\frac{z}{r_y}\right)dz \\
N_{yx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy}\left(1-\frac{z}{r_x}\right)dz
\end{cases}
$$

Bending Moments

$$
\mathbf{M} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\sigma} z\left(1-\frac{z}{r}\right)dz \qquad
\begin{cases}
M_{xx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xx} z\left(1-\frac{z}{r_y}\right)dz \\
M_{yy} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{yy} z\left(1-\frac{z}{r_x}\right)dz \\
M_{xy} = -\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} z\left(1-\frac{z}{r_y}\right)dz \\
M_{yx} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{xy} z\left(1-\frac{z}{r_x}\right)dz
\end{cases} \qquad (2.47)
$$

Transverse Shear Forces

$$
\mathbf{Q} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \boldsymbol{\tau}\left(1-\frac{z}{r}\right)dz \qquad
\begin{cases}
Q_{x} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz}\left(1-\frac{z}{r_y}\right)dz \\
Q_{y} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{yz}\left(1-\frac{z}{r_x}\right)dz
\end{cases}
$$

2.5.2 Plates

38 Considering an arbitrary plate, the stresses and resulting forces are shown in Fig. 2.8, and resultants

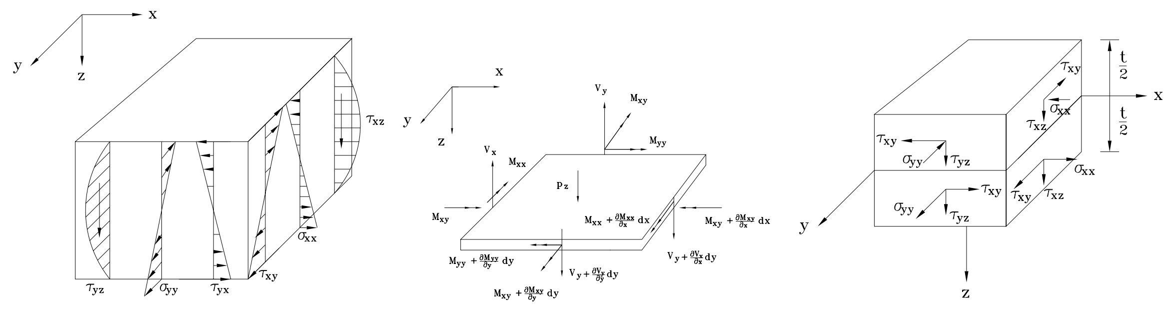


Figure 2.8: Stresses and Resulting Forces in a Plate

per unit width are given by

$$\begin{aligned}
\textbf{Membrane Force} \quad \mathbf{N} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} dz \quad \left\{ \begin{array}{lcl} N_{xx} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} dz \\ N_{yy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} dz \\ N_{xy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} dz \end{array} \right. \\
\textbf{Bending Moments} \quad \mathbf{M} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\sigma} z dz \quad \left\{ \begin{array}{lcl} M_{xx} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx} z dz \\ M_{yy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy} z dz \\ M_{xy} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xy} z dz \end{array} \right. \\
\textbf{Transverse Shear Forces} \quad \mathbf{V} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \boldsymbol{\tau} dz \quad \left\{ \begin{array}{lcl} V_{x} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} dz \\ V_{y} &=& \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} dz \end{array} \right.
\end{aligned} \tag{2.48-a}$$

39 Note that in plate theory, we ignore the effect of the membrane forces, those in turn will be accounted for in shells.

Chapter 3

MATHEMATICAL PRELIMINARIES; Part II VECTOR DIFFERENTIATION

3.1 Introduction

1 A **field** is a function defined over a continuous region. This includes, **Scalar Field** (such as temperature) $g(\mathbf{x})$, **Vector Field** (such as gravity or magnetic) $\mathbf{v}(\mathbf{x})$, Fig. 3.1 or **Tensor Field** $\mathbf{T}(\mathbf{x})$.

2 We first introduce the **differential vector operator** "Nabla" denoted by $\boldsymbol{\nabla}$

$$\boxed{\boldsymbol{\nabla} \equiv \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}} \tag{3.1}$$

3 We also note that there are as many ways to differentiate a vector field as there are ways of multiplying vectors, the analogy being given by Table 3.1.

Multiplication		Differentiation		Tensor Order
$\mathbf{u}\cdot\mathbf{v}$	dot	$\boldsymbol{\nabla}\cdot\mathbf{v}$	divergence	$\downarrow$
$\mathbf{u}\times\mathbf{v}$	cross	$\boldsymbol{\nabla}\times\mathbf{v}$	curl	$\rightarrow$
$\mathbf{u}\otimes\mathbf{v}$	tensor	$\boldsymbol{\nabla}\mathbf{v}$	gradient	$\uparrow$

Table 3.1: Similarities Between Multiplication and Differentiation Operators

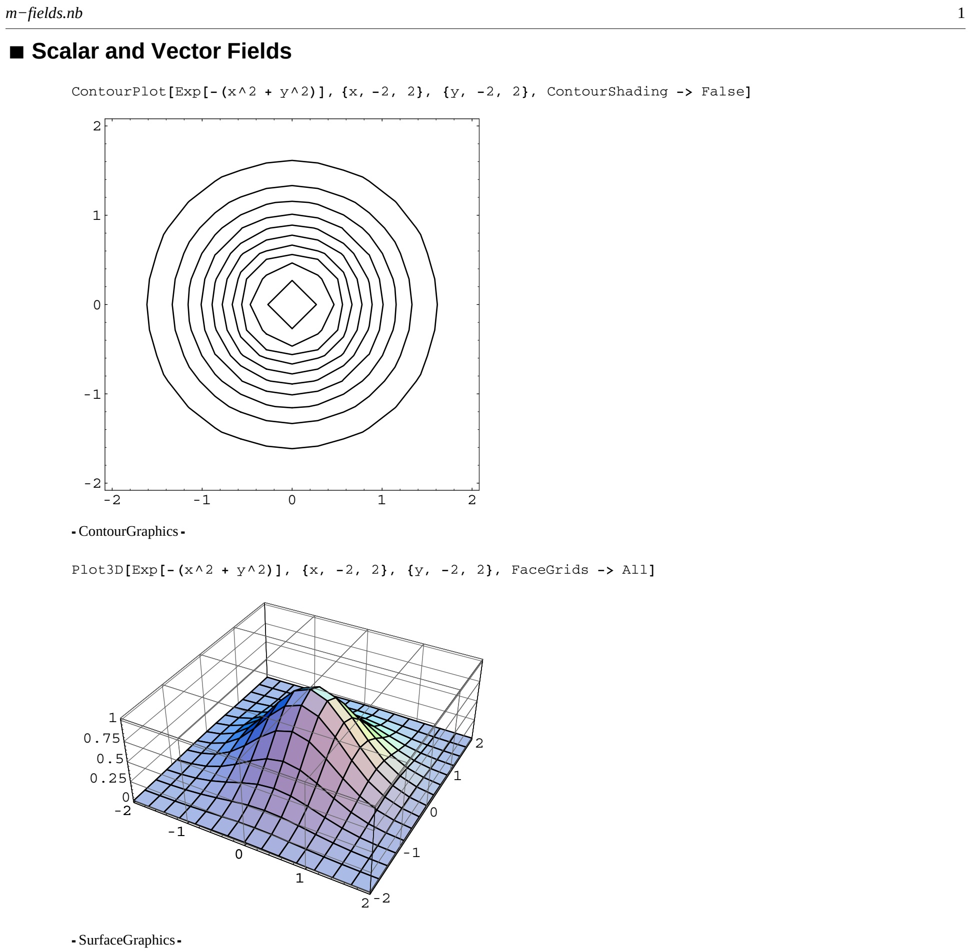
3.2 Derivative WRT to a Scalar

4 The derivative of a vector $\mathbf{p}(u)$ with respect to a scalar u, Fig. 3.2 is defined by

$$\frac{d\mathbf{p}}{du} \equiv \lim_{\Delta u \to 0} \frac{\mathbf{p}(u+\Delta u) - \mathbf{p}(u)}{\Delta u} \tag{3.2}$$

5 If $\mathbf{p}(u)$ is a **position vector** $\mathbf{p}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, then

$$\frac{d\mathbf{p}}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k} \tag{3.3}$$



m−fields.nb 1

■ Scalar and Vector Fields

```
ContourPlot[Exp[-(x^2 + y^2)], {x, -2, 2}, {y, -2, 2}, ContourShading -> False]
```


- ContourGraphics -

```
Plot3D[Exp[-(x^2 + y^2)], {x, -2, 2}, {y, -2, 2}, FaceGrids -> All]
```


- SurfaceGraphics -

Figure 3.1: Examples of a Scalar and Vector Fields

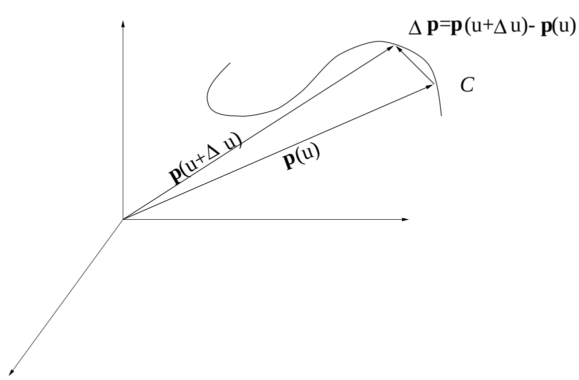


Figure 3.2: Differentiation of position vector **p**

is a vector along the tangent to the curve.

6 If u is the time t, then $\frac{d\mathbf{p}}{dt}$ is the velocity

7 †In **differential geometry**, if we consider a curve $\mathcal{C}$ defined by the function $\mathbf{p}(u)$ then $\frac{d\mathbf{p}}{du}$ is a vector tangent ot $\mathcal{C}$, and if u is the curvilinear coordinate s measured from any point along the curve, then $\frac{d\mathbf{p}}{ds}$ is a unit tangent vector to $\mathcal{C}$ $\mathbf{T}$, Fig. 3.3. and we have the following relations

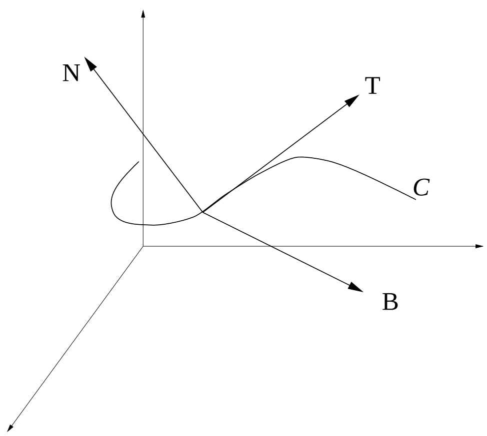


Figure 3.3: Curvature of a Curve

$$\frac{d\mathbf{p}}{ds} = \mathbf{T} \quad (3.4)$$
$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \quad (3.5)$$
$$\mathbf{B} = \mathbf{T}\times\mathbf{N} \quad (3.6)$$
$$\kappa \quad \text{curvature} \quad (3.7)$$
$$\rho = \frac{1}{\kappa} \quad \text{Radius of Curvature} \quad (3.8)$$

we also note that $\mathbf{p}\cdot\frac{d\mathbf{p}}{ds} = 0$ if $\left|\frac{d\mathbf{p}}{ds}\right| \neq 0$.

■ **Example 3-1: Tangent to a Curve**

Determine the unit vector tangent to the curve: $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$ for $t = 2$.

Solution:

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left[(t^2+1)\mathbf{i} + (4t-3)\mathbf{j} + (2t^2-6t)\mathbf{k}\right] = 2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k} \quad \text{(3.9-a)}$$
$$\left|\frac{d\mathbf{p}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2} \quad \text{(3.9-b)}$$
$$\mathbf{T} = \frac{2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}} \quad \text{(3.9-c)}$$
$$= \frac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (4)^2 + (2)^2}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \quad \text{for} \quad t = 2 \quad \text{(3.9-d)}$$

Mathematica solution is shown in Fig. 3.4 ■

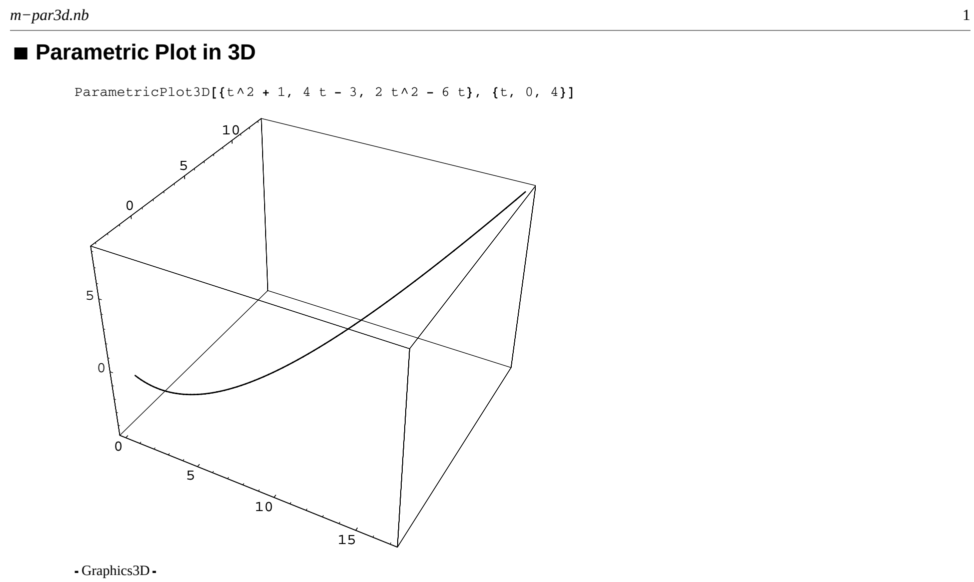


Figure 3.4: Mathematica Solution for the Tangent to a Curve in 3D

3.3 Divergence

3.3.1 Vector

8 The **divergence** of a vector field of a body $\mathcal{B}$ with boundary Ω, Fig. 3.5 is defined by considering that each point of the surface has a normal $\mathbf{n}$, and that the body is surrounded by a vector field $\mathbf{v}(\mathbf{x})$. The volume of the body is $v(\mathcal{B})$.

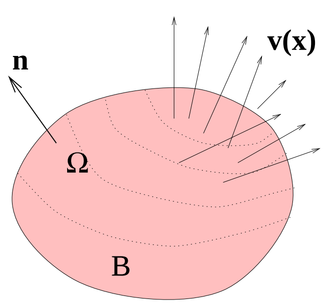


Figure 3.5: Vector Field Crossing a Solid Region

9 The divergence of the vector field is thus defined as

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{v} = \text{div }\mathbf{v}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0}\frac{1}{v(\mathcal{B})}\int_{\Omega}\mathbf{v}\cdot\mathbf{n}dA} \tag{3.10}$$

where $\mathbf{v}.\mathbf{n}$ is often referred as the **flux** and represents the total volume of "fluid" that passes through dA in unit time, Fig. 3.6 This volume is then equal to the base of the cylinder dA times the height of

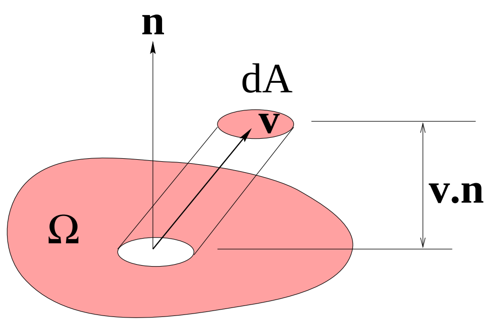


Figure 3.6: Flux Through Area dA

the cylinder $\mathbf{v}\cdot\mathbf{n}$. We note that the streamlines which are tangent to the boundary do not let any fluid out, while those normal to it let it out most efficiently.

10 The divergence thus measure the rate of change of a vector field.

11 †The definition is clearly independent of the shape of the solid region, however we can gain an insight into the divergence by considering a rectangular parallelepiped with sides Δx_1, Δx_2, and Δx_3, and with normal vectors pointing in the directions of the coordinate axies, Fig. 3.7. If we also consider the corner

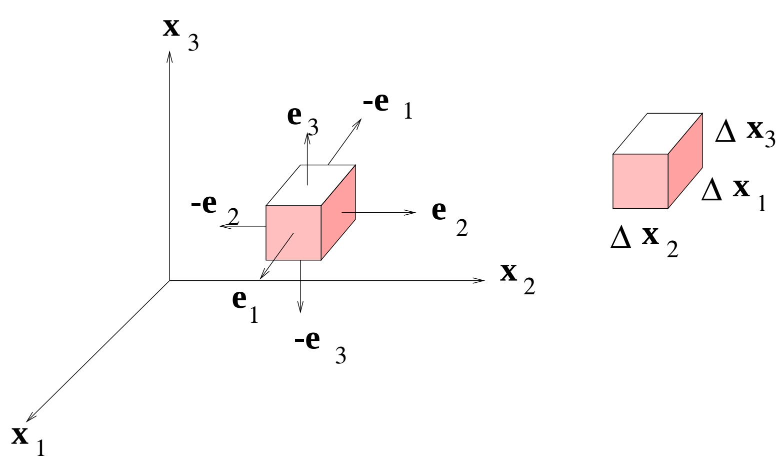


Figure 3.7: Infinitesimal Element for the Evaluation of the Divergence

closest to the origin as located at $\mathbf{x}$, then the contribution (from Eq. 3.10) of the two surfaces with normal vectors $\mathbf{e}_1$ and $-\mathbf{e}_1$ is

$$\lim_{\Delta x_1,\Delta x_2,\Delta x_3\to 0}\frac{1}{\Delta x_1\Delta x_2\Delta x_3}\int_{\Delta x_2\Delta x_3}[\mathbf{v}(\mathbf{x}+\Delta x_1\mathbf{e}_1)\cdot\mathbf{e}_1+\mathbf{v}(x)\cdot(-\mathbf{e}_1)]dx_2dx_3 \tag{3.11}$$

or

$$\lim_{\Delta x_1,\Delta x_2,\Delta x_3\to 0}\frac{1}{\Delta x_2\Delta x_3}\int_{\Delta x_2\Delta x_3}\frac{\mathbf{v}(\mathbf{x}+\Delta x_1\mathbf{e}_1)-\mathbf{v}(\mathbf{x})}{\Delta x_1}\cdot\mathbf{e}_1dx_2dx_3 = \lim_{\Delta x_1\to 0}\frac{\Delta\mathbf{v}}{\Delta x_1}\cdot\mathbf{e}_1 \tag{3.12-a}$$

$$= \frac{\partial\mathbf{v}}{\partial x_1}\cdot\mathbf{e}_1 \tag{3.12-b}$$

hence, we can generalize

$$\boxed{\text{div }\mathbf{v}(\mathbf{x})=\frac{\partial\mathbf{v}(\mathbf{x})}{\partial x_i}\cdot\mathbf{e}_i} \tag{3.13}$$

12 or alternatively

$$\text{div } \mathbf{v} = \nabla\cdot\mathbf{v} = (\frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \frac{\partial}{\partial x_3}\mathbf{e}_3)\cdot(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3) \quad (3.14)$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_i}{\partial x_i} = \partial_i v_i = v_{i,i} \quad (3.15)$$

13 The divergence of a vector is a **scalar**.

14 We note that the **Laplacian Operator** is defined as

$$\nabla^2 F \equiv \nabla\nabla F = F_{,ii} \quad (3.16)$$

■ **Example 3-2: Divergence**

Determine the divergence of the vector $\mathbf{A} = x^2z\mathbf{i} - 2y^3z^2\mathbf{j} + xy^2z\mathbf{k}$ at point $(1, -1, 1)$.
Solution:

$$\nabla\cdot\mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(x^2z\mathbf{i} - 2y^3z^2\mathbf{j} + xy^2z\mathbf{k}) \quad (3.17\text{-a})$$

$$= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z} \quad (3.17\text{-b})$$

$$= 2xz - 6y^2z^2 + xy^2 \quad (3.17\text{-c})$$

$$= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = -3 \text{ at } (1, -1, 1) \quad (3.17\text{-d})$$

Mathematica solution is shown in Fig. 3.8 ■

3.3.2 Second-Order Tensor

15 By analogy to Eq. 3.10, the divergence of a second-order tensor field $\mathbf{T}$ is

$$\nabla\cdot\mathbf{T} = \text{div } \mathbf{T}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0} \frac{1}{v(\mathcal{B})}\int_\Omega \mathbf{T}\cdot\mathbf{n}dA \quad (3.18)$$

which is the **vector field**

$$\nabla\cdot\mathbf{T} = \frac{\partial T_{pq}}{\partial x_p}\mathbf{e}_q \quad (3.19)$$

3.4 Gradient

3.4.1 Scalar

16 The **gradient** of a scalar field $g(\mathbf{x})$ is a vector field grad $g(x)$ or $\nabla g(\mathbf{x})$ such that for any unit vector $\mathbf{v}$, the directional derivative dg/ds in the direction of $\mathbf{v}$ is given by the scalar product

$$\frac{dg}{ds} = \nabla g\cdot\mathbf{v} \quad (3.20)$$

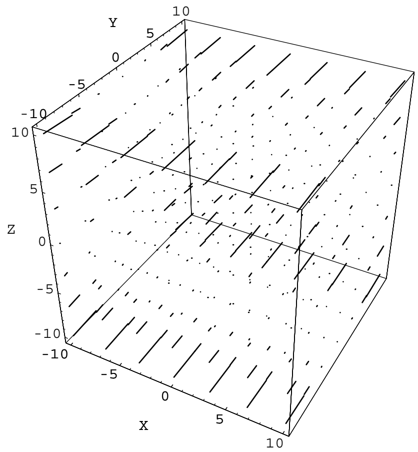
m−diver.nb 1

■ **Divergence of a Vector**

```
<< Calculus'VectorAnalysis'

V = {x^2 z, -2 y^3 z^2, x y^2 z};

Div[V, Cartesian[x, y, z]]
```

$-6z^2y^2 + xy^2 + 2xz$

```
<< Graphics'PlotField3D'

PlotVectorField3D[{x^2 z, -2 y^3 z^2, x y^2 z}, {x, -10, 10}, {y, -10, 10}, {z, -10, 10},
 Axes -> Automatic, AxesLabel -> {"X", "Y", "Z"}]
```



- Graphics3D -

```
Div[Curl[V, Cartesian[x, y, z]], Cartesian[x, y, z]]
```

0

Figure 3.8: Mathematica Solution for the Divergence of a Vector

where $\mathbf{v} = \frac{d\mathbf{p}}{ds}$ We note that the definition made no reference to any coordinate system. The gradient is thus a **vector invariant**.

17 To find the components in any rectangular Cartesian coordinate system we substitute

$$\mathbf{v} = \frac{d\mathbf{p}}{ds} = \frac{dx_i}{ds}\mathbf{e}_i \tag{3.21}$$

into Eq. 3.20 yielding

$$\frac{dg}{ds} = \boldsymbol{\nabla} g \frac{dx_i}{ds}\mathbf{e}_i \tag{3.22}$$

But from the chain rule we have

$$\frac{dg}{ds} = \frac{\partial g}{\partial x_i}\frac{dx_i}{ds} \tag{3.23}$$

thus

$$\left[(\nabla g)_i - \frac{\partial g}{\partial x_i}\right]\frac{\mathrm{d}x_i}{\mathrm{d}s} = 0 \tag{3.24}$$

or

$$\boxed{\boldsymbol{\nabla} g = \frac{\partial g}{\partial x_i}\mathbf{e}_i} \tag{3.25}$$

which is equivalent to

$$\boldsymbol{\nabla}\phi \equiv \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi \tag{3.26-a}$$

$$= \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \tag{3.26-b}$$

and note that it defines a **vector field**.

18 The physical significance of the gradient of a scalar field is that it points in the direction in which the field is changing most rapidly (for a three dimensional surface, the gradient is pointing along the normal to the plane tangent to the surface). The length of the vector $||\boldsymbol{\nabla} g(\mathbf{x})||$ is perpendicular to the contour lines.

19 $\boldsymbol{\nabla} g(\mathbf{x})\cdot\mathbf{n}$ gives the rate of change of the scalar field in the direction of $\mathbf{n}$.

■ Example 3-3: Gradient of a Scalar

Determine the gradient of $\phi = x^2yz + 4xz^2$ at point $(1, -2, -1)$ along the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.
Solution:

$$\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 8xz)\mathbf{k} \tag{3.27-a}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \text{ at } (1, -2, -1) \tag{3.27-b}$$

$$\mathbf{n} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \tag{3.27-c}$$

$$\boldsymbol{\nabla}\phi\cdot\mathbf{n} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k})\cdot\left(\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3} \tag{3.27-d}$$

Since this last value is positive, ϕ increases along that direction. ■

■ Example 3-4: Stress Vector normal to the Tangent of a Cylinder

The stress tensor throughout a continuum is given with respect to Cartesian axes as

$$\boldsymbol{\sigma} = \begin{bmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3^2 \\ 0 & 2x_3 & 0 \end{bmatrix} \tag{3.28}$$

Determine the stress vector (or traction) at the point $P(2, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface $x_2^2 + x_3^2 = 4$ at P, Fig. 3.9.
Solution:

At point P, the stress tensor is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{bmatrix} \tag{3.29}$$

The unit normal to the surface at P is given from

$$\boldsymbol{\nabla}(x_2^2 + x_3^2 - 4) = 2x_2\mathbf{e}_2 + 2x_3\mathbf{e}_3 \tag{3.30}$$

At point P,

$$\boldsymbol{\nabla}(x_2^2 + x_3^2 - 4) = 2\mathbf{e}2_2 + 2\sqrt{3}\mathbf{e}_3 \tag{3.31}$$

and thus the unit normal at P is

$$\mathbf{n} = \frac{1}{2}\mathbf{e}_2 + \frac{\sqrt{3}}{2}\mathbf{e}_3 \tag{3.32}$$

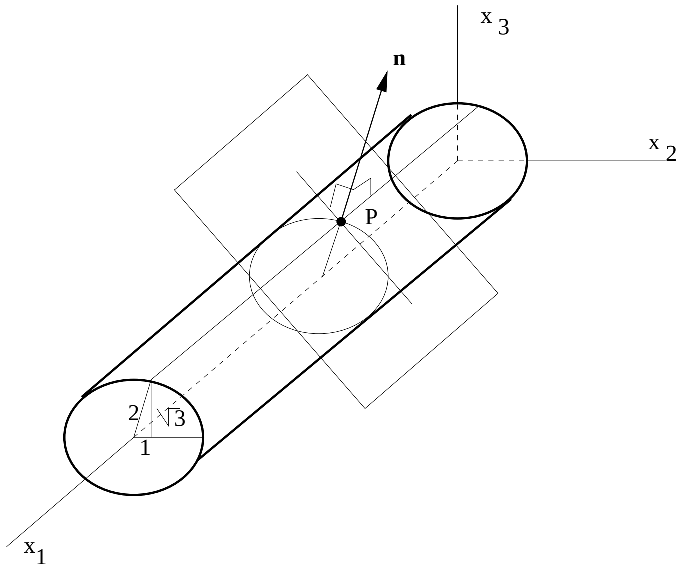


Figure 3.9: Radial Stress vector in a Cylinder

Thus the traction vector will be determined from

$$\boldsymbol{\sigma} = \begin{bmatrix} 6 & 5 & 0 \\ 5 & 0 & 2\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{Bmatrix} = \begin{Bmatrix} 5/2 \\ 3 \\ \sqrt{3} \end{Bmatrix} \tag{3.33}$$

or $\mathbf{t}^{\mathbf{n}} = \frac{5}{2}\mathbf{e}_1 + 3\mathbf{e}_2 + \sqrt{3}\mathbf{e}_3$ ■

3.4.2 Vector

20 We can also define the gradient of a vector field. If we consider a solid domain $\mathcal{B}$ with boundary Ω, Fig. 3.5, then the gradient of the vector field $\mathbf{v}(\mathbf{x})$ is a second order tensor defined by

$$\boxed{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}(\mathbf{x}) \equiv \lim_{v(\mathcal{B})\to 0} \frac{1}{v(\mathcal{B})}\int_\Omega \mathbf{v}\otimes\mathbf{n}dA} \tag{3.34}$$

and with a construction similar to the one used for the divergence, it can be shown that

$$\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}(\mathbf{x}) = \frac{\partial v_i(\mathbf{x})}{\partial x_j}[\mathbf{e}_i\otimes\mathbf{e}_j] \tag{3.35}$$

where summation is implied for both i and j.

21 The components of $\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}$ are simply the various partial derivatives of the component functions with respect to the coordinates:

$$[\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.36}$$

$$[\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}}] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.37}$$

that is $[\nabla \mathbf{v}]_{ij}$ gives the rate of change of the ith component of $\mathbf{v}$ with respect to the jth coordinate axis.

22 Note the diference between $\mathbf{v}\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{x}}\mathbf{v}$. In matrix representation, one is the transpose of the other.

23 The gradient of a vector is a tensor of order 2.

24 We can interpret the gradient of a vector geometrically, Fig. 3.10. If we consider two points a and b that are near to each other (i.e Δs is very small), and let the unit vector $\mathbf{m}$ points in the direction from a to b. The value of the vector field at a is $\mathbf{v}(\mathbf{x})$ and the value of the vector field at b is $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})$. Since the vector field changes with position in the domain, those two vectors are different both in length and orientation. If we now transport a copy of $\mathbf{v}(\mathbf{x})$ and place it at b, then we compare the differences between those two vectors. The vector connecting the heads of $\mathbf{v}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})$ is $\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})-\mathbf{v}(\mathbf{x})$, the change in vector. Thus, if we divide this change by Δs, then we get the rate of change as we move in the specified direction. Finally, taking the limit as Δs goes to zero, we obtain

$$\lim_{\Delta s\to 0}\frac{\mathbf{v}(\mathbf{x}+\Delta s\mathbf{m})-\mathbf{v}(\mathbf{x})}{\Delta s}\equiv D\mathbf{v}(\mathbf{x})\cdot\mathbf{m} \tag{3.38}$$

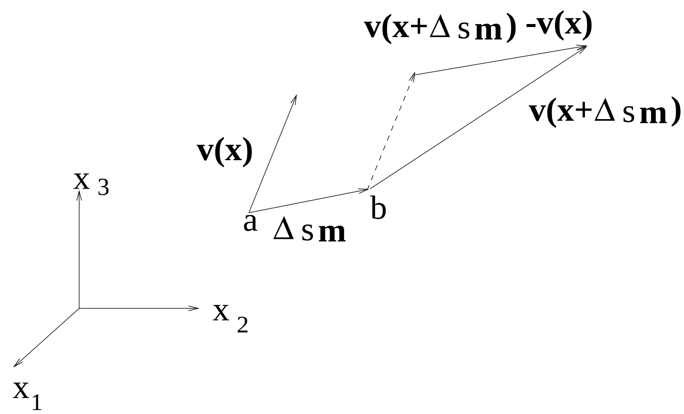


Figure 3.10: Gradient of a Vector

The quantity $D\mathbf{v}(\mathbf{x})\cdot\mathbf{m}$ is called the **directional derivative** because it gives the rate of change of the vector field as we move in the direction $\mathbf{m}$.

■ Example 3-5: Gradient of a Vector Field

Determine the gradient of the following vector field $\mathbf{v}(\mathbf{x})=x_1x_2x_3(x_1\mathbf{e}_1+x_2\mathbf{e}_2+x_3\mathbf{e}_3)$.
Solution:

$$\begin{aligned}\nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x}) &= 2x_1x_2x_3[\mathbf{e}_1\otimes\mathbf{e}_1]+x_1^2x_3[\mathbf{e}_1\otimes\mathbf{e}_2]+x_1^2x_2[\mathbf{e}_1\otimes\mathbf{e}_3]\\ &\quad +x_2^2x_3[\mathbf{e}_2\otimes\mathbf{e}_1]+2x_1x_2x_3[\mathbf{e}_2\otimes\mathbf{e}_2]+x_1x_2^2[\mathbf{e}_2\otimes\mathbf{e}_3] \qquad (3.39\text{-a})\\ &\quad +x_2x_3^2[\mathbf{e}_3\otimes\mathbf{e}_1]+x_1x_3^2[\mathbf{e}_3\otimes\mathbf{e}_2]+2x_1x_2x_3[\mathbf{e}_3\otimes\mathbf{e}_3]\\ &= x_1x_2x_3\begin{bmatrix}2 & x_1/x_2 & x_1/x_3\\ x_2/x_1 & 2 & x_2/x_3\\ x_3/x_1 & x_3/x_2 & 2\end{bmatrix} \qquad (3.39\text{-b})\end{aligned}$$

■

3.4.3 Mathematica Solution

25 Mathematica solution of the two preceding examples is shown in Fig. 3.11.

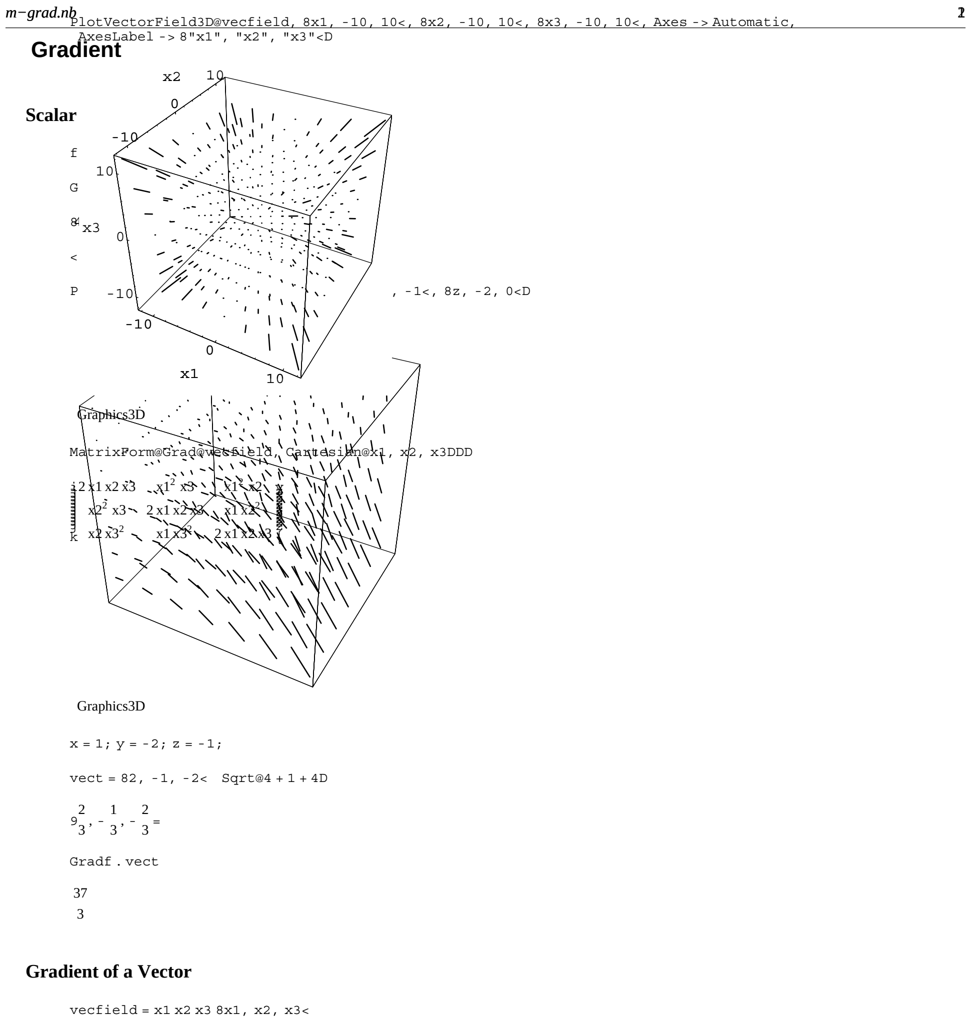


Figure 3.11: Mathematica Solution for the Gradients of a Scalar and of a Vector

Chapter 4

KINEMATIC

Or on How Bodies Deform

4.1 Elementary Definition of Strain

1 We begin our detailed coverage of strain by a simplified and elementary set of definitions for the 1D and 2D cases. Following this a mathematically rigorous derivation of the various expressions for strain will follow.

4.1.1 Small and Finite Strains in 1D

2 We begin by considering an elementary case, an axial rod with initial lenght l_0, and subjected to a deformation Δl into a final deformed length of l, Fig. 4.1.

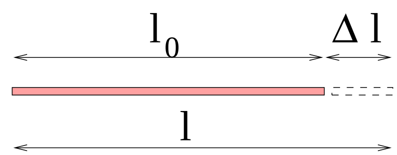


Figure 4.1: Elongation of an Axial Rod

3 We seek to quantify the deformation of the rod and even though we only have 2 variables (l_0 and l), there are different possibilities to introduce the notion of **strain**. We first define the **stretch** of the rod as

$$\lambda \equiv \frac{l}{l_0} \tag{4.1}$$

This stretch is one in the undeformed case, and greater than one when the rod is elongated.

4 Using l_0, l and λ we next introduce four possible definitions of the strain in 1D:

$$\textbf{Engineering Strain} \quad \varepsilon \quad \equiv \frac{l - l_0}{l_0} = \lambda - 1 \tag{4.2}$$

$$\textbf{Natural Strain} \quad \eta \quad \equiv \frac{l - l_0}{l} = 1 - \frac{1}{\lambda} \tag{4.3}$$

$$\textbf{Lagrangian Strain} \quad E \quad \equiv \frac{1}{2}\left(\frac{l^2 - l_0^2}{l_0^2}\right) = \frac{1}{2}(\lambda^2 - 1) \tag{4.4}$$

$$\textbf{Eulerian Strain} \quad E^* \quad \equiv \frac{1}{2}\left(\frac{l^2 - l_0^2}{l^2}\right) = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right) \tag{4.5}$$

we note the strong analogy between the Lagrangian and the engineering strain on the one hand, and the Eulerian and the natural strain on the other.

5 The choice of which strain definition to use is related to the stress-strain relation (or constitutive law) that we will later adopt.

4.1.2 Small Strains in 2D

6 The elementary definition of strains in 2D is illustrated by Fig. 4.2 and are given by

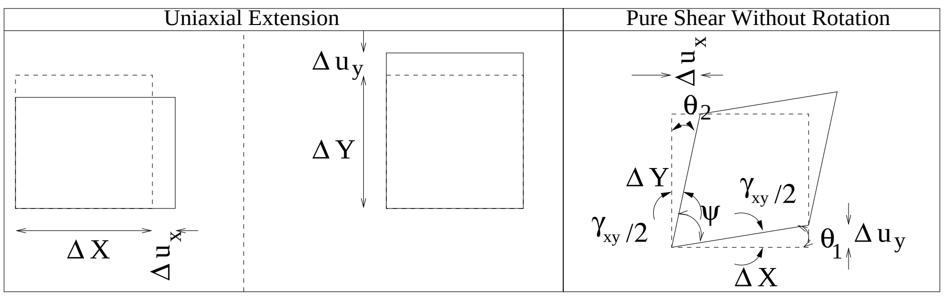


Figure 4.2: Elementary Definition of Strains in 2D

$$\varepsilon_{xx} \approx \frac{\Delta u_x}{\Delta X} \tag{4.6-a}$$

$$\varepsilon_{yy} \approx \frac{\Delta u_y}{\Delta Y} \tag{4.6-b}$$

$$\gamma_{xy} = \frac{\pi}{2} - \psi = \theta_2 + \theta_1 = \frac{\Delta u_x}{\Delta Y} + \frac{\Delta u_y}{\Delta X} \tag{4.6-c}$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} \approx \frac{1}{2}\left(\frac{\Delta u_x}{\Delta Y} + \frac{\Delta u_y}{\Delta X}\right) \tag{4.6-d}$$

In the limit as both ΔX and ΔY approach zero, then

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial X}; \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial Y}; \quad \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial Y} + \frac{\partial u_y}{\partial X}\right) \tag{4.7}$$

We note that in the expression of the shear strain, we used $\tan\theta \approx \theta$ which is applicable as long as θ is small compared to one radian.

7 We have used capital letters to represent the coordinates in the initial state, and lower case letters for the final or current position coordinates ($x = X + u_x$). This corresponds to the Lagrangian strain representation.

4.2 Strain Tensor

8 Following the simplified (and restrictive) introduction to strain, we now turn our attention to a rigorous presentation of this important deformation tensor.

9 The approach we will take in this section is as follows:

1. Define **Material** (fixed, X_i) and **Spatial** (moving, x_i) coordinate systems.
2. Introduce the notion of a **position** and of a **displacement** vector, $\mathbf{U}, \mathbf{u}$, (with respect to either coordinate system).
3. Introduce Lagrangian and Eulerian descriptions.
4. Introduce the notion of a **material deformation gradient** and **spatial deformation gradient**
5. Introduce the notion of a **material displacement gradient** and **spatial displacement gradient**.
6. Define **Cauchy's** and **Green's deformation tensors** (in terms of $(\mathrm{d}X)^2$ and $(dx)^2$ respectively.
7. Introduce the notion of **strain tensor** in terms of $(dx)^2 - (\mathrm{d}X)^2$ as a measure of **deformation** in terms of either spatial coordinates or in terms of displacements.

4.2.1 Position and Displacement Vectors; $(\mathbf{x}, \mathbf{X})$

10 We consider in Fig. 4.3 the undeformed configuration of a material continuum at time $t = 0$ together with the deformed configuration at coordinates for each configuration.

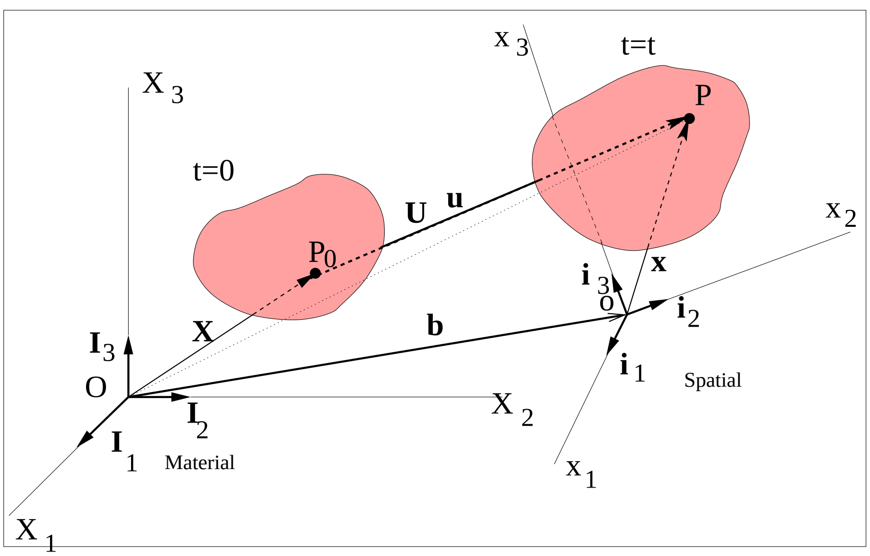


Figure 4.3: Position and Displacement Vectors

11 In the initial configuration P_0 has the **position vector**

$$\mathbf{X} = X_1\mathbf{I}_1 + X_2\mathbf{I}_2 + X_3\mathbf{I}_3 \tag{4.8}$$

which is here expressed in terms of the **material coordinates** (X_1, X_2, X_3).

12 In the deformed configuration, the particle P_0 has now moved to the new position P and has the following position vector

$$\mathbf{x} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3 \tag{4.9}$$

which is expressed in terms of the **spatial coordinates**.

13 Note certain similarity with Fig. 4.1, and Eq. 4.2-4.5 where the strains are defined in terms of l and l_0 rather than the displacement Δl.

14 The relative orientation of the material axes $(OX_1X_2X_3)$ and the spatial axes $(ox_1x_2x_3)$ is specified through the direction cosines $a_{\mathbf{x}}^{\mathbf{X}}$.

15 The displacement vector $\mathbf{u}$ connecting P_0 and P is the **displacement vector** which can be expressed in both the material or spatial coordinates

$$\mathbf{U} = U_K\mathbf{I}_K \tag{4.10-a}$$
$$\mathbf{u} = u_k\mathbf{i}_k \tag{4.10-b}$$

again U_k and u_k are interrelated through the direction cosines $\mathbf{i}_k = a_k^K\mathbf{I}_K$. Substituting above we obtain

$$\mathbf{u} = u_k(a_k^K\mathbf{I}_K) = U_K\mathbf{I}_K = \mathbf{U} \Rightarrow U_K = a_k^K u_k \tag{4.11}$$

16 The vector $\mathbf{b}$ relates the origin o with respect to O. From geometry $\mathbf{X}+\mathbf{u} = \mathbf{b}+\mathbf{x}$, thus $\mathbf{u} = \mathbf{b}+\mathbf{x}-\mathbf{X}$ or if the origins are the same (superimposed axis), Fig. 4.4:

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \tag{4.12-a}$$
$$u_k = x_k - \underbrace{a_k^K}_{\delta_{kK}} X_K \tag{4.12-b}$$
$$u_k = x_k - X_k \tag{4.12-c}$$

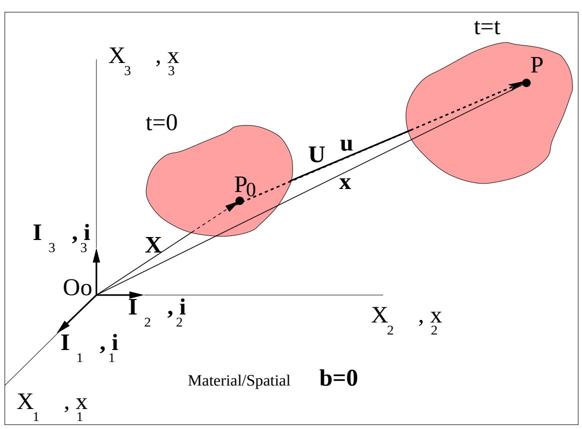


Figure 4.4: Position and Displacement Vectors, $\mathbf{b} = 0$

■ **Example 4-1: Displacement Vectors in Material and Spatial Forms**

With respect to superposed material axis X_i and spatial axes x_i, the displacement field of a continuum body is given by: $x_1 = X_1$, $x_2 = X_2 + AX_3$, and $x_3 = AX_2 + X_3$ where A is constant.

1. Determine the displacement vector components, $\mathbf{u}$, in both the material and spatial form.
2. Determine the displacements of the edges of a cube with edges along the coordinate axes of length $\mathrm{d}X_i = \mathrm{d}X$, and sketch the displaced configuration for $A = 1/2$.
3. Determine the displaced location of material particles which originally comprises the plane circular surface $X_1 = 0$, $X_2^2 + X_3^2 = 1/(1 - A^2)$ if $A = 1/2$.

Solution:

1. From Eq. 4.12-c the displacement field can be written in material coordinates as

$$u_1 = x_1 - X_1 = 0 \quad (4.13\text{-a})$$
$$u_2 = x_2 - X_2 = AX_3 \quad (4.13\text{-b})$$
$$u_3 = x_3 - X_3 = AX_2 \quad (4.13\text{-c})$$

2. The position vector can be written in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \quad (4.14)$$

or upon inversion

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{1}{1 - A^2} \begin{bmatrix} 1 - A^2 & 0 & 0 \\ 0 & 1 & -A \\ 0 & -A & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad (4.15)$$

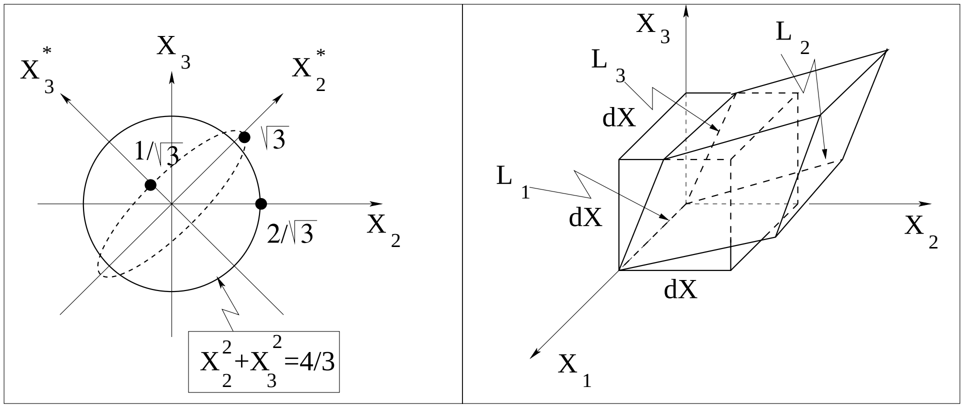
that is $X_1 = x_1$, $X_2 = (x_2 - Ax_3)/(1 - A^2)$, and $X_3 = (x_3 - Ax_2)/(1 - A^2)$.

3. The displacement field can now be written in spatial coordinates as

$$u_1 = x_1 - X_1 = 0 \quad (4.16\text{-a})$$
$$u_2 = x_2 - X_2 = \frac{A(x_3 - Ax_2)}{1 - A^2} \quad (4.16\text{-b})$$
$$u_3 = x_3 - X_3 = \frac{A(x_2 - Ax_3)}{a - A^2} \quad (4.16\text{-c})$$

4. The displacements for the edge of the cube are determined as follows:
   (a) L_1: Edge $X_1 = X_1, X_2 = X_3 = 0$, $u_1 = u_2 = u_3 = 0$
   (b) L_2:$X_1 = X_3 = 0, X_2 = X_2$, $u_1 = u_2 = 0, u_3 = AX_2$.
   (c) L_3: Edge $X_1 = X_2 = 0, X_3 = X_3$, $u_1 = u_3 = 0, u_2 = AX_3$, thus points along this edge are displaced in the X_2 direction proportionally to their distance from the origin.

5. For the circular surface, and by direct substitution of $X_2 = (x_2 - Ax_3)/(1 - A^2)$, and $X_3 = (x_3 - Ax_2)/(1 - A^2)$ in $X_2^2 + X_3^2 = 1/(1 - A^2)$, the circular surface becomes the elliptical surface $(1 + A^2)x_2^2 - 4Ax_2x_3 + (1 + A^2)x_3^2 = (1 - A^2)$ or for $A = 1/2$, $\boxed{5x_2^2 - 8x_2x_3 + 5x_3^2 = 3}$. When expressed in its principal axes, X_i^* (at $\pi/4$), it has the equation $x_2^{*2} + 9x_3^{*2} = 3$



■

4.2.1.1 Lagrangian and Eulerian Descriptions; $\mathbf{x}(\mathbf{X}, t), \mathbf{X}(\mathbf{x}, t)$

17 When the continuum undergoes deformation (or flow), the particles in the continuum move along various paths which can be expressed in either the material coordinates or in the spatial coordinates system giving rise to two different formulations:

Lagrangian Formulation: gives the present location x_i of the particle that occupied the point $(X_1 X_2 X_3)$ at time $t = 0$, and is a mapping of the initial configuration into the current one.

$$\boxed{x_i = x_i(X_1, X_2, X_3, t) \quad \text{or} \quad \mathbf{x} = \mathbf{x}(\mathbf{X}, t)} \tag{4.17}$$

Eulerian Formulation: provides a tracing of its original position of the particle that now occupies the location (x_1, x_2, x_3) at time t, and is a mapping of the current configuration into the initial one.

$$\boxed{X_i = X_i(x_1, x_2, x_3, t) \quad \text{or} \quad \mathbf{X} = \mathbf{X}(\mathbf{x}, t)} \tag{4.18}$$

and the independent variables are the coordinates x_i and t.

18 $(\mathbf{X}, t)$ and $(\mathbf{x}, t)$ are the Lagrangian and Eulerian variables respectivly.

19 If $X(x, t)$ is linear, then the deformation is said to be **homogeneous** and plane sections remain plane.

20 For both formulation to constitute a one-to-one mapping, with continuous partial derivatives, they must be the unique inverses of one another. A necessary and unique condition for the inverse functions to exist is that the determinant of the **Jacobian** should not vanish

$$|J| = \left| \frac{\partial x_i}{\partial X_i} \right| \neq 0 \tag{4.19}$$

For example, the Lagrangian description given by

$$x_1 = X_1 + X_2(e^t - 1); \quad x_2 = X_1(e^{-t} - 1) + X_2; \quad x_3 = X_3 \tag{4.20}$$

has the inverse Eulerian description given by

$$X_1 = \frac{-x_1 + x_2(e^t - 1)}{1 - e^t - e^{-t}}; \quad X_2 = \frac{x_1(e^{-t} - 1) - x_2}{1 - e^t - e^{-t}}; \quad X_3 = x_3 \tag{4.21}$$

■ **Example 4-2: Lagrangian and Eulerian Descriptions**

The Lagrangian description of a deformation is given by $x_1 = X_1 + X_3(e^2 - 1)$, $x_2 = X_2 + X_3(e^2 - e^{-2})$, and $x_3 = e^2 X_3$ where e is a constant. Show that the jacobian does not vanish and determine the Eulerian equations describing the motion.
Solution:

The Jacobian is given by

$$\begin{vmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2-e^{-2}) \\ 0 & 0 & e^2 \end{vmatrix} = e^2 \neq 0 \tag{4.22}$$

Inverting the equation

$$\begin{bmatrix} 1 & 0 & (e^2-1) \\ 0 & 1 & (e^2-e^{-2}) \\ 0 & 0 & e^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & (e^{-2}-1) \\ 0 & 1 & (e^{-4}-1) \\ 0 & 0 & e^{-2} \end{bmatrix} \Rightarrow \begin{cases} X_1 &= x_1 + (e^{-2}-1)x_3 \\ X_2 &= x_2 + (e^{-4}-1)x_3 \\ X_3 &= e^{-2}x_3 \end{cases} \tag{4.23}$$

■

4.2.2 Gradients

4.2.2.1 Deformation; $(\mathbf{x}\nabla_{\mathbf{X}}, \mathbf{X}\nabla_{\mathbf{x}})$

21 Partial differentiation of Eq. 4.17 with respect to X_j produces the tensor $\partial x_i/\partial X_j$ which is the **material deformation gradient**. In symbolic notation $\partial x_i/\partial X_j$ is represented by the dyadic

$$\boxed{\mathbf{F} \equiv \mathbf{x}\nabla_{\mathbf{X}} = \frac{\partial \mathbf{x}}{\partial X_1}\mathbf{e}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\mathbf{e}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\mathbf{e}_3 = \frac{\partial x_i}{\partial X_j}} \tag{4.24}$$

The matrix form of $\mathbf{F}$ is

$$\mathcal{F} = \left\{ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right\} \lfloor \begin{array}{ccc} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} \end{array} \rfloor = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \left[\frac{\partial x_i}{\partial X_j}\right] \tag{4.25}$$

22 Similarly, differentiation of Eq. 4.18 with respect to x_j produces the **spatial deformation gradient**

$$\boxed{\mathbf{H} = \mathbf{X}\nabla_{\mathbf{x}} \equiv \frac{\partial \mathbf{X}}{\partial x_1}\mathbf{e}_1 + \frac{\partial \mathbf{X}}{\partial x_2}\mathbf{e}_2 + \frac{\partial \mathbf{X}}{\partial x_3}\mathbf{e}_3 = \frac{\partial X_i}{\partial x_j}} \tag{4.26}$$

The matrix form of $\mathbf{H}$ is

$$\mathcal{H} = \left\{ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \right\} \lfloor \begin{array}{ccc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{array} \rfloor = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} = \left[\frac{\partial X_i}{\partial x_j}\right] \tag{4.27}$$

23 The material and spatial deformation tensors are interrelated through the chain rule

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k} = \frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k} = \delta_{ik} \tag{4.28}$$

and thus $\mathcal{F}^{-1} = \mathcal{H}$ or

$$\boxed{\mathbf{H} = \mathbf{F}^{-1}} \tag{4.29}$$

24 The deformation gradient characterizes the rate of change of deformation with respect to coordinates. It reflects the stretching and rotation of the domain in the infinitesimal neighborhood at point $\mathbf{x}$ (or $\mathbf{X}$).

25 The deformation gradient is often called a **two point tensor** because the basis $\mathbf{e}_i \otimes \mathbf{E}_j$ has one leg in the spatial (deformed), and the other in the material (undeformed) configuration.

26 $\mathbf{F}$ is a tensor of order two which when operating on a unit tangent vector in the undeformed configuration will produce a tangent vector in the deformed configuration. Similarly $\mathbf{H}$ is a tensor of order two which when operating on a unit tangent vector in the deformed configuration will produce a tangent vector in the undeformed configuration.

4.2.2.1.1 † Change of Area Due to Deformation 27 In order to facilitate the derivation of the **Piola-Kirchoff** stress tensor later on, we need to derive an expression for the change in area due to deformation.

28 If we consider two material element $\mathrm{d}\mathbf{X}^{(1)} = \mathrm{d}X_1\mathbf{e}_1$ and $\mathrm{d}\mathbf{X}^{(2)} = \mathrm{d}X_2\mathbf{e}_2$ emanating from $\mathbf{X}$, the rectangular area formed by them at the reference time t_0 is

$$\mathrm{d}\mathbf{A}_0 = \mathrm{d}\mathbf{X}^{(1)} \times \mathrm{d}\mathbf{X}^{(2)} = \mathrm{d}X_1 \mathrm{d}X_2 \mathbf{e}_3 = \mathrm{d}A_0 \mathbf{e}_3 \tag{4.30}$$

29 At time t, $\mathrm{d}\mathbf{X}^{(1)}$ deforms into $d\mathbf{x}^{(1)} = \mathbf{F}\mathrm{d}\mathbf{X}^{(1)}$ and $\mathrm{d}\mathbf{X}^{(2)}$ into $d\mathbf{x}^{(2)} = \mathbf{F}\mathrm{d}\mathbf{X}^{(2)}$, and the new area is

$$\mathrm{d}\mathbf{A} = \mathbf{F}\mathrm{d}\mathbf{X}^{(1)} \times \mathbf{F}\mathrm{d}\mathbf{X}^{(2)} = \mathrm{d}X_1 \mathrm{d}X_2 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 = \mathrm{d}A_0 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \tag{4.31-a}$$

$$= \mathrm{d}A\mathbf{n} \tag{4.31-b}$$

where the orientation of the deformed area is normal to $\mathbf{F}\mathbf{e}_1$ and $\mathbf{F}\mathbf{e}_2$ which is denoted by the unit vector $\mathbf{n}$. Thus,

$$\mathbf{F}\mathbf{e}_1 \cdot \mathrm{d}A\mathbf{n} = \mathbf{F}\mathbf{e}_2 \cdot \mathrm{d}A\mathbf{n} = 0 \tag{4.32}$$

and recalling that $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$ is equal to the determinant whose rows are components of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{F}\mathbf{e}_3 \cdot d\mathbf{A} = \mathrm{d}A_0 \underbrace{(\mathbf{F}\mathbf{e}_3 \cdot \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2)}_{\det(\mathbf{F})} \tag{4.33}$$

or

$$\mathbf{e}_3 \cdot \mathbf{F}^T \mathbf{n} = \frac{\mathrm{d}A_0}{\mathrm{d}A} \det(\mathbf{F}) \tag{4.34}$$

and $\mathbf{F}^T\mathbf{n}$ is in the direction of $\mathbf{e}_3$ so that

$$\mathbf{F}^T \mathbf{n} = \frac{\mathrm{d}A_0}{\mathrm{d}A} \det \mathbf{F}\mathbf{e}_3 \Rightarrow \mathrm{d}A\mathbf{n} = \mathrm{d}A_0 \det(\mathbf{F})(\mathbf{F}^{-1})^T \mathbf{e}_3 \tag{4.35}$$

which implies that the deformed area has a normal in the direction of $(\mathbf{F}^{-1})^T\mathbf{e}_3$. A generalization of the preceding equation would yield

$$\boxed{\mathrm{d}A\mathbf{n} = \mathrm{d}A_0 \det(\mathbf{F})(\mathbf{F}^{-1})^T \mathbf{n}_0} \tag{4.36}$$

4.2.2.1.2 † Change of Volume Due to Deformation 30 If we consider an infinitesimal element it has the following volume in material coordinate system:

$$\mathrm{d}\Omega_0 = (\mathrm{d}X_1 \mathbf{e}_1 \times \mathrm{d}X_2 \mathbf{e}_2) \cdot \mathrm{d}X_3 \mathbf{e}_3 = \mathrm{d}X_1 \mathrm{d}X_2 \mathrm{d}X_3 \tag{4.37}$$

in spatial cordiantes:

$$\mathrm{d}\Omega = (\mathrm{d}x_1 \mathbf{e}_1 \times \mathrm{d}x_2 \mathbf{e}_2) \cdot \mathrm{d}x_3 \mathbf{e}_3 \tag{4.38}$$

If we define

$$\mathbf{F}_i = \frac{\partial x_i}{\partial X_j}\mathbf{e}_i \tag{4.39}$$

then the deformed volume will be

$$\mathrm{d}\Omega = (\mathbf{F}_1\mathrm{d}X_1 \times \mathbf{F}_2\mathrm{d}X_2)\cdot\mathbf{F}_3\mathrm{d}X_3 = (\mathbf{F}_1 \times \mathbf{F}_2 \cdot \mathbf{F}_3)\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3 \tag{4.40}$$

or

$$\boxed{d\Omega = \det \mathbf{F}\mathrm{d}\Omega_0} \tag{4.41}$$

and J is called the **Jacobian** and is the determinant of the deformation gradient **F**

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix} \tag{4.42}$$

and thus the **Jacobian is a measure of deformation.**

31 We observe that if a material is **incompressible** than $\det \mathbf{F} = 1$.

■ Example 4-3: Change of Volume and Area

For the following deformation: $x_1 = \lambda_1 X_1$, $x_2 = -\lambda_3 X_3$, and $x_3 = \lambda_2 X_2$, find the deformed volume for a unit cube and the deformed area of the unit square in the $X_1 - X_2$ plane.
Solution:

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \tag{4.43-a}$$

$$\det \mathbf{F} = \lambda_1\lambda_2\lambda_3 \tag{4.43-b}$$

$$\Delta V = \lambda_1\lambda_2\lambda_3 \tag{4.43-c}$$

$$\Delta A_0 = 1 \tag{4.43-d}$$

$$\mathbf{n}_0 = -\mathbf{e}_3 \tag{4.43-e}$$

$$\Delta A\mathbf{n} = (1)(\det \mathbf{F})(\mathbf{F}^{-1})^T \tag{4.43-f}$$

$$= \lambda_1\lambda_2\lambda_3 \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda_3} \\ 0 & \frac{1}{\lambda_2} & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \lambda_1\lambda_2 \\ 0 \end{Bmatrix} \tag{4.43-g}$$

$$\Delta A\mathbf{n} = \lambda_1\lambda_2\mathbf{e}_2 \tag{4.43-h}$$

■

4.2.2.2 Displacements; $(\mathbf{u}\nabla_{\mathbf{X}}, \mathbf{u}\nabla_{\mathbf{x}})$

32 We now turn our attention to the displacement vector u_i as given by Eq. 4.12-c. Partial differentiation of Eq. 4.12-c with respect to X_j produces the **material displacement gradient**

$$\boxed{\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij} \quad \text{or} \quad \mathbf{J} \equiv \mathbf{u}\nabla_{\mathbf{X}} = \mathbf{F} - \mathbf{I}} \tag{4.44}$$

The matrix form of $\mathbf{J}$ is

$$\mathcal{J} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \lfloor \frac{\partial}{\partial X_1} \quad \frac{\partial}{\partial X_2} \quad \frac{\partial}{\partial X_3} \rfloor = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} = \left[\frac{\partial u_i}{\partial X_j} \right] \tag{4.45}$$

33 Similarly, differentiation of Eq. 4.12-c with respect to x_j produces the **spatial displacement gradient**

$$\boxed{\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j} \quad \text{or} \quad \mathbf{K} \equiv \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = \mathbf{I} - \mathbf{H}} \tag{4.46}$$

The matrix form of $\mathbf{K}$ is

$$\mathcal{K} = \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right\} \lfloor \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \frac{\partial}{\partial x_3} \rfloor = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \left[\frac{\partial u_i}{\partial x_j} \right] \tag{4.47}$$

4.2.2.3 Examples

■ Example 4-4: Material Deformation and Displacement Gradients

A displacement field is given by $\mathbf{u} = X_1 X_3^2 \mathbf{e}_1 + X_1^2 X_2 \mathbf{e}_2 + X_2^2 X_3 \mathbf{e}_3$, determine the material deformation gradient $\mathbf{F}$ and the material displacement gradient $\mathbf{J}$, and verify that $\mathbf{J} = \mathbf{F} - \mathbf{I}$.

Solution:

1. Since $\mathbf{x} = \mathbf{u} + \mathbf{X}$, the displacement field is given by

$$\mathbf{x} = \underbrace{X_1(1 + X_3^2)}_{x_1} \mathbf{e}_1 + \underbrace{X_2(1 + X_1^2)}_{x_2} \mathbf{e}_2 + \underbrace{X_3(1 + X_2^2)}_{x_3} \mathbf{e}_3 \tag{4.48}$$

2. Thus

$$\mathbf{F} = \mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial X_1}\mathbf{e}_1 + \frac{\partial \mathbf{x}}{\partial X_2}\mathbf{e}_2 + \frac{\partial \mathbf{x}}{\partial X_3}\mathbf{e}_3 = \frac{\partial x_i}{\partial X_j} \tag{4.49-a}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \tag{4.49-b}$$

$$= \begin{bmatrix} 1 + X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & 1 + X_1^2 & 0 \\ 0 & 2X_2X_3 & 1 + X_2^2 \end{bmatrix} \tag{4.49-c}$$

3. The material deformation gradient is:

$$\frac{\partial u_i}{\partial X_j} = \mathbf{J} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = = \begin{bmatrix} \frac{\partial u_{X_1}}{\partial X_1} & \frac{\partial u_{X_1}}{\partial X_2} & \frac{\partial u_{X_1}}{\partial X_3} \\ \frac{\partial u_{X_2}}{\partial X_1} & \frac{\partial u_{X_2}}{\partial X_2} & \frac{\partial u_{X_2}}{\partial X_3} \\ \frac{\partial u_{X_3}}{\partial X_1} & \frac{\partial u_{X_3}}{\partial X_2} & \frac{\partial u_{X_3}}{\partial X_3} \end{bmatrix} \tag{4.50-a}$$

$$= \begin{bmatrix} X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2 & 0 \\ 0 & 2X_2X_3 & X_2^2 \end{bmatrix} \tag{4.50-b}$$

We observe that the two second order tensors are related by $\mathbf{J} = \mathbf{F} - \mathbf{I}$.

■

4.2.3 Deformation Tensors

34 The deofrmation gradients, previously presented, can not be used to determine strains as embedded in them is rigid body motion.

35 Having derived expressions for $\frac{\partial x_i}{\partial X_j}$ and $\frac{\partial X_i}{\partial x_j}$ we now seek to determine dx^2 and dX^2 where dX and dx correspond to the distance between points P and Q in the undeformed and deformed cases respectively.

36 We consider next the initial (undeformed) and final (deformed) configuration of a continuum in which the material OX_1, X_2, X_3 and spatial coordinates $ox_1x_2x_3$ are superimposed. Neighboring particles P_0 and Q_0 in the initial configurations moved to P and Q respectively in the final one, Fig. 4.5.

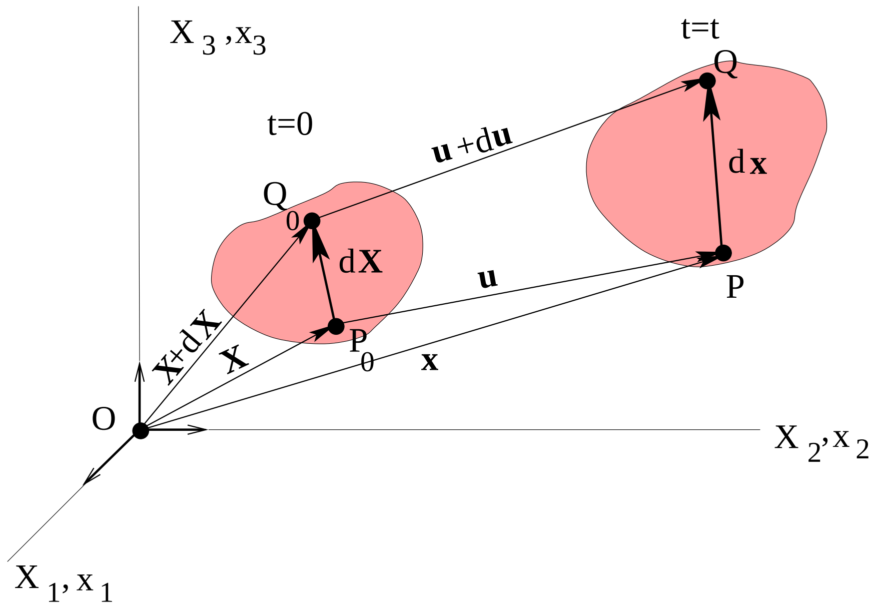


Figure 4.5: Undeformed and Deformed Configurations of a Continuum

4.2.3.1 Cauchy's Deformation Tensor; $(dX)^2$

37 The Cauchy deformation tensor, introduced by Cauchy in 1827, $\mathbf{B}^{-1}$ (alternatively denoted as $\mathbf{c}$) gives the initial square length $(dX)^2$ of an element $d\mathbf{x}$ in the deformed configuration.

38 This tensor is the inverse of the tensor $\mathbf{B}$ which will not be introduced until Sect. 4.3.2.

39 The square of the differential element connecting P_o and Q_0 is

$$(dX)^2 = d\mathbf{X}\cdot d\mathbf{X} = dX_i dX_i \tag{4.51}$$

however from Eq. 4.18 the distance differential dX_i is

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j \quad \text{or} \quad d\mathbf{X} = \mathbf{H}\cdot d\mathbf{x} \tag{4.52}$$

thus the squared length $(dX)^2$ in Eq. 4.51 may be rewritten as

$$(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j = B^{-1}_{ij} dx_i dx_j \tag{4.53-a}$$

$$= d\mathbf{x}\cdot\mathbf{B}^{-1}\cdot d\mathbf{x} \tag{4.53-b}$$

in which the second order tensor

$$\boxed{B_{ij}^{-1} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} \text{ or } \mathbf{B}^{-1} = \underbrace{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{X}\cdot\mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}}}_{\mathbf{H}_c\cdot\mathbf{H}}} \tag{4.54}$$

is **Cauchy's deformation tensor**. It relates $(\mathrm{d}X)^2$ to $(\mathrm{d}x)^2$.

4.2.3.2 Green's Deformation Tensor; $(\mathrm{d}x)^2$

40 The Green deformation tensor, introduced by Green in 1841, **C** (alternatively denoted as $\mathbf{B}^{-1}$), referred to in the undeformed configuration, gives the new square length $(\mathrm{d}x)^2$ of the element $\mathrm{d}\mathbf{X}$ is deformed.

41 The square of the differential element connecting P_o and Q_0 is now evaluated in terms of the spatial coordinates

$$(\mathrm{d}x)^2 = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}x_i\mathrm{d}x_i \tag{4.55}$$

however from Eq. 4.17 the distance differential $\mathrm{d}x_i$ is

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j \text{ or } \mathrm{d}\mathbf{x} = \mathbf{F}\cdot\mathrm{d}\mathbf{X} \tag{4.56}$$

thus the squared length $(\mathrm{d}x)^2$ in Eq. 4.55 may be rewritten as

$$(\mathrm{d}x)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}\mathrm{d}X_i\mathrm{d}X_j = C_{ij}\mathrm{d}X_i\mathrm{d}X_j \tag{4.57-a}$$

$$= \mathrm{d}\mathbf{X}\cdot\mathbf{C}\cdot\mathrm{d}\mathbf{X} \tag{4.57-b}$$

in which the second order tensor

$$\boxed{C_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} \text{ or } \mathbf{C} = \underbrace{\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x}\cdot\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}}}_{\mathbf{F}_c\cdot\mathbf{F}}} \tag{4.58}$$

is **Green's deformation tensor** also known as **metric tensor**, or **deformation tensor** or **right Cauchy-Green deformation tensor**. It relates $(\mathrm{d}x)^2$ to $(\mathrm{d}X)^2$.

42 Inspection of Eq. 4.54 and Eq. 4.58 yields

$$\boxed{\mathbf{C}^{-1} = \mathbf{B}^{-1} \text{ or } \mathbf{B}^{-1} = (\mathbf{F}^{-1})^T\cdot\mathbf{F}^{-1}} \tag{4.59}$$

■ **Example 4-5: Green's Deformation Tensor**

A continuum body undergoes the deformation $x_1 = X_1$, $x_2 = X_2 + AX_3$, and $x_3 = X_3 + AX_2$ where A is a constant. Determine the deformation tensor **C**.
Solution:

From Eq. 4.58 $\mathbf{C} = \mathbf{F}_c\cdot\mathbf{F}$ where **F** was defined in Eq. 4.24 as

$$\mathbf{F} = \frac{\partial x_i}{\partial X_j} \tag{4.60-a}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} \tag{4.60-b}$$

and thus

$$\mathbf{C} = \mathbf{F}_c{\cdot}\mathbf{F} \tag{4.61-a}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} \tag{4.61-b}$$

■

4.2.4 Strains; $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$

43 With $(\mathrm{d}x)^2$ and $(\mathrm{d}X)^2$ defined we can now finally introduce the concept of strain through $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$.

4.2.4.1 Finite Strain Tensors

44 We start with the most general case of finite strains where no constraints are imposed on the deformation (small).

4.2.4.1.1 Lagrangian/Green's Strain Tensor

45 The difference $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$ for two neighboring particles in a continuum is used as the **measure of deformation**. Using Eqs. 4.57-a and 4.51 this difference is expressed as

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)\mathrm{d}X_i\mathrm{d}X_j = 2E_{ij}\mathrm{d}X_i\mathrm{d}X_j \tag{4.62-a}$$

$$= \mathrm{d}\mathbf{X}{\cdot}(\mathbf{F}_c{\cdot}\mathbf{F} - \mathbf{I}){\cdot}\mathrm{d}\mathbf{X} = 2\mathrm{d}\mathbf{X}{\cdot}\mathbf{E}{\cdot}d\mathbf{X} \tag{4.62-b}$$

in which the second order tensor

$$E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\nabla_{\mathbf{X}}\mathbf{x}{\cdot}\mathbf{x}\nabla_{\mathbf{X}}}_{\mathbf{F}_c\cdot\mathbf{F}=\mathbf{C}} - \mathbf{I}) \tag{4.63}$$

is called the **Lagrangian (or Green's) finite strain tensor** which was introduced by Green in 1841 and St-Venant in 1844.

46 The Lagrangian stress tensor is one half the difference between the Green deformation tensor and **I**.

47 Note similarity with Eq. 4.4 where the Lagrangian strain (in 1D) was defined as the difference between the square of the deformed length and the square of the original length divided by twice the square of the original length ($E \equiv \frac{1}{2}\left(\frac{l^2-l_0^2}{l_0^2}\right)$). Eq. 4.62-a can be rewritten as

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = 2E_{ij}\mathrm{d}X_i\mathrm{d}X_j \Rightarrow E_{ij} = \frac{(\mathrm{d}x)^2 - (\mathrm{d}X)^2}{2\mathrm{d}X_i\mathrm{d}X_j} \tag{4.64}$$

which gives a clearer physical meaning to the Lagrangian Tensor.

48 To express the Lagrangian tensor in terms of the displacements, we substitute Eq. 4.44 in the preceding equation, and after some simple algebraic manipulations, the Lagrangian finite strain tensor can be rewritten as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\mathbf{u}\nabla_{\mathbf{X}} + \nabla_{\mathbf{X}}\mathbf{u}}_{\mathbf{J}+\mathbf{J}_c} + \underbrace{\nabla_{\mathbf{X}}\mathbf{u}{\cdot}\mathbf{u}\nabla_{\mathbf{X}}}_{\mathbf{J}_c\cdot\mathbf{J}}) \tag{4.65}$$

or:

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right] \quad (4.66\text{-a})$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) + \frac{1}{2}\left[\frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right] \quad (4.66\text{-b})$$

$$\cdots = \cdots \quad (4.66\text{-c})$$

■ Example 4-6: Lagrangian Tensor

Determine the Lagrangian finite strain tensor **E** for the deformation of example 4.2.3.2.
Solution:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} \quad (4.67\text{-a})$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \quad (4.67\text{-b})$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{bmatrix} \quad (4.67\text{-c})$$

Note that the matrix is symmetric. ■

4.2.4.1.2 Eulerian/Almansi's Tensor

49 Alternatively, the difference $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$ for the two neighboring particles in the continuum can be expressed in terms of Eqs. 4.55 and 4.53-b this same difference is now equal to

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)\mathrm{d}x_i\mathrm{d}x_j = 2E^*_{ij}\mathrm{d}x_i\mathrm{d}x_j \quad (4.68\text{-a})$$

$$= \mathrm{d}\mathbf{x}\cdot(\mathbf{I} - \mathbf{H}_c\cdot\mathbf{H})\cdot\mathrm{d}\mathbf{x} = 2\mathrm{d}\mathbf{x}\cdot\mathbf{E}^*\cdot d\mathbf{x} \quad (4.68\text{-b})$$

in which the second order tensor

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}(\mathbf{I} - \underbrace{\boldsymbol{\nabla}_\mathbf{x}\mathbf{X}\cdot\mathbf{X}\boldsymbol{\nabla}_\mathbf{x}}_{\mathbf{H}_c\cdot\mathbf{H}=\mathbf{B}^{-1}})} \quad (4.69)$$

is called the **Eulerian (or Almansi) finite strain tensor**.

50 The Eulerian strain tensor is one half the difference between **I** and the Cauchy deformation tensor.

51 Note similarity with Eq. 4.5 where the Eulerian strain (in 1D) was defined as the difference between the square of the deformed length and the square of the original length divided by twice the square of the deformed length ($E^* \equiv \frac{1}{2}\left(\frac{l^2-l_0^2}{l^2}\right)$). Eq. 4.68-a can be rewritten as

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = 2E^*_{ij}\mathrm{d}x_i\mathrm{d}x_j \Rightarrow E^*_{ij} = \frac{(\mathrm{d}x)^2 - (\mathrm{d}X)^2}{2\mathrm{d}x_i\mathrm{d}x_j} \quad (4.70)$$

which gives a clearer physical meaning to the Eulerian Tensor.

52 For infinitesimal strain it was introduced by Cauchy in 1827, and for finite strain by Almansi in 1911.

53 To express the Eulerian tensor in terms of the displacements, we substitute 4.46 in the preceding equation, and after some simple algebraic manipulations, the Eulerian finite strain tensor can be rewritten as

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}}_{\mathbf{K}+\mathbf{K}_c} - \underbrace{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\cdot\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}}_{\mathbf{K}_c\cdot\mathbf{K}})} \tag{4.71}$$

54 Expanding

$$E^*_{11} = \frac{\partial u_1}{\partial x_1} - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{4.72-a}$$

$$E^*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) - \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right] \tag{4.72-b}$$

$$\cdots = \cdots \tag{4.72-c}$$

4.2.4.2 Infinitesimal Strain Tensors; Small Deformation Theory

55 The **small deformation theory** of continuum mechanics has as basic condition the requirement that the displacement gradients be small compared to unity. The fundamental measure of deformation is the difference $(\mathrm{d}x)^2 - (\mathrm{d}X)^2$, which may be expressed in terms of the displacement gradients by inserting Eq. 4.65 and 4.71 into 4.62-b and 4.68-b respectively. If the displacement gradients are small, the finite strain tensors in Eq. 4.62-b and 4.68-b reduce to **infinitesimal strain tensors** and the resulting equations represent **small deformations**.

56 For instance, if we were to evaluate $\epsilon + \epsilon^2$, for $\epsilon = 10^{-3}$ and 10^{-1}, then we would obtain $0.001001 \approx 0.001$ and 0.11 respectively. In the first case ϵ^2 is "negligible" compared to ϵ, in the other it is not.

4.2.4.2.1 Lagrangian Infinitesimal Strain Tensor

57 In Eq. 4.65 if the displacement gradient components $\frac{\partial u_i}{\partial X_j}$ are each small compared to unity, then the third term are negligible and may be dropped. The resulting tensor is the **Lagrangian infinitesimal strain tensor** denoted by

$$\boxed{E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}}_{\mathbf{J}+\mathbf{J}_c})} \tag{4.73}$$

or:

$$E_{11} = \frac{\partial u_1}{\partial X_1} \tag{4.74-a}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) \tag{4.74-b}$$

$$\cdots = \cdots \tag{4.74-c}$$

Note the similarity with Eq. 4.7.

4.2.4.2.2 Eulerian Infinitesimal Strain Tensor

58 Similarly, inn Eq. 4.71 if the displacement gradient components $\frac{\partial u_i}{\partial x_j}$ are each small compared to unity, then the third term are negligible and may be dropped. The resulting tensor is the **Eulerian infinitesimal strain tensor** denoted by

$$\boxed{E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \mathbf{E}^* = \frac{1}{2}(\underbrace{\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}}_{\mathbf{K}+\mathbf{K}_c})} \tag{4.75}$$

59 Expanding

$$E^*_{11} = \frac{\partial u_1}{\partial x_1} \tag{4.76-a}$$

$$E^*_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{4.76-b}$$

$$\cdots = \cdots \tag{4.76-c}$$

4.2.4.3 Examples

■ **Example 4-7: Lagrangian and Eulerian Linear Strain Tensors**

A displacement field is given by $x_1 = X_1 + AX_2, x_2 = X_2 + AX_3, x_3 = X_3 + AX_1$ where A is constant. Calculate the Lagrangian and the Eulerian linear strain tensors, and compare them for the case where A is very small.

Solution:

The displacements are obtained from Eq. 4.12-c $u_k = x_k - X_k$ or

$$u_1 = x_1 - X_1 = X_1 + AX_2 - X_1 = AX_2 \tag{4.77-a}$$

$$u_2 = x_2 - X_2 = X_2 + AX_3 - X_2 = AX_3 \tag{4.77-b}$$

$$u_3 = x_3 - X_3 = X_3 + AX_1 - X_3 = AX_1 \tag{4.77-c}$$

then from Eq. 4.44

$$\mathbf{J} \equiv \mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix} \tag{4.78}$$

From Eq. 4.73:

$$2\mathbf{E} = (\mathbf{J} + \mathbf{J}_c) = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A \\ A & 0 & 0 \\ 0 & A & 0 \end{bmatrix} \tag{4.79-a}$$

$$= \begin{bmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{bmatrix} \tag{4.79-b}$$

To determine the Eulerian tensor, we need the displacement u in terms of x, thus inverting the displacement field given above:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} 1 & A & 0 \\ 0 & 1 & A \\ A & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} \Rightarrow \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \frac{1}{1+A^3} \begin{bmatrix} 1 & -A & A^2 \\ A^2 & 1 & -A \\ -A & A^2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{4.80}$$

thus from Eq. 4.12-c $u_k = x_k - X_k$ we obtain

$$u_1 = x_1 - X_1 = x_1 - \frac{1}{1+A^3}(x_1 - Ax_2 + A^2x_3) = \frac{A(A^2x_1 + x_2 - Ax_3)}{1+A^3} \quad (4.81\text{-a})$$

$$u_2 = x_2 - X_2 = x_2 - \frac{1}{1+A^3}(A^2x_1 + x_2 - Ax_3) = \frac{A(-Ax_1 + A^2x_2 + x_3)}{1+A^3} \quad (4.81\text{-b})$$

$$u_3 = x_3 - X_3 = x_3 - \frac{1}{1+A^3}(-Ax_1 + A^2x_2 + x_3) = \frac{A(x_1 - Ax_2 + A^2x_3)}{1+A^3} \quad (4.81\text{-c})$$

From Eq. 4.46

$$\mathbf{K} \equiv \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = \frac{A}{1+A^3}\begin{bmatrix} A^2 & 1 & -A \\ -A & A^2 & 1 \\ 1 & -A & A^2 \end{bmatrix} \quad (4.82)$$

Finally, from Eq. 4.71

$$2\mathbf{E}^* = \mathbf{K} + \mathbf{K}_c \quad (4.83\text{-a})$$

$$= \frac{A}{1+A^3}\begin{bmatrix} A^2 & 1 & -A \\ -A & A^2 & 1 \\ 1 & -A & A^2 \end{bmatrix} + \frac{A}{1+A^3}\begin{bmatrix} A^2 & -A & 1 \\ 1 & A^2 & -A \\ -A & 1 & A^2 \end{bmatrix} \quad (4.83\text{-b})$$

$$= \frac{A}{1+A^3}\begin{bmatrix} 2A^2 & 1-A & 1-A \\ 1-A & 2A^2 & 1-A \\ 1-A & 1-A & 2A^2 \end{bmatrix} \quad (4.83\text{-c})$$

as A is very small, A^2 and higher power may be neglected with the results, then $\mathbf{E}^* \to \mathbf{E}$. ■

4.2.5 †Physical Interpretation of the Strain Tensor

4.2.5.1 Small Strain

60 We finally show that the linear lagrangian tensor in small deformation E_{ij} is nothing else than the strain as was defined earlier in Eq.4.7.

61 We rewrite Eq. 4.62-b as

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = (\mathrm{d}x - \mathrm{d}X)(\mathrm{d}x + \mathrm{d}X) = 2E_{ij}\mathrm{d}X_i\mathrm{d}X_j \quad (4.84\text{-a})$$

or

$$(\mathrm{d}x)^2 - (\mathrm{d}X)^2 = (\mathrm{d}x - \mathrm{d}X)(\mathrm{d}x + \mathrm{d}X) = \mathrm{d}\mathbf{X}{\cdot}2\mathbf{E}{\cdot}\mathrm{d}\mathbf{X} \quad (4.84\text{-b})$$

but since $\mathrm{d}x \approx \mathrm{d}X$ under current assumption of small deformation, then the previous equation can be rewritten as

$$\frac{\overbrace{\mathrm{d}x - \mathrm{d}X}^{\mathrm{d}u}}{\mathrm{d}X} = E_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}X}\frac{\mathrm{d}X_j}{\mathrm{d}X} = E_{ij}\xi_i\xi_j = \boldsymbol{\xi}{\cdot}\mathbf{E}{\cdot}\boldsymbol{\xi} \quad (4.85)$$

62 We recognize that the left hand side is nothing else than the change in length per unit original length, and is called the **normal strain** for the line element having direction cosines $\frac{\mathrm{d}X_i}{\mathrm{d}X}$.

63 With reference to Fig. 4.6 we consider two cases: normal and shear strain.

Normal Strain: When Eq. 4.85 is applied to the differential element P_0Q_0 which lies along the X_2 axis, the result will be the normal strain because since $\frac{\mathrm{d}X_1}{\mathrm{d}X} = \frac{\mathrm{d}X_3}{\mathrm{d}X} = 0$ and $\frac{\mathrm{d}X_2}{\mathrm{d}X} = 1$. Therefore, Eq. 4.85 becomes (with $u_i = x_i - X_i$):

$$\boxed{\frac{\mathrm{d}x - \mathrm{d}X}{\mathrm{d}X} = E_{22} = \frac{\partial u_2}{\partial X_2}} \quad (4.86)$$

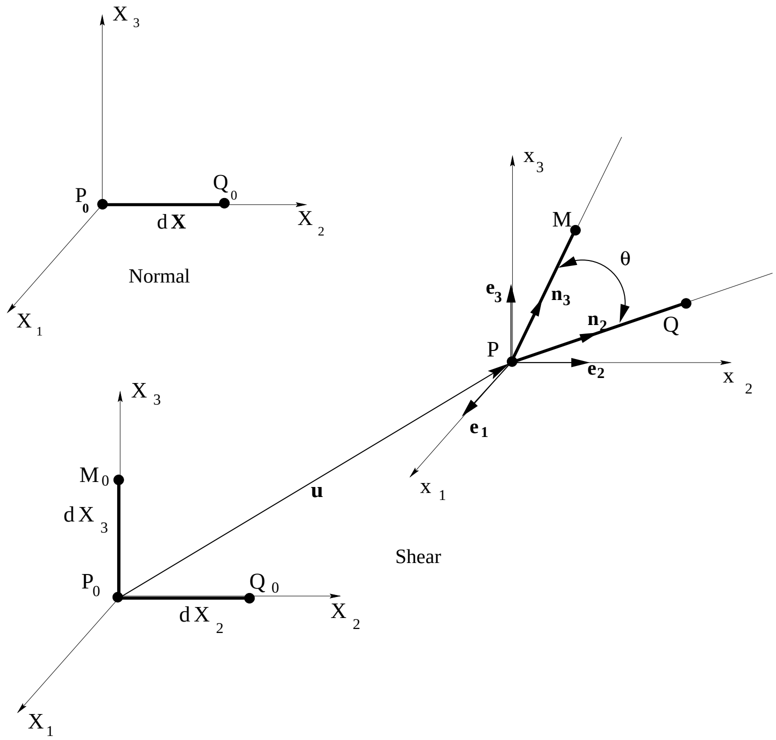


Figure 4.6: Physical Interpretation of the Strain Tensor

Likewise for the other 2 directions. Hence the diagonal terms of the linear strain tensor represent normal strains in the coordinate system.

Shear Strain: For the diagonal terms E_{ij} we consider the two line elements originally located along the X_2 and the X_3 axes before deformation. After deformation, the original right angle between the lines becomes the angle θ. From Eq. 4.101 ($\mathrm{d}u_i = \left(\frac{\partial u_i}{\partial X_j}\right)_{P_0} \mathrm{d}X_j$) a first order approximation gives the unit vector at P in the direction of Q, and M as:

$$\mathbf{n}_2 = \frac{\partial u_1}{\partial X_2}\mathbf{e}_1 + \mathbf{e}_2 + \frac{\partial u_3}{\partial X_2}\mathbf{e}_3 \tag{4.87-a}$$

$$\mathbf{n}_3 = \frac{\partial u_1}{\partial X_3}\mathbf{e}_1 + \frac{\partial u_2}{\partial X_3}\mathbf{e}_2 + \mathbf{e}_3 \tag{4.87-b}$$

and from the definition of the dot product:

$$\cos\theta = \mathbf{n}_2 \cdot \mathbf{n}_3 = \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{4.88}$$

or neglecting the higher order term

$$\boxed{\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2E_{23}} \tag{4.89}$$

64 Finally taking the change in right angle between the elements as $\gamma_{23} = \pi/2 - \theta$, and recalling that for small strain theory γ_{23} is very small it follows that

$$\boxed{\gamma_{23} \approx \sin\gamma_{23} = \sin(\pi/2 - \theta) = \cos\theta = 2E_{23}.} \tag{4.90}$$

Therefore the off diagonal terms of the linear strain tensor represent one half of the angle change between two line elements originally at right angles to one another. These components are called the **shear strains**.

64 The **Engineering shear strain** is defined as one half the tensorial shear strain, and the resulting tensor is written as

$$\boxed{E_{ij} = \begin{bmatrix} \varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{12} & \varepsilon_{22} & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_{33} \end{bmatrix}} \tag{4.91}$$

65 We note that a similar development paralleling the one just presented can be made for the linear Eulerian strain tensor (where the straight lines and right angle will be in the deformed state).

4.2.5.2 Finite Strain; Stretch Ratio

66 The simplest and most useful measure of the extensional strain of an infinitesimal element is the **stretch** or **stretch ratio** as $\frac{\mathrm{d}x}{\mathrm{d}X}$ which may be defined at point P_0 in the undeformed configuration or at P in the deformed one (Refer to the original definition given by Eq, 4.1).

67 Hence, from Eq. 4.57-a, and Eq. 4.63 the squared stretch at P_0 for the line element along the unit vector $\mathbf{m} = \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}X}$ is given by

$$\Lambda^2_{\mathbf{m}} \equiv \left(\frac{\mathrm{d}x}{\mathrm{d}X}\right)^2_{P_0} = C_{ij}\frac{\mathrm{d}X_i}{\mathrm{d}X}\frac{\mathrm{d}X_j}{\mathrm{d}X} \quad \text{or} \quad \Lambda^2_{\mathbf{m}} = \mathbf{m}{\cdot}\mathbf{C}{\cdot}\mathbf{m} \tag{4.92}$$

Thus for an element originally along X_2, Fig. 4.6, $\mathbf{m} = \mathbf{e}_2$ and therefore $\mathrm{d}X_1/\mathrm{d}X = \mathrm{d}X_3/\mathrm{d}X = 0$ and $\mathrm{d}X_2/\mathrm{d}X = 1$, thus Eq. 4.92 (with Eq. **??**) yields

$$\Lambda^2_{\mathbf{e}_2} = C_{22} = 1 + 2E_{22} \tag{4.93}$$

and similar results can be obtained for $\Lambda^2_{\mathbf{e}_1}$ and $\Lambda^2_{\mathbf{e}_3}$.

68 Similarly from Eq. 4.53-b, the reciprocal of the squared stretch for the line element at P along the unit vector $\mathbf{n} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}x}$ is given by

$$\frac{1}{\lambda^2_{\mathbf{n}}} \equiv \left(\frac{\mathrm{d}X}{\mathrm{d}x}\right)^2_{P} = B^{-1}_{ij}\frac{\mathrm{d}x_i}{\mathrm{d}x}\frac{\mathrm{d}x_j}{\mathrm{d}x} \quad \text{or} \quad \frac{1}{\lambda^2_{\mathbf{n}}} = \mathbf{n}{\cdot}\mathbf{B}^{-1}{\cdot}\mathbf{n} \tag{4.94}$$

Again for an element originally along X_2, Fig. 4.6, we obtain

$$\frac{1}{\lambda^2_{\mathbf{e}_2}} = 1 - 2E^*_{22} \tag{4.95}$$

69 we note that in general $\Lambda_{\mathbf{e}_2} \neq \lambda_{\mathbf{e}_2}$ since the element originally along the X_2 axis will not be along the x_2 after deformation. Furthermore Eq. 4.92 and 4.94 show that in the matrices of rectangular cartesian components the diagonal elements of both $\mathbf{C}$ and $\mathbf{B}^{-1}$ must be positive, while the elements of $\mathbf{E}$ must be greater than $-\frac{1}{2}$ and those of $\mathbf{E}^*$ must be greater than $+\frac{1}{2}$.

70 The unit extension of the element is

$$\frac{\mathrm{d}x - \mathrm{d}X}{\mathrm{d}X} = \frac{\mathrm{d}x}{\mathrm{d}X} - 1 = \Lambda_{\mathbf{m}} - 1 \tag{4.96}$$

and for the element P_0Q_0 along the X_2 axis, the **unit extension** is

$$\frac{\mathrm{d}x - \mathrm{d}X}{\mathrm{d}X} = E_{(2)} = \Lambda_{\mathbf{e}_2} - 1 = \sqrt{1 + 2E_{22}} - 1 \tag{4.97}$$

for small deformation theory $E_{22} << 1$, and

$$\frac{\mathrm{d}x - \mathrm{d}X}{\mathrm{d}X} = E_{(2)} = (1+2E_{22})^{\frac{1}{2}} - 1 \simeq 1 + \frac{1}{2}2E_{22} - 1 \simeq E_{22} \tag{4.98}$$

which is identical to Eq. 4.86.

71 For the two differential line elements of Fig. 4.6, the change in angle $\gamma_{23} = \frac{\pi}{2} - \theta$ is given in terms of both $\Lambda_{\mathbf{e}_2}$ and $\Lambda_{\mathbf{e}_3}$ by

$$\sin\gamma_{23} = \frac{2E_{23}}{\Lambda_{\mathbf{e}_2}\Lambda_{\mathbf{e}_3}} = \frac{2E_{23}}{\sqrt{1+2E_{22}}\sqrt{1+2E_{33}}} \tag{4.99}$$

Again, when deformations are small, this equation reduces to Eq. 4.90.

4.3 Strain Decomposition

72 In this section we first seek to express the relative displacement vector as the **sum** of the linear (Lagrangian or Eulerian) strain tensor and the linear (Lagrangian or Eulerian) rotation tensor. This is restricted to small strains.

73 For finite strains, the former additive decomposition is no longer valid, instead we shall consider the strain tensor as a **product** of a rotation tensor and a stretch tensor.

4.3.1 †Linear Strain and Rotation Tensors

74 Strain components are quantitative measures of certain type of relative displacement between neighboring parts of the material. A solid material will resist such relative displacement giving rise to internal stresses.

75 Not all kinds of relative motion give rise to strain (and stresses). If a body moves as a **rigid body**, the rotational part of its motion produces relative displacement. Thus the general problem is to express the strain in terms of the displacements by separating off that part of the displacement distribution which does not contribute to the strain.

4.3.1.1 Small Strains

76 From Fig. 4.7 the displacements of two neighboring particles are represented by the vectors u^{P_0} and u^{Q_0} and the vector

$$\mathrm{d}u_i = u_i^{Q_0} - u_i^{P_0} \quad \text{or} \quad \mathrm{d}\mathbf{u} = \mathbf{u}^{Q_0} - \mathbf{u}^{P_0} \tag{4.100}$$

is called the **relative displacement vector** of the particle originally at Q_0 with respect to the one originally at P_0.

4.3.1.1.1 Lagrangian Formulation

77 Neglecting higher order terms, and through a Taylor expansion

$$\mathrm{d}u_i = \left(\frac{\partial u_i}{\partial X_j}\right)_{P_0} \mathrm{d}X_j \quad \text{or} \quad \mathrm{d}\mathbf{u} = (\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}})_{P_0}\,\mathrm{d}\mathbf{X} \tag{4.101}$$

78 We also define a **unit relative displacement vector** $\mathrm{d}u_i/\mathrm{d}X$ where $\mathrm{d}X$ is the magnitude of the differential distance $\mathrm{d}X_i$, or $\mathrm{d}X_i = \xi_i \mathrm{d}X$, then

$$\frac{\mathrm{d}u_i}{\mathrm{d}X} = \frac{\partial u_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j}\xi_j \quad \text{or} \quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}X} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}\cdot\boldsymbol{\xi} = \mathbf{J}\cdot\boldsymbol{\xi} \tag{4.102}$$

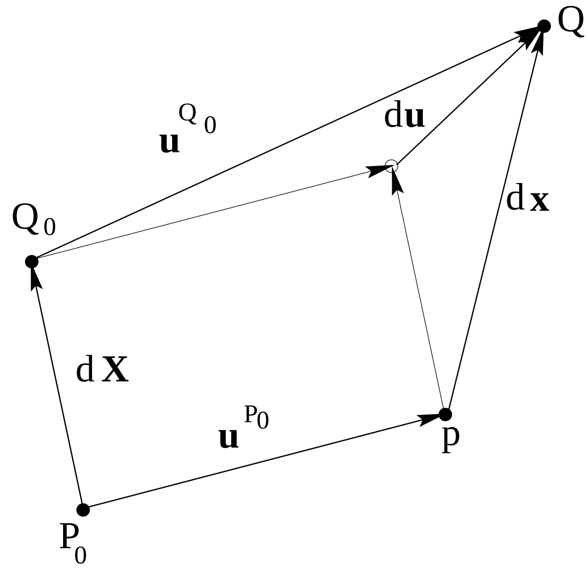


Figure 4.7: Relative Displacement d**u** of Q relative to P

79 The material displacement gradient $\frac{\partial u_i}{\partial X_j}$ can be decomposed uniquely into a symmetric and an anti-symetric part, we rewrite the previous equation as

$$\mathrm{d}u_i = \Bigg[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}+\frac{\partial u_j}{\partial X_i}\right)}_{E_{ij}}+\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}-\frac{\partial u_j}{\partial X_i}\right)}_{W_{ij}}\Bigg]\mathrm{d}X_j \tag{4.103-a}$$

or

$$\mathrm{d}\mathbf{u} = \Bigg[\underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}+\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)}_{\mathbf{E}}+\underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}-\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)}_{\mathbf{W}}\Bigg]\cdot\mathrm{d}\mathbf{X} \tag{4.103-b}$$

or

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3}+\frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3}+\frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3}+\frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3}+\frac{\partial u_3}{\partial X_2}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{4.104}$$

We thus introduce the **linear lagrangian rotation tensor**

$$\boxed{W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j}-\frac{\partial u_j}{\partial X_i}\right) \quad \text{or} \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}}-\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\right)} \tag{4.105}$$

in matrix form:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}-\frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3}-\frac{\partial u_3}{\partial X_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_2}-\frac{\partial u_2}{\partial X_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3}-\frac{\partial u_3}{\partial X_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial X_3}-\frac{\partial u_3}{\partial X_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial X_3}-\frac{\partial u_3}{\partial X_2}\right) & 0 \end{bmatrix} \tag{4.106}$$

80 In a displacement for which E_{ij} is zero in the vicinity of a point P_0, the relative displacement at that point will be an infinitesimal **rigid body rotation**. It can be shown that this rotation is given by the

linear Lagrangian rotation vector

$$\boxed{w_i = \frac{1}{2}\epsilon_{ijk}W_{kj} \quad \text{or} \quad \mathbf{w} = \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{X}}\times\mathbf{u}} \tag{4.107}$$

or

$$\mathbf{w} = -W_{23}\mathbf{e}_1 - W_{31}\mathbf{e}_2 - W_{12}\mathbf{e}_3 \tag{4.108}$$

4.3.1.1.2 Eulerian Formulation

81 The derivation in an Eulerian formulation parallels the one for Lagrangian formulation. Hence,

$$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\mathrm{d}x_j \quad \text{or} \quad \mathrm{d}\mathbf{u} = \mathbf{K}\cdot\mathrm{d}\mathbf{x} \tag{4.109}$$

82 The **unit relative displacement vector** will be

$$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\eta_j \quad \text{or} \quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}\cdot\boldsymbol{\eta} = \mathbf{K}\cdot\beta \tag{4.110}$$

83 The decomposition of the Eulerian displacement gradient $\frac{\partial u_i}{\partial x_j}$ results in

$$\mathrm{d}u_i = \left[\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{E^*_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}\right]\mathrm{d}x_j \tag{4.111-a}$$

or

$$\mathrm{d}\mathbf{u} = \left[\underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}+\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)}_{\mathbf{E}^*} + \underbrace{\frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}-\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)}_{\boldsymbol{\Omega}}\right]\cdot\mathrm{d}\mathbf{x} \tag{4.111-b}$$

or

$$\mathbf{E} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \tag{4.112}$$

84 We thus introduced the **linear Eulerian rotation tensor**

$$\boxed{w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \boldsymbol{\Omega} = \frac{1}{2}\left(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}}-\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\right)} \tag{4.113}$$

in matrix form:

$$\mathbf{W} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix} \tag{4.114}$$

and the **linear Eulerian rotation vector** will be

$$\boxed{\omega_i = \frac{1}{2}\epsilon_{ijk}\omega_{kj} \quad \text{or} \quad \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{x}}\times\mathbf{u}} \tag{4.115}$$

4.3.1.2 Examples

■ Example 4-8: Relative Displacement along a specified direction

A displacement field is specified by $\mathbf{u} = X_1^2 X_2 \mathbf{e}_1 + (X_2 - X_3^2)\mathbf{e}_2 + X_2^2 X_3 \mathbf{e}_3$. Determine the relative displacement vector $\mathrm{d}\mathbf{u}$ in the direction of the $-X_2$ axis at $P(1, 2, -1)$. Determine the relative displacements $\mathbf{u}_{Q_i} - \mathbf{u}_P$ for $Q_1(1, 1, -1)$, $Q_2(1, 3/2, -1)$, $Q_3(1, 7/4, -1)$ and $Q_4(1, 15/8, -1)$ and compute their directions with the direction of $\mathrm{d}\mathbf{u}$.

Solution:

From Eq. 4.44, $\mathbf{J} = \mathbf{u}\nabla_{\mathbf{X}}$ or

$$\frac{\partial u_i}{\partial X_j} = \begin{bmatrix} 2X_1X_2 & X_1^2 & 0 \\ 0 & 1 & -2X_3 \\ 0 & 2X_2X_3 & X_2^2 \end{bmatrix} \quad (4.116)$$

thus from Eq. 4.101 $\mathrm{d}\mathbf{u} = (\mathbf{u}\nabla_{\mathbf{X}})_P\,\mathrm{d}\mathbf{X}$ in the direction of $-X_2$ or

$$\{\mathrm{d}\mathbf{u}\} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ 4 \end{Bmatrix} \quad (4.117)$$

By direct calculation from $\mathbf{u}$ we have

$$\mathbf{u}_P = 2\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3 \quad (4.118\text{-a})$$
$$\mathbf{u}_{Q_1} = \mathbf{e}_1 - \mathbf{e}_3 \quad (4.118\text{-b})$$

thus

$$\mathbf{u}_{Q_1} - \mathbf{u}_P = -\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3 \quad (4.119\text{-a})$$
$$\mathbf{u}_{Q_2} - \mathbf{u}_P = \frac{1}{2}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.5\mathbf{e}_3) \quad (4.119\text{-b})$$
$$\mathbf{u}_{Q_3} - \mathbf{u}_P = \frac{1}{4}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.75\mathbf{e}_3) \quad (4.119\text{-c})$$
$$\mathbf{u}_{Q_4} - \mathbf{u}_P = \frac{1}{8}(-\mathbf{e}_1 - \mathbf{e}_2 + 3.875\mathbf{e}_3) \quad (4.119\text{-d})$$

and it is clear that as Q_i approaches P, the direction of the relative displacements of the two particles approaches the limiting direction of $\mathrm{d}\mathbf{u}$. ■

■ Example 4-9: Linear strain tensor, linear rotation tensor, rotation vector

Under the restriction of small deformation theory $\mathbf{E} = \mathbf{E}^*$, a displacement field is given by $\mathbf{u} = (x_1 - x_3)^2\mathbf{e}_1 + (x_2 + x_3)^2\mathbf{e}_2 - x_1x_2\mathbf{e}_3$. Determine the linear strain tensor, the linear rotation tensor and the rotation vector at point $P(0, 2, -1)$.

Solution:

the matrix form of the displacement gradient is

$$[\frac{\partial u_i}{\partial x_j}] = \begin{bmatrix} 2(x_1 - x_3) & 0 & -2(x_1 - x_3) \\ 0 & 2(x_2 + x_3) & 2(x_2 + x_3) \\ -x_2 & -x_1 & 0 \end{bmatrix} \quad (4.120\text{-a})$$
$$\left[\frac{\partial u_i}{\partial x_j}\right]_P = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix} \quad (4.120\text{-b})$$

Decomposing this matrix into symmetric and antisymmetric components give:

$$[E_{ij}] + [w_{ij}] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{4.121}$$

and from Eq. Eq. 4.108

$$\mathbf{w} = -W_{23}\mathbf{e}_1 - W_{31}\mathbf{e}_2 - W_{12}\mathbf{e}_3 = -1\mathbf{e}_1 \tag{4.122}$$

4.3.2 Finite Strain; Polar Decomposition

85 When the displacement gradients are finite, then we no longer can decompose $\frac{\partial u_i}{\partial X_j}$ (Eq. 4.101) or $\frac{\partial u_i}{\partial x_j}$ (Eq. 4.109) into a unique sum of symmetric and skew parts (pure strain and pure rotation).

86 Thus in this case, rather than having an **additive** decomposition, we will have a **multiplicative** decomposition.

87 we call this a **polar decomposition** and it should decompose the deformation gradient in the product of two tensors, one of which represents a rigid-body rotation, while the other is a symmetric positive-definite tensor.

88 We apply this decomposition to the deformation gradient $\mathbf{F}$:

$$\boxed{F_{ij} \equiv \frac{\partial x_i}{\partial X_j} = R_{ik}U_{kj} = V_{ik}R_{kj} \quad \text{or} \quad \mathbf{F} = \mathbf{R}{\cdot}\mathbf{U} = \mathbf{V}{\cdot}\mathbf{R}} \tag{4.123}$$

where $\mathbf{R}$ is the **orthogonal rotation tensor**, and $\mathbf{U}$ and $\mathbf{V}$ are positive symmetric tensors known as the **right stretch tensor** and the **left stretch tensor** respectively.

89 The interpretation of the above equation is obtained by inserting the above equation into $\mathrm{d}x_i = \frac{\partial x_i}{\partial X_j}\mathrm{d}X_j$

$$\mathrm{d}x_i = R_{ik}U_{kj}\mathrm{d}X_j = V_{ik}R_{kj}\mathrm{d}X_j \quad \text{or} \quad \mathrm{d}\mathbf{x} = \mathbf{R}{\cdot}\mathbf{U}{\cdot}\mathrm{d}\mathbf{X} = \mathbf{V}{\cdot}\mathbf{R}{\cdot}\mathrm{d}\mathbf{X} \tag{4.124}$$

and we observe that in the first form the deformation consists of a sequential stretching (by $\mathbf{U}$) and rotation ($\mathbf{R}$) to be followed by a rigid body displacement to $\mathbf{x}$. In the second case, the orders are reversed, we have first a rigid body translation to $\mathbf{x}$, followed by a rotation ($\mathbf{R}$) and finally a stretching (by $\mathbf{V}$).

90 To determine the stretch tensor from the deformation gradient

$$\mathbf{F}^T\mathbf{F} = (\mathbf{RU})^T(\mathbf{RU}) = \mathbf{U}^T\mathbf{R}^T\mathbf{RU} = \mathbf{U}^T\mathbf{U} \tag{4.125}$$

Recalling that $\mathbf{R}$ is an orthonormal matrix, and thus $\mathbf{R}^T = \mathbf{R}^{-1}$ then we can compute the various tensors from

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \tag{4.126}$$

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \tag{4.127}$$

$$\mathbf{V} = \mathbf{F}\mathbf{R}^T \tag{4.128}$$

91 It can be shown that

$$\boxed{\mathbf{U} = \mathbf{C}^{1/2} \quad \text{and} \quad \mathbf{V} = \mathbf{B}^{1/2}} \tag{4.129}$$

■ **Example 4-10: Polar Decomposition I**

Given $x_1 = X_1$, $x_2 = -3X_3$, $x_3 = 2X_2$, find the deformation gradient **F**, the right stretch tensor **U**, the rotation tensor **R**, and the left stretch tensor **V**.
Solution:

From Eq. 4.25

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \tag{4.130}$$

From Eq. 4.126

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \tag{4.131}$$

thus

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{4.132}$$

From Eq. 4.127

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.133}$$

Finally, from Eq. 4.128

$$\mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{4.134}$$

■

■ Example 4-11: Polar Decomposition II

For the following deformation: $x_1 = \lambda_1 X_1$, $x_2 = -\lambda_3 X_3$, and $x_3 = \lambda_2 X_2$, find the rotation tensor.
Solution:

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \tag{4.135}$$

$$[\mathbf{U}]^2 = [\mathbf{F}]^T[\mathbf{F}] \tag{4.136}$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{4.137}$$

$$[\mathbf{U}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{4.138}$$

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.139}$$

Thus we note that **R** corresponds to a 90^o rotation about the $\mathbf{e}_1$ axis. ■

In[4]:=
```
{v1, v2, v3} = N[Eigenvectors[CST], 4]
```

Determine U and U^{-1} with respect to the e_i basis

Pola

Out[4]= $\begin{pmatrix} 0 & 0 & 1. \end{pmatrix}$

In[10]:=
```
U_e = N[vnormalized . Ueigen . vnormalized, 3]
```

Given x_1 ...

matrix U ...

obtain th ...

In[5]:=

Out[10]= $\begin{pmatrix} 0.707 & 0.707 & 0. \\ 0.707 & 2.12 & 0. \\ 0. & 0. & 1. \end{pmatrix}$

Determ

In[6]:=
```
vnormalized = GramSchmidt[{v3, -v2, v1}]
```

In[11]:=
```
U_einverse = N[Inverse[%], 3]
```

In[1]:=

$\begin{pmatrix} 0.382683 & 0.92388 & 0 \end{pmatrix}$

Out[6]=

Out[11]= $\begin{pmatrix} 2.12 & -0.707 & 0. \\ -0.707 & 0.707 & 0. \\ 0. & 0. & 1. \end{pmatrix}$

Out[1]=

In[7]:=
```
CSTeigen = Chop[N[vnormalized . CST . vnormalized, 4]]
```

Determine R with respect to the e_i basis

Solve

Out[7]= $\begin{pmatrix} 5.828 & 0 & 0 \\ 0 & 0.1716 & 0 \end{pmatrix}$

In[12]:=
```
R = N[F . %, 3]
```

In[2]:=

Out[12]= $\begin{pmatrix} 0.707 & 0.707 & 0. \\ -0.707 & 0.707 & 0. \\ 0. & 0. & 1. \end{pmatrix}$

Determ

Out[2]=

In[8]:=
```
Ueigen = N[Sqrt[CSTeigen], 4]
```

Out[8]= $\begin{pmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1. \end{pmatrix}$

Determ

In[3]:=
```
N[Eigenvalues[CST]]
```

In[9]:=
```
Ueigenminus1 = Inverse[Ueigen]
```

Out[3]= {1., 0.171573, 5.82843}

Out[9]= $\begin{pmatrix} 0.414214 & 0. & 0. \\ 0. & 2.41421 & 0. \\ 0. & 0. & 1. \end{pmatrix}$

■

4.4 Summary and Discussion

92 From the above, we deduce the following observations:

1. If both the displacement gradients and the displacements themselves are small, then $\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j}$ and thus the Eulerian and the Lagrangian infinitesimal strain tensors may be taken as equal $E_{ij} = E^*_{ij}$.
2. If the displacement gradients are small, but the displacements are large, we should use the Eulerian infinitesimal representation.
3. If the displacements gradients are large, but the displacements are small, use the Lagrangian finite strain representation.
4. If both the displacement gradients and the displacements are large, use the Eulerian finite strain representation.

4.5 Compatibility Equation

93 If $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ then we have six differential equations (in 3D the strain tensor has a total of 9 terms, but due to symmetry, there are 6 independent ones) for determining (upon integration) three unknowns displacements u_i. Hence the system is overdetermined, and there must be some linear relations between the strains.

94 It can be shown (through appropriate successive differentiation of the strain expression) that the compatibility relation for strain reduces to:

$$\boxed{\frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_j} + \frac{\partial^2 \varepsilon_{jj}}{\partial x_i \partial x_k} - \frac{\partial^2 \varepsilon_{jk}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ij}}{\partial x_j \partial x_k} = 0. \quad \text{or} \quad \boldsymbol{\nabla}_{\mathbf{x}} \times \mathbf{L} \times \boldsymbol{\nabla}_{\mathbf{x}} = 0} \tag{4.140}$$

There are 81 equations in all, but only six are distinct

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \tag{4.141-a}$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} \tag{4.141-b}$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} = 2\frac{\partial^2 \varepsilon_{31}}{\partial x_3 \partial x_1} \tag{4.141-c}$$

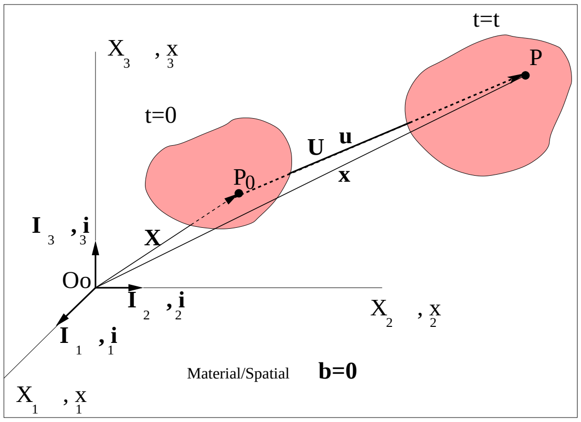
$$\frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} \tag{4.141-d}$$

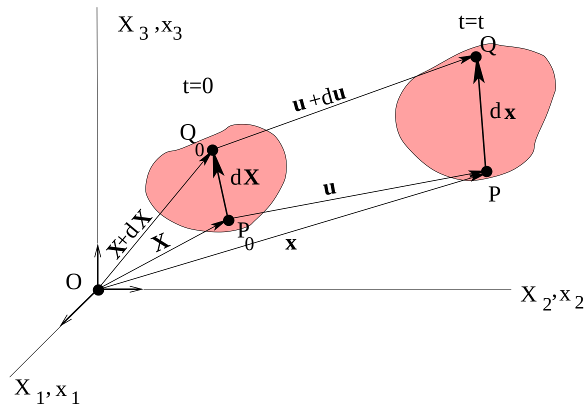
$$\frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} \tag{4.141-e}$$

$$\frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right) = \frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} \tag{4.141-f}$$

In 2D, this results in (by setting $i = 2$, $j = 1$ and $l = 2$):

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2}} \tag{4.142}$$





	LAGRANGIAN Material	EULERIAN Spatial
Position Vector	$\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$	$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$
	GRADIENTS	
Deformation	$\mathbf{F} = \mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}} \equiv \frac{\partial x_i}{\partial X_j}$	$\mathbf{H} = \mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}} \equiv \frac{\partial X_i}{\partial x_j}$
	$\mathbf{H} = \mathbf{F}^{-1}$	
Displacement	$\frac{\partial u_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} - \delta_{ij}$ or $\mathbf{J} = \mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} = \mathbf{F} - \mathbf{I}$	$\frac{\partial u_i}{\partial x_j} = \delta_{ij} - \frac{\partial X_i}{\partial x_j}$ or $\mathbf{K} \equiv \mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} = \mathbf{I} - \mathbf{H}$
	TENSOR	
Deformation	$\mathrm{d}X^2 = \mathrm{d}\mathbf{x}\cdot\mathbf{B}^{-1}\cdot\mathrm{d}\mathbf{x}$	$\mathrm{d}x^2 = \mathrm{d}\mathbf{X}\cdot\mathbf{C}\cdot\mathrm{d}\mathbf{X}$
	Cauchy $B^{-1}_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}$ or $\mathbf{B}^{-1} = \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{X}\cdot\mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}} = \mathbf{H}_c\cdot\mathbf{H}$	Green $C_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}$ or $\mathbf{C} = \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x}\cdot\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}} = \mathbf{F}_c\cdot\mathbf{F}$
	$\mathbf{C}^{-1} = \mathbf{B}^{-1}$	
	STRAINS	
	Lagrangian	**Eulerian/Almansi**
Finite Strain	$\mathrm{d}x^2 - \mathrm{d}X^2 = \mathrm{d}\mathbf{X}\cdot 2\mathbf{E}\cdot\mathrm{d}\mathbf{X}$	$\mathrm{d}x^2 - \mathrm{d}X^2 = \mathrm{d}\mathbf{x}\cdot 2\mathbf{E}^*\cdot\mathrm{d}\mathbf{x}$
	$E_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$ or $\mathbf{E} = \frac{1}{2}(\underbrace{\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{x}\cdot\mathbf{x}\boldsymbol{\nabla}_{\mathbf{X}}}_{\mathbf{F}_c\cdot\mathbf{F}} - \mathbf{I})$	$E^*_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)$ or $\mathbf{E}^* = \frac{1}{2}(\mathbf{I} - \underbrace{\boldsymbol{\nabla}_{\mathbf{x}}\mathbf{X}\cdot\mathbf{X}\boldsymbol{\nabla}_{\mathbf{x}}}_{\mathbf{H}_c\cdot\mathbf{H}})$
	$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$ or $\mathbf{E} = \frac{1}{2}\underbrace{(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}\cdot\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}})}_{\mathbf{J}+\mathbf{J}_c+\mathbf{J}_c\cdot\mathbf{J}}$	$E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$ or $\mathbf{E}^* = \frac{1}{2}\underbrace{(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u} - \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}\cdot\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}})}_{\mathbf{K}+\mathbf{K}_c-\mathbf{K}_c\cdot\mathbf{K}}$
Small Deformation	$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ $\mathbf{E} = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u}) = \frac{1}{2}(\mathbf{J} + \mathbf{J}_c)$	$E^*_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}) = \frac{1}{2}(\mathbf{K} + \mathbf{K}_c)$
	ROTATION TENSORS	
Small deformation	$[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)]\mathrm{d}X_j$ $[\underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} + \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u})}_{\mathbf{E}} + \underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{X}} - \boldsymbol{\nabla}_{\mathbf{X}}\mathbf{u})}_{\mathbf{W}}]\cdot\mathrm{d}\mathbf{X}$	$\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right]\mathrm{d}x_j$ $[\underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u})}_{\mathbf{E}^*} + \underbrace{\frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u})}_{\boldsymbol{\Omega}}]\cdot\mathrm{d}\mathbf{x}$
Finite Strain	$\mathbf{F} = \mathbf{R}\cdot\mathbf{U} = \mathbf{V}\cdot\mathbf{R}$	
	STRESS TENSORS	
	Piola-Kirchoff	**Cauchy**
First Second	$\mathbf{T}_0 = (\det\mathbf{F})\mathbf{T}\left(\mathbf{F}^{-1}\right)^T$ $\tilde{\mathbf{T}} = (\det\mathbf{F})\left(\mathbf{F}^{-1}\right)\mathbf{T}\left(\mathbf{F}^{-1}\right)^T$	

Table 4.1: Summary of Major Equations

(recall that $2\varepsilon_{12} = \gamma_{12}$.)

95 When he compatibility equation is written in term of the stresses, it yields:

$$\frac{\partial^2 \sigma_{11}}{\partial x_2^2} - \nu \frac{\partial \sigma_{22}}{\partial x_2^2}^2 + \frac{\partial^2 \sigma_{22}}{\partial x_1^2} - \nu \frac{\partial^2 \sigma_{11}}{\partial x_1^2} = 2(1+\nu) \frac{\partial^2 \sigma_{21}}{\partial x_1 \partial x_2} \tag{4.143}$$

■ **Example 4-13: Strain Compatibility**

For the following strain field

$$\begin{bmatrix} -\frac{X_2}{X_1^2+X_2^2} & \frac{X_1}{2(X_1^2+X_2^2)} & 0 \\ \frac{X_1}{2(X_1^2+X_2^2)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.144}$$

does there exist a single-valued continuous displacement field?
Solution:

$$\frac{\partial E_{11}}{\partial X_2} = -\frac{(X_1^2+X_2^2) - X_2(2X_2)}{(X_1^2+X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2+X_2^2)^2} \tag{4.145-a}$$

$$2\frac{\partial E_{12}}{\partial X_1} = \frac{(X_1^2+X_2^2) - X_1(2X_1)}{(X_1^2+X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2+X_2^2)^2} \tag{4.145-b}$$

$$\frac{\partial E_{22}}{\partial X_1^2} = 0 \tag{4.145-c}$$

$$\Rightarrow \frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2\frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \surd \tag{4.145-d}$$

Actually, it can be easily verified that the unique displacement field is given by

$$u_1 = \arctan \frac{X_2}{X_1}; \quad u_2 = 0; \quad u_3 = 0 \tag{4.146}$$

to which we could add the rigid body displacement field (if any). ■

4.6 Lagrangian Stresses; Piola Kirchoff Stress Tensors

96 In Sect. 2.2 the discussion of stress applied to the deformed configuration dA (using spatial coordiantes $\mathbf{x}$), that is the one where equilibrium must hold. The deformed configuration being the natural one in which to characterize stress. Hence we had

$$d\mathbf{f} = \mathbf{t} dA \tag{4.147-a}$$

$$\mathbf{t} = \mathbf{Tn} \tag{4.147-b}$$

(note the use of $\mathbf{T}$ instead of $\boldsymbol{\sigma}$). Hence the Cauchy stress tensor was really defined in the Eulerian space.

97 However, there are certain advantages in referring all quantities back to the undeformed configuration (Lagrangian) of the body because often that configuration has geometric features and symmetries that are lost through the deformation.

98 Hence, if we were to define the strain in material coordinates (in terms of $\mathbf{X}$), we need also to express the stress as a function of the material point $\mathbf{X}$ in material coordinates.

4.6.1 First

99 The first Piola-Kirchoff stress tensor $\mathbf{T}_0$ is defined in the undeformed geometry in such a way that it results in the **same total force** as the traction in the deformed configuration (where Cauchy's stress tensor was defined). Thus, we define

$$\mathrm{d}\mathbf{f} \equiv \mathbf{t}_0 \mathrm{d}A_0 \tag{4.148}$$

where $\mathbf{t}_0$ is a **pseudo-stress vector** in that being based on the undeformed area, it does not describe the actual intensity of the force, however it has the same direction as Cauchy's stress vector $\mathbf{t}$.

100 The first Piola-Kirchoff stress tensor (also known as **Lagrangian Stress Tensor**) is thus the linear transformation $\mathbf{T}_0$ such that

$$\mathbf{t}_0 = \mathbf{T}_0\mathbf{n}_0 \tag{4.149}$$

and for which

$$\mathrm{d}\mathbf{f} = \mathbf{t}_0\mathrm{d}A_0 = \mathbf{t}\mathrm{d}A \Rightarrow \mathbf{t}_0 = \frac{\mathrm{d}A}{\mathrm{dd}A_0}\mathbf{t} \tag{4.150}$$

using Eq. 4.147-b and 4.149 the preceding equation becomes

$$\mathbf{T}_0\mathbf{n}_0 = \frac{\mathrm{d}A}{\mathrm{d}A_0}\mathbf{T}\mathbf{n} = \mathbf{T}\frac{\mathrm{d}A}{\mathrm{d}A_0}\mathbf{n} \tag{4.151}$$

and using Eq. 4.36 $\mathrm{d}A\mathbf{n} = \mathrm{d}A_0(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T\mathbf{n}_0$ we obtain

$$\mathbf{T}_0\mathbf{n}_0 = \mathbf{T}(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T\mathbf{n}_0 \tag{4.152}$$

the above equation is true for all $\mathbf{n}_0$, therefore

$$\mathbf{T}_0 = (\det\mathbf{F})\mathbf{T}\left(\mathbf{F}^{-1}\right)^T \tag{4.153}$$

$$\mathbf{T} = \frac{1}{(\det\mathbf{F})}\mathbf{T}_0\mathbf{F}^T \quad \text{or} \quad T_{ij} = \frac{1}{(\det\mathbf{F})}(T_0)_{im}F_{jm} \tag{4.154}$$

101 The first Piola-Kirchoff stress tensor is **not symmetric** in general, and is not energitically correct. That is multiplying this stress tensor with the Green-Lagrange tensor will not be equal to the product of the Cauchy stress tensor multiplied by the deformation strain tensor.

102 To determine the corresponding stress vector, we solve for $\mathbf{T}_0$ first, then for $\mathrm{d}A_0$ and $\mathbf{n}_0$ from $\mathrm{d}A_0\mathbf{n}_0 = \frac{1}{\det\mathbf{F}}\mathbf{F}^T\mathbf{n}$ (assuming unit area $\mathrm{d}A$), and finally $\mathbf{t}_0 = \mathbf{T}_0\mathbf{n}_0$.

4.6.2 Second

103 The second Piola-Kirchoff stress tensor, $\tilde{\mathbf{T}}$ is formulated differently. Instead of the actual force $d\mathbf{f}$ on $\mathrm{d}A$, it gives the force $d\tilde{\mathbf{f}}$ related to the force $d\mathbf{f}$ in the same way that a material vector $\mathrm{d}\mathbf{X}$ at $\mathbf{X}$ is related by the deformation to the corresponding spatial vector $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$. Thus, if we let

$$d\tilde{\mathbf{f}} = \tilde{\mathbf{t}}\mathrm{d}A_0 \tag{4.155-a}$$

and

$$d\mathbf{f} = \mathbf{F}d\tilde{\mathbf{f}} \tag{4.155-b}$$

where $d\tilde{\mathbf{f}}$ is the pseudo differential force which transforms, under the deformation gradient $\mathbf{F}$, the (actual) differential force $d\mathbf{f}$ at the deformed position (note similarity with $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$). Thus, the pseudo vector $\mathbf{t}$ is in general in a diffent direction than that of the Cauchy stress vector $\mathbf{t}$.

104 The second Piola-Kirchoff stress tensor is a linear transformation $\tilde{\mathbf{T}}$ such that

$$\tilde{\mathbf{t}} = \tilde{\mathbf{T}}n_0 \tag{4.156}$$

thus the preceding equations can be combined to yield

$$d\mathbf{f} = \mathbf{F}\tilde{\mathbf{T}}n_0 \mathrm{d}A_0 \tag{4.157}$$

we also have from Eq. 4.148 and 4.149

$$d\mathbf{f} = \mathbf{t}_0 \mathrm{d}A_0 = \mathbf{T}_0\mathbf{n}_0 \mathrm{d}\mathrm{d}A_0 \tag{4.158}$$

and comparing the last two equations we note that

$$\boxed{\tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{T}_0} \tag{4.159}$$

which gives the relationship between the first Piola-Kirchoff stress tensor $\mathbf{T}_0$ and the second Piola-Kirchoff stress tensor $\tilde{\mathbf{T}}$.

105 Finally the relation between the second Piola-Kirchoff stress tensor and the Cauchy stress tensor can be obtained from the preceding equation and Eq. 4.153

$$\boxed{\tilde{\mathbf{T}} = (\det \mathbf{F}) \left(\mathbf{F}^{-1}\right) \mathbf{T} \left(\mathbf{F}^{-1}\right)^T} \tag{4.160}$$

and we note that this second Piola-Kirchoff stress tensor is always symmetric (if the Cauchy stress tensor is symmetric). It can also be shown that it is energitically correct.

106 To determine the corresponding stress vector, we solve for $\tilde{\mathbf{T}}$ first, then for $\mathrm{d}A_0$ and $\mathbf{n}_0$ from $\mathrm{d}A_0\mathbf{n}_0 = \frac{1}{\det \mathbf{F}}\mathbf{F}^T\mathbf{n}$ (assuming unit area $\mathrm{d}A$), and finally $\tilde{\mathbf{t}} = \tilde{\mathbf{T}}\mathbf{n}_0$.

■ **Example 4-14: Piola-Kirchoff Stress Tensors**

■

■

4.7 Hydrostatic and Deviatoric Strain

85 The lagrangian and Eulerian **linear** strain tensors can each be split into **spherical** and **deviator** tensor as was the case for the stresses. Hence, if we define

$$\frac{1}{3}e = \frac{1}{3}\mathrm{tr}\ \mathbf{E} \tag{4.161}$$

then the components of the strain deviator $\mathbf{E}'$ are given by

$$\boxed{E'_{ij} = E_{ij} - \frac{1}{3}e\delta_{ij} \quad \text{or} \quad \mathbf{E}' = \mathbf{E} - \frac{1}{3}e\mathbf{1}} \tag{4.162}$$

We note that $\mathbf{E}'$ measures the change in shape of an element, while the **spherical** or **hydrostatic** strain $\frac{1}{3}e\mathbf{1}$ represents the volume change.

■ Second Piola−Kirchoff Stress Tensor

The defo... ...ess tensor is given by ... calculate the respective...

```
MatrixForm[Transpose[F] . n / detF]
```

$$\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

Thus $n = e_2$ and using $t_0 = T_0\, n_0$ we obtain

{{0, 0, 0}, {0, 25/4, 0}, {0, 0, 0}}

```
t01st = MatrixForm[Tfirst . {0, 1, 0}]
```

■ F tensor

$$\begin{pmatrix} 0 \\ 0 \\ 25 \end{pmatrix}$$

We note that this vector is in the same direction as the Cauchy stress vector, its magnitude is one fourth of that of the Cauchy stress vector, because the undeformed area is 4 times that of the deformed area

■ Cuchy stress vector

■ Pseudo−Stress vector associated with the Second Piola−Kirchoff stress tensor

```
F = {{1/2, 0, 0}, {0, 0, -1/2}, {0, 4, 0}}
```

Can be obtained from t=CST n

$$\left\{\left\{\frac{1}{2}, 0, 0\right\}, \left\{0, 0, -\frac{1}{2}\right\}, \{0, 4, 0\}\right\}$$

```
t0second = MatrixForm[Tsecond . {0, 1, 0}]
```

$$\begin{pmatrix} 0 \\ \frac{25}{4} \\ 0 \end{pmatrix}$$

We see that this pseudo stress vector is in a different direction from that of the Cauchy stress vector (and we note that the tensor F transforms e_2 into e_3).

■ Pseudo−Stress vector associated with the First Piola−Kirchoff stress tensor

■ First Piola−Kirchoff Stress Tensor

For a unit area in the deformed state in the e_3 direction, its undeformed area $dA_0\, n_0$ is given by $dA_0\, \mathbf{n}_0 = \frac{F^T\, n}{\det F}$

```
detF = Det[F]
```

1

```
MatrixForm[σ]
n = {0, 0, 1}
```

{0, 0, 1}

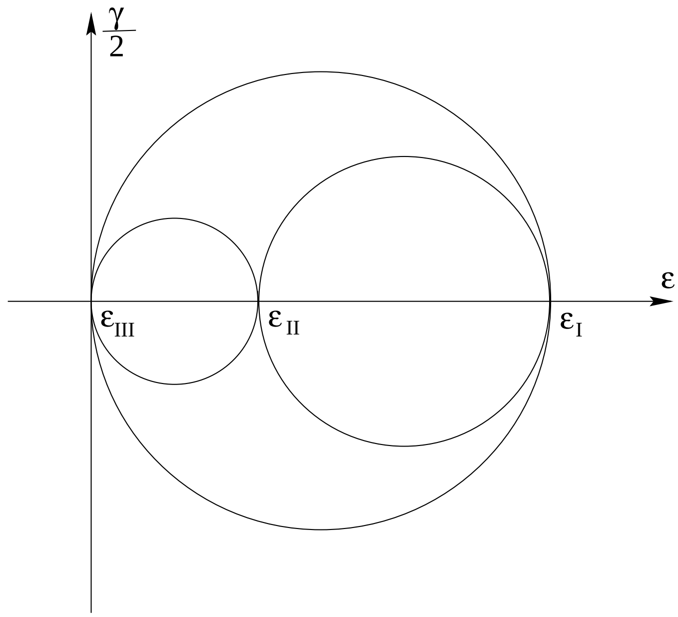


Figure 4.8: Mohr Circle for Strain

4.8 Principal Strains, Strain Invariants, Mohr Circle

86 Determination of the principal strains ($E_{(3)} < E_{(2)} < E_{(1)}$, strain invariants and the Mohr circle for strain parallel the one for stresses (Sect. 2.3) and will not be repeated here.

$$\boxed{\lambda^3 - I_E\lambda^2 - II_E\lambda - III_E = 0} \tag{4.163}$$

where the symbols I_E, II_E and III_E denote the following scalar expressions in the strain components:

$$\begin{aligned}
I_E &= E_{11} + E_{22} + E_{33} = E_{ii} = \text{tr } \mathbf{E} & (4.164)\\
II_E &= -(E_{11}E_{22} + E_{22}E_{33} + E_{33}E_{11}) + E_{23}^2 + E_{31}^2 + E_{12}^2 & (4.165)\\
&= \frac{1}{2}(E_{ij}E_{ij} - E_{ii}E_{jj}) = \frac{1}{2}E_{ij}E_{ij} - \frac{1}{2}I_E^2 & (4.166)\\
&= \frac{1}{2}(\mathbf{E} : \mathbf{E} - I_E^2) & (4.167)\\
III_E &= \det\mathbf{E} = \frac{1}{6}e_{ijk}e_{pqr}E_{ip}E_{jq}E_{kr} & (4.168)
\end{aligned}$$

87 In terms of the principal strains, those invariants can be simplified into

$$\begin{aligned}
I_E &= E_{(1)} + E_{(2)} + E_{(3)} & (4.169)\\
II_E &= -(E_{(1)}E_{(2)} + E_{(2)}E_{(3)} + E_{(3)}E_{(1)}) & (4.170)\\
III_E &= E_{(1)}E_{(2)}E_{(3)} & (4.171)
\end{aligned}$$

88 The Mohr circle uses the **Engineering shear strain** definition of Eq. 4.91, Fig. 4.8

■ **Example 4-15: Strain Invariants & Principal Strains**

Determine the planes of principal strains for the following strain tensor

$$\begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.172}$$

Solution:

The strain invariants are given by

$$I_E = E_{ii} = 2 \tag{4.173-a}$$

$$II_E = \frac{1}{2}(E_{ij}E_{ij} - E_{ii}E_{jj}) = -1 + 3 = +2 \tag{4.173-b}$$

$$III_E = |E_{ij}| = -3 \tag{4.173-c}$$

The principal strains by

$$E_{ij} - \lambda\delta_{ij} = \begin{bmatrix} 1-\lambda & \sqrt{3} & 0 \\ \sqrt{3} & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \tag{4.174-a}$$

$$= (1-\lambda)\left(\lambda - \frac{1+\sqrt{13}}{2}\right)\left(\lambda - \frac{1-\sqrt{13}}{2}\right) \tag{4.174-b}$$

$$E_{(1)} = \lambda_{(1)} = \frac{1+\sqrt{13}}{2} = 2.3 \tag{4.174-c}$$

$$E_{(2)} = \lambda_{(2)} = 1 \tag{4.174-d}$$

$$E_{(3)} = \lambda_{(3)} = \frac{1-\sqrt{13}}{2} = -1.3 \tag{4.174-e}$$

The eigenvectors for $E_{(1)} = \frac{1+\sqrt{13}}{2}$ give the principal directions $\mathbf{n}^{(1)}$:

$$\begin{bmatrix} 1-\frac{1+\sqrt{13}}{2} & \sqrt{3} & 0 \\ \sqrt{3} & -\frac{1+\sqrt{13}}{2} & 0 \\ 0 & 0 & 1-\frac{1+\sqrt{13}}{2} \end{bmatrix} \begin{Bmatrix} n_1^{(1)} \\ n_2^{(1)} \\ n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} \left(1-\frac{1+\sqrt{13}}{2}\right)n_1^{(1)} + \sqrt{3}n_2^{(1)} \\ \sqrt{3}n_1^{(1)} - \left(\frac{1+\sqrt{13}}{2}\right)n_2^{(1)} \\ \left(1-\frac{1+\sqrt{13}}{2}\right)n_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.175}$$

which gives

$$n_1^{(1)} = \frac{1+\sqrt{13}}{2\sqrt{3}}n_2^{(1)} \tag{4.176-a}$$

$$n_3^{(1)} = 0 \tag{4.176-b}$$

$$\mathbf{n}^{(1)}\cdot\mathbf{n}^{(1)} = \left(\frac{1+2\sqrt{13}+13}{12}+1\right)\left(n_2^{(1)}\right)^2 = 1 \Rightarrow n_2^1 = 0.8; \tag{4.176-c}$$

$$\Rightarrow \mathbf{n}^{(1)} = \lfloor\ 0.8 \quad 0.6 \quad 0\ \rfloor \tag{4.176-d}$$

For the second eigenvector $\lambda_{(2)} = 1$:

$$\begin{bmatrix} 1-1 & \sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{Bmatrix} n_1^{(2)} \\ n_2^{(2)} \\ n_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} \sqrt{3}n_2^{(2)} \\ \sqrt{3}n_1^{(2)} - n_2^{(2)} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.177}$$

which gives (with the requirement that $\mathbf{n}^{(2)}\cdot\mathbf{n}^{(2)} = 1$)

$$\mathbf{n}^{(2)} = \lfloor\ 0 \quad 0 \quad 1\ \rfloor \tag{4.178}$$

Finally, the third eigenvector can be obrained by the same manner, but more easily from

$$\mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)} = \det \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.6\mathbf{e}_1 - 0.8\mathbf{e}_2 \tag{4.179}$$

Therefore

$$a_i^j = \left\{ \begin{array}{c} \mathbf{n}^{(1)} \\ \mathbf{n}^{(2)} \\ \mathbf{n}^{(3)} \end{array} \right\} = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \tag{4.180}$$

and this results can be checked via

$$[\mathbf{a}][\mathbf{E}][\mathbf{a}]^T = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.8 & 0 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0 & 0.6 \\ 0.6 & 0 & -0.8 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2.3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1.3 \end{bmatrix} \tag{4.181}$$

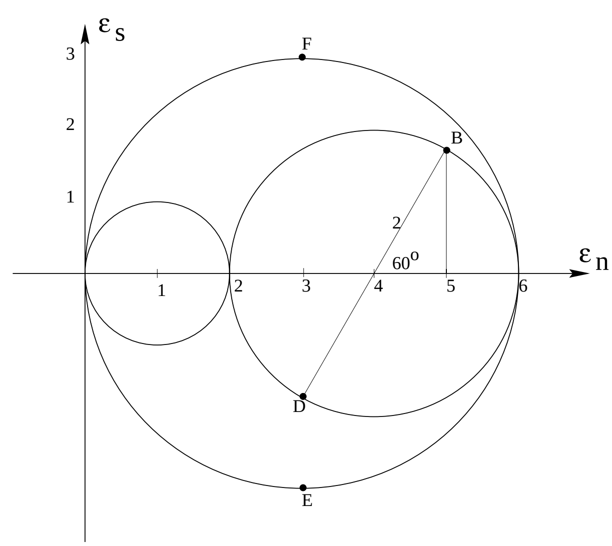
■

■ **Example 4-16: Mohr's Circle**

Construct the Mohr's circle for the following plane strain case:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{bmatrix} \tag{4.182}$$

Solution:



■

We note that since $E_{(1)} = 0$ is a principal value for plane strain, ttwo of the circles are drawn as shown.

4.9 Initial or Thermal Strains

89 Initial (or thermal strain) in 2D:

$$\varepsilon_{ij} = \underbrace{\begin{bmatrix} \alpha\Delta T & 0 \\ 0 & \alpha\Delta T \end{bmatrix}}_{\text{Plane Stress}} = \underbrace{(1+\nu)\begin{bmatrix} \alpha\Delta T & 0 \\ 0 & \alpha\Delta T \end{bmatrix}}_{\text{Plane Strain}} \tag{4.183}$$

note there is no shear strains caused by thermal expansion.

4.10 † Experimental Measurement of Strain

90 Typically, the transducer to measure strains in a material is the strain gage. The most common type of strain gage used today for stress analysis is the bonded resistance strain gage shown in Figure 4.9.

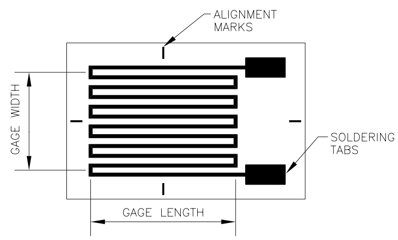


Figure 4.9: Bonded Resistance Strain Gage

91 These gages use a grid of fine wire or a metal foil grid encapsulated in a thin resin backing. The gage is glued to the carefully prepared test specimen by a thin layer of epoxy. The epoxy acts as the carrier matrix to transfer the strain in the specimen to the strain gage. As the gage changes in length, the tiny wires either contract or elongate depending upon a tensile or compressive state of stress in the specimen. The cross sectional area will increase for compression and decrease in tension. Because the wire has an electrical resistance that is proportional to the inverse of the cross sectional area, $R\,\alpha\,\frac{1}{A}$, a measure of the change in resistance can be converted to arrive at the strain in the material.

92 Bonded resistance strain gages are produced in a variety of sizes, patterns, and resistance. One type of gage that allows for the complete state of strain at a point in a plane to be determined is a strain gage rosette. It contains three gages aligned radially from a common point at different angles from each other, as shown in Figure 4.10. The strain transformation equations to convert from the three strains a t any angle to the strain at a point in a plane are:

$$\epsilon_a = \epsilon_x \cos^2\theta_a + \epsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \tag{4.184}$$

$$\epsilon_b = \epsilon_x \cos^2\theta_b + \epsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \tag{4.185}$$

$$\epsilon_c = \epsilon_x \cos^2\theta_c + \epsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c \tag{4.186}$$

93 When the measured strains ϵ_a, ϵ_b, and ϵ_c, are measured at their corresponding angles from the reference axis and substituted into the above equations the state of strain at a point may be solved, namely, ϵ_x, ϵ_y, and γ_{xy}. In addition the principal strains may then be computed by Mohr's circle or the principal strain equations.

94 Due to the wide variety of styles of gages, many factors must be considered in choosing the right gage for a particular application. Operating temperature, state of strain, and stability of installation all influence gage selection. Bonded resistance strain gages are well suited for making accurate and practical strain measurements because of their high sensitivity to strains, low cost, and simple operation.

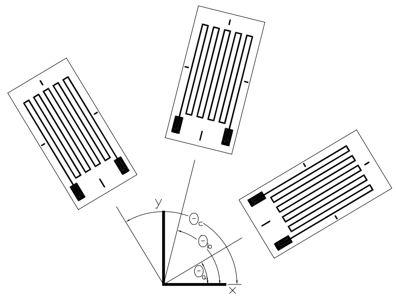


Figure 4.10: Strain Gage Rosette

95 The measure of the change in electrical resistance when the strain gage is strained is known as the gage factor. The gage factor is defined as the fractional change in resistance divided by the fractional change in length along the axis of the gage. $GF = \frac{\frac{\Delta R}{R}}{\frac{\Delta L}{L}}$ Common gage factors are in the range of 1.5-2 for most resistive strain gages.

96 Common strain gages utilize a grid pattern as opposed to a straight length of wire in order to reduce the gage length. This grid pattern causes the gage to be sensitive to deformations transverse to the gage length. Therefore, corrections for transverse strains should be computed and applied to the strain data. Some gages come with the tranverse correction calculated into the gage factor. The transverse sensitivity factor, K_t, is defined as the transverse gage factor divided by the longitudinal gage factor. $K_t = \frac{GF_{transverse}}{GF_{longitudinal}}$ These sensitivity values are expressed as a percentage and vary from zero to ten percent.

97 A final consideration for maintaining accurate strain measurement is temperature compensation. The resistance of the gage and the gage factor will change due to the variation of resistivity and strain sensitivity with temperature. Strain gages are produced with different temperature expansion coefficients. In order to avoid this problem, the expansion coefficient of the strain gage should match that of the specimen. If no large temperature change is expected this may be neglected.

98 The change in resistance of bonded resistance strain gages for most strain measurements is very small. From a simple calculation, for a strain of 1 $\mu\epsilon$ ($\mu = 10^{-6}$) with a 120 Ω gage and a gage factor of 2, the change in resistance produced by the gage is $\Delta R = 1 \times 10^{-6} \times 120 \times 2 = 240 \times 10^{-6}\Omega$. Furthermore, it is the fractional change in resistance that is important and the number to be measured will be in the order of a couple of μ ohms. For large strains a simple multi-meter may suffice, but in order to acquire sensitive measurements in the $\mu\Omega$ range a Wheatstone bridge circuit is necessary to amplify this resistance. The Wheatstone bridge is described next.

4.10.1 Wheatstone Bridge Circuits

99 Due to their outstanding sensitivity, Wheatstone bridge circuits are very advantageous for the measurement of resistance, inductance, and capacitance. Wheatstone bridges are widely used for strain measurements. A Wheatstone bridge is shown in Figure 4.11. It consists of 4 resistors arranged in a diamond orientation. An input DC voltage, or excitation voltage, is applied between the top and bottom of the diamond and the output voltage is measured across the middle. When the output voltage is zero, the bridge is said to be balanced. One or more of the legs of the bridge may be a resistive transducer, such as a strain gage. The other legs of the bridge are simply completion resistors with resistance equal to that of the strain gage(s). As the resistance of one of the legs changes, by a change in strain from a resistive strain gage for example, the previously balanced bridge is now unbalanced. This unbalance causes a voltage to appear across the middle of the bridge. This induced voltage may be measured with a voltmeter or the resistor in the opposite leg may be adjusted to re-balance the bridge. In either case the change in resistance that caused the induced voltage may be measured and converted to obtain the

engineering units of strain.

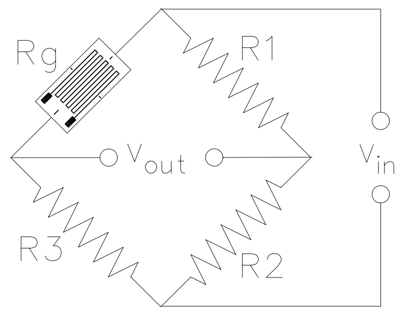


Figure 4.11: Quarter Wheatstone Bridge Circuit

4.10.2 Quarter Bridge Circuits

100 If a strain gage is oriented in one leg of the circuit and the other legs contain fixed resistors as shown in Figure 4.11, the circuit is known as a quarter bridge circuit. The circuit is balanced when $\frac{R_1}{R_2} = \frac{R_{gage}}{R_3}$. When the circuit is unbalanced $V_{out} = V_{in}(\frac{R_1}{R_1+R_2} - \frac{R_{gage}}{R_{gage}+R_3})$.

101 Wheatstone bridges may also be formed with two or four legs of the bridge being composed of resistive transducers and are called a half bridge and full bridge respectively. Depending upon the type of application and desired results, the equations for these circuits will vary as shown in Figure 4.12. Here E_0 is the output voltage in mVolts, E is the excitation voltage in Volts, ϵ is strain and ν is Poisson's ratio.

102 In order to illustrate how to compute a calibration factor for a particular experiment, suppose a single active gage in uniaxial compression is used. This will correspond to the upper Wheatstone bridge configuration of Figure 4.12. The formula then is

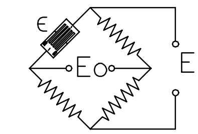
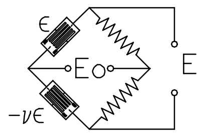
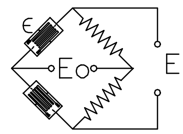
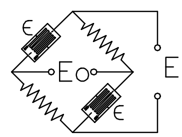
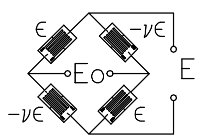
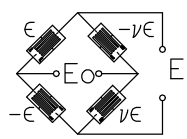
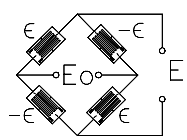
Bridge	Description	Eo/E in mV/V; ϵ in microStrain
	Single active gage in uniaxial compression or tension.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{(4+2F\epsilon(10^{-6}))}$
	Two active gages in uniaxial stress field. One "Poisson Gage" & one gage aligned with max– imum principal strain.	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{(4+2F\epsilon(1-\nu)(10^{-6}))}$
	Two active gages with equal and opposite. Common for bending beam test.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{2}$
	Two active gages with equal strains of the same sign. Bending cancellation arrangement.	$\frac{E}{Eo} = \frac{F\epsilon(10^{-3})}{(2+F\epsilon(10^{-6}))}$
	Four active gages in uniaxial stress field. Two "Poisson Gages" & two aligned with maximum principal strain. Column test.	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{(2+F\epsilon(1-\nu)(10^{-6}))}$
	Four active gages in uniaxial stress field. Two "Poisson Gages" & two aligned with maxium principal strain. Beam Test	$\frac{E}{Eo} = \frac{F\epsilon(1+\nu)(10^{-3})}{2}$
	Four active gages with pairs subjected to equal and opposite strains. Typical of a beam in bending or a shaft in torsion.	$\frac{E}{Eo} = F\epsilon(10^{-3})$

Figure 4.12: Wheatstone Bridge Configurations

$$\frac{E_0}{E} = \frac{F\epsilon(10^{-3})}{4 + 2F\epsilon(10^{-6})} \tag{4.187}$$

103 The extra term in the denominator $2F\epsilon(10^{-6})$ is a correction factor for non-linearity. Because this term is quite small compared to the other term in the denominator it will be ignored. For most measurements a gain is necessary to increase the output voltage from the Wheatstone bridge. The gain relation for the output voltage may be written as $V = GE_0(10^3)$, where V is now in Volts. so Equation 4.187 becomes

$$\begin{aligned} \frac{V}{EG(10^3)} &= \frac{F\epsilon(10^{-3})}{4} \\ \frac{\epsilon}{V} &= \frac{4}{FEG} \end{aligned} \tag{4.188}$$

104 Here, Equation 4.188 is the calibration factor in units of strain per volt. For common values where $F = 2.07$, $G = 1000$, $E = 5$, the calibration factor is simply $\frac{4}{(2.07)(1000)(5)}$ or 386.47 microstrain per volt.

Chapter 5

MATHEMATICAL PRELIMINARIES; Part III VECTOR INTEGRALS

In a preceding chapter, we have reviewed vector differentiation. This was necessary to properly introduce the concept of strain tensor. This time, we will review some of the basic vector integral theorems. Those will be needed to properly express the conservation laws in the next chapter.

5.1 Integral of a Vector

1 The integral of a vector $\mathbf{R}(u) = R_1(u)\mathbf{e}_1 + R_2(u)\mathbf{e}_2 + R_3(u)\mathbf{e}_3$ is defined as

$$\int \mathbf{R}(u)\mathrm{d}u = \mathbf{e}_1 \int R_1(u)\mathrm{d}u + \mathbf{e}_2 \int R_2(u)\mathrm{d}u + \mathbf{e}_3 \int R_3(u)\mathrm{d}u \tag{5.1}$$

if a vector $\mathbf{S}(u)$ exists such that $\mathbf{R}(u) = \frac{d}{\mathrm{d}u}(\mathbf{S}(u))$, then

$$\int \mathbf{R}(u)\mathrm{d}u = \int \frac{d}{\mathrm{d}u}(\mathbf{S}(u))\,\mathrm{d}u = \mathbf{S}(u) + c \tag{5.2}$$

5.2 Line Integral

2 Given $\mathbf{r}(u) = x(u)\mathbf{e}_1 + y(u)\mathbf{e}_2 + z(u)\mathbf{e}_3$ where $\mathbf{r}(u)$ is a position vector defining a curve $\mathcal{C}$ connecting point P_1 to P_2 where $u = u_1$ and $u = u_2$ respectively, anf given $A(x,y,z) = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$ being a vectorial function defined and continuous along $\mathcal{C}$, then the integral of the tangential component of $\mathbf{A}$ along $\mathcal{C}$ from P_1 to P_2 is given by

$$\int_{P_1}^{P_2} \mathbf{A}\cdot\mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r} = \int_{\mathcal{C}} A_1\mathrm{d}x + A_2\mathrm{d}y + A_3\mathrm{d}z \tag{5.3}$$

If $\mathbf{A}$ were a force, then this integral would represent the corresponding work.

3 If the contour is closed, then we define the **contour integral** as

$$\oint_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{r} = \int_{\mathcal{C}} A_1\mathrm{d}x + A_2\mathrm{d}y + A_3\mathrm{d}z \tag{5.4}$$

4 †It can be shown that if $\mathbf{A} = \boldsymbol{\nabla}\phi$ (i.e a vector), then

$$\int_{P_1}^{P_2} \mathbf{A}{\cdot}\mathrm{d}\mathbf{r} \qquad \text{is independent of the path } \mathcal{C} \text{ connecting } P_1 \text{ to } P_2 \tag{5.5-a}$$

$$\oint_{\mathcal{C}} \mathbf{A}{\cdot}\mathrm{d}\mathbf{r} = 0 \qquad \text{along a closed contour line} \tag{5.5-b}$$

5.3 Integration by Parts

5 The integration by part formula is

$$\boxed{\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)\mathrm{d}x} \tag{5.6}$$

5.4 Gauss; Divergence Theorem

6 In the most general case we have

$$\int_\Omega \delta F = \int_{\delta\Omega} F \tag{5.7}$$

7 The divergence theorem (also known as Ostrogradski's Theorem) comes repeatedly in solid mechanics and can be stated as follows:

$$\boxed{\int_\Omega \boldsymbol{\nabla}{\cdot}\mathbf{v}d\Omega = \int_\Gamma \mathbf{v}.\mathbf{n}d\Gamma \quad \text{or} \quad \int_\Omega v_{i,i}d\Omega = \int_\Gamma v_i n_i d\Gamma} \tag{5.8}$$

The flux of a vector function through some closed surface equals the integral of the divergence of that function over the volume enclosed by the surface.

8 For 2D-1D transformations, we have

$$\boxed{\int_A \boldsymbol{\nabla}{\cdot}\mathbf{q}dA = \oint_s \mathbf{q}^T\mathbf{n}ds} \tag{5.9}$$

9 This theorem is sometime refered to as Green's theorem in space.

5.4.1 †Green-Gauss

$$\boxed{\int_\Omega \Phi\boldsymbol{\nabla}{\cdot}\mathbf{v}d\Omega = \int_\Gamma \Phi\mathbf{v}^T\mathbf{n}d\Gamma - \int_\Omega (\boldsymbol{\nabla}\Phi)^T\mathbf{v}d\Omega} \tag{5.10}$$

If we select $\mathbf{v}^T = [\ \Psi \quad 0 \quad 0\ ]$, we obtain

$$\int_\Omega \Phi\frac{\partial\Psi}{\partial x}d\Omega = \int_\Gamma \Phi\Psi n_x d\Gamma - \int_\Omega \frac{\partial\Phi}{\partial x}\Psi d\Omega \tag{5.11}$$

5.5 Stoke's Theorem

10 Stoke's theorem states that

$$\oint_{\mathcal{C}} \mathbf{A}\cdot\mathrm{d}\mathbf{r} = \int\int_{\mathcal{S}} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{n}\mathrm{d}S = \int\int_{\mathcal{S}} (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathrm{d}\mathbf{S} \tag{5.12}$$

where $\mathcal{S}$ is an open surface with two faces confined by $\mathcal{C}$ Stoke's theorem says that the line integral of the tangential component of a vector function over some closed path equals the surface integral of the normal component of the curl of that function integrated over any capping surface of the path.

5.5.1 Green; Gradient Theorem

11 Green's theorem in plane is a special case of Stoke's theorem.

$$\oint (R\mathrm{d}x + S\mathrm{d}y) = \int_{\Gamma} \left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y} \right) \mathrm{d}x\mathrm{d}y \tag{5.13}$$

■ **Example 5-1: Physical Interpretation of the Divergence Theorem**

Provide a physical interpretation of the Divergence Theorem.

Solution:

A fluid has a velocity field $\mathbf{v}(x, y, z)$ and we first seek to determine the net inflow per unit time per unit volume in a parallelepiped centered at $P(x, y, z)$ with dimensions $\Delta x, \Delta y, \Delta z$, Fig. 5.1-a.

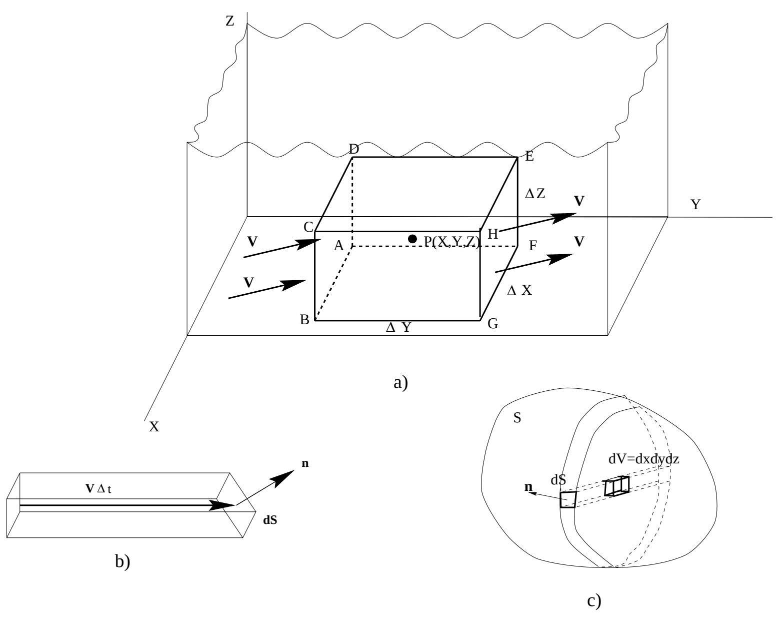


Figure 5.1: Physical Interpretation of the Divergence Theorem

$$v_x \,|_{x,y,z} \quad \approx \quad v_x \tag{5.14-a}$$

$$v_x\Big|_{x-\Delta x/2,y,z} \approx v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x \quad \text{AFED} \tag{5.14-b}$$

$$v_x\Big|_{x+\Delta x/2,y,z} \approx v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x \quad \text{GHCB} \tag{5.14-c}$$

The net inflow per unit time across the x planes is

$$\Delta V_x = \left(v_x + \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\Delta z - \left(v_x - \frac{1}{2}\frac{\partial v_x}{\partial x}\Delta x\right)\Delta y\Delta z \tag{5.15-a}$$

$$= \frac{\partial v_x}{\partial x}\Delta x\Delta y\Delta z \tag{5.15-b}$$

Similarly

$$\Delta V_y = \frac{\partial v_y}{\partial y}\Delta x\Delta y\Delta z \tag{5.16-a}$$

$$\Delta V_z = \frac{\partial v_z}{\partial z}\Delta x\Delta y\Delta z \tag{5.16-b}$$

Hence, the total increase per unit volume and unit time will be given by

$$\frac{\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z} = \text{div }\mathbf{v} = \boldsymbol{\nabla}\cdot\mathbf{v} \tag{5.17}$$

Furthermore, if we consider the total of fluid crossing dS during Δt, Fig. 5.1-b, it will be given by $(\mathbf{v}\Delta t)\cdot\mathbf{n}dS = \mathbf{v}\cdot\mathbf{n}dS\Delta t$ or the volume of fluid crossing dS per unit time is $\mathbf{v}\cdot\mathbf{n}dS$.

Thus for an arbitrary volume, Fig. 5.1-c, the total amount of fluid crossing a closed surface S per unit time is $\int_S \mathbf{v}\cdot\mathbf{n}dS$. But this is equal to $\int_V \boldsymbol{\nabla}\cdot\mathbf{v}dV$ (Eq. 5.17), thus

$$\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \boldsymbol{\nabla}\cdot\mathbf{v}dV \tag{5.18}$$

which is the divergence theorem. ■

Chapter 6

FUNDAMENTAL LAWS of CONTINUUM MECHANICS

6.1 Introduction

1 We have thus far studied the stress tensors (Cauchy, Piola Kirchoff), and several other tensors which describe strain at a point. In general, those tensors will vary from point to point and represent a **tensor field**.

2 We have also obtained only one differential equation, that was the compatibility equation.

3 In this chapter, we will derive additional differential equations governing the way stress and deformation vary at a point and with time. They will apply to any continuous medium, and yet at the end we will still not have enough equations to determine unknown tensor fields. For that we need to wait for the next chapter where constitututive laws relating stress and strain will be introduced. Only with constitutive equations and boundary and initial conditions would we be able to obtain a well defined mathematical problem to solve for the stress and deformation distribution or the displacement or velocity fields (i.e. identical number of variables and equations).

4 In this chapter we shall derive differential equations expressing locally the conservation of mass, momentum and energy. These differential equations of balance will be derived from integral forms of the equation of balance expressing the fundamental postulates of continuum mechanics.

6.1.1 Conservation Laws

5 Conservation laws constitute a fundamental component of classical physics. A conservation law establishes a balance of a scalar or tensorial quantity in voulme V bounded by a surface S. In its most general form, such a law may be expressed as

$$\boxed{\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathcal{A}(\mathbf{x},t)\mathrm{d}V}_{\text{Rate of variation}} + \underbrace{\int_S \boldsymbol{\alpha}(\mathbf{x},t,\mathbf{n})\mathrm{d}S}_{\text{Exchange by Diffusion}} = \underbrace{\int_V \mathbf{A}(\mathbf{x},t)\mathrm{d}V}_{\text{Source}}} \tag{6.1}$$

where $\mathcal{A}$ is the volumetric density of the quantity of interest (mass, linear momentum, energy, ...), $\mathbf{A}$ is the rate of volumetric density of what is provided from the outside, and $\boldsymbol{\alpha}$ is the rate of surface density of what is lost through the surface S of V and will be a function of the normal to the surface $\mathbf{n}$.

6 Hence, we read the previous equation as: *The input quantity (provided by the right hand side) is equal to what is lost across the boundary, and to modify $\mathcal{A}$ which is the quantity of interest.*

7 †The dimensions of various quantities are given by

$$\dim(\mathcal{A}) = \dim(\mathcal{A}L^{-3}) \tag{6.2-a}$$
$$\dim(\boldsymbol{\alpha}) = \dim(\mathcal{A}L^{-2}t^{-1}) \tag{6.2-b}$$
$$\dim(\mathbf{A}) = \dim(\mathcal{A}L^{-3}t^{-1}) \tag{6.2-c}$$

8 Hence this chapter will apply the conservation law to mass, momentum, and energy. The resulting differential equations will provide additional interesting relation with regard to the imcompressibiltiy of solids (important in classical hydrodynamics and plasticity theories), equilibrium and symmetry of the stress tensor, and the first law of thermodynamics.

9 The enunciation of the preceding three conservation laws plus the second law of thermodynamics, constitute what is commonly known as the **fundamental laws of continuum mechanics**.

6.1.2 Fluxes

10 Prior to the enunciation of the first conservation law, we need to define the concept of flux across a bounding surface.

11 The **flux** across a surface can be graphically defined through the consideration of an imaginary surface fixed in space with continuous "medium" flowing through it. If we assign a positive side to the surface, and take **n** in the positive sense, then the volume of "material" flowing through the infinitesimal surface area dS in time dt is equal to the volume of the cylinder with base dS and slant height vdt parallel to the velocity vector **v**, Fig. 6.1. The altitude of the cylinder is $v_n\mathrm{d}t = \mathbf{v}\cdot\mathbf{n}\mathrm{d}t$, hence the volume at time dt is $v_n\mathrm{d}t\mathrm{d}S$, (If $\mathbf{v}\cdot\mathbf{n}$ is negative, then the flow is in the negative direction), and the total flux of the

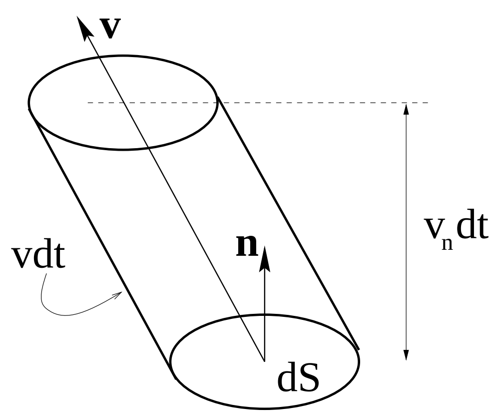


Figure 6.1: Flux Through Area dS

volume is

$$\boxed{\text{Volume Flux } = \int_S \mathbf{v}\cdot\mathbf{n}\mathrm{d}S = \int_S v_j n_j \mathrm{d}S} \tag{6.3}$$

where the last form is for rectangular cartesian components.

12 We can generalize this definition and define the following fluxes per unit area through dS:

$$\text{Mass Flux} = \int_S \rho\mathbf{v}\cdot\mathbf{n}\mathrm{d}S = \int_S \rho v_j n_j \mathrm{d}S \tag{6.4-a}$$

$$\text{Momentum Flux} = \int_S \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\mathrm{d}S = \int_S \rho v_k v_j n_j \mathrm{d}S \quad (6.4\text{-b})$$

$$\text{Kinetic Energy Flux} = \int_S \frac{1}{2}\rho v^2(\mathbf{v}\cdot\mathbf{n})\mathrm{d}S = \int_S \frac{1}{2}\rho v_i v_i v_j n_j \mathrm{d}S \quad (6.4\text{-c})$$

$$\text{Heat flux} = \int_S \mathbf{q}\cdot\mathbf{n}\mathrm{d}S = \int_S q_j n_j \mathrm{d}S \quad (6.4\text{-d})$$

$$\text{Electric flux} = \int_S \mathbf{J}\cdot\mathbf{n}\mathrm{d}S = \int_S J_j n_j \mathrm{d}S \quad (6.4\text{-e})$$

6.1.3 †Spatial Gradient of the Velocity

13 We define **L** as the **spatial gradient of the velocity** and in turn this gradient can be decomposed into a symmetric **rate of deformation tensor D** (or **stretching tensor**) and a skew-symmeteric tensor **W** called the **spin tensor** or **vorticity tensor**[1].

$$L_{ij} = v_{i,j} \quad \text{or} \quad \mathbf{L} = \mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} \quad (6.5\text{-a})$$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \quad (6.5\text{-b})$$

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}\right) \quad \text{and} \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{v}\boldsymbol{\nabla}_{\mathbf{x}} - \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}\right) \quad (6.5\text{-c})$$

these terms will be used in the derivation of the first principle.

6.2 †Conservation of Mass; Continuity Equation

14 If we consider an arbitrary volume V, fixed in space, and bounded by a surface S. If a continuous medium of density ρ fills the volume at time t, then the total mass in V is

$$M = \int_V \rho(\mathbf{x}, t)\mathrm{d}V \quad (6.6)$$

where $\rho(\mathbf{x}, t)$ is a continuous function called the **mass density**. We note that this spatial form in terms of $\mathbf{x}$ is most common in fluid mechanics.

15 The rate of increase of the total mass in the volume is

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t}\mathrm{d}V \quad (6.7)$$

16 The **Law of conservation of mass** requires that the mass of a specific portion of the continuum remains constant. Hence, if no mass is created or destroyed inside V, then the preceding equation must eqaul the **inflow of mass** (of **flux**) through the surface. The outflow is equal to $\mathbf{v}\cdot\mathbf{n}$, thus the inflow will be equal to $-\mathbf{v}\cdot\mathbf{n}$.

$$\int_S (-\rho v_n)\mathrm{d}S = -\int_S \rho\mathbf{v}\cdot\mathbf{n}\mathrm{d}S = -\int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\mathrm{d}V \quad (6.8)$$

must be equal to $\frac{\partial M}{\partial t}$. Thus

$$\int_V \left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})\right]\mathrm{d}V = 0 \quad (6.9)$$

since the integral must hold for any arbitrary choice of $\mathrm{d}V$, then we obtain

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) \quad \text{or} \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \quad (6.10)$$

[1]Note similarity with Eq. 4.111-b.

17 The chain rule will in turn give

$$\frac{\partial(\rho v_i)}{\partial x_i} = \rho\frac{\partial v_i}{\partial x_i} + v_i\frac{\partial \rho}{\partial x_i} \tag{6.11}$$

18 It can be shown that the rate of change of the density in the neighborhood of a particle instantaneously at **x** by

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\rho = \frac{\partial \rho}{\partial t} + v_i\frac{\partial \rho}{\partial x_i} \tag{6.12}$$

where the first term gives the local rate of change of the density in the neighborhood of the place of **x**, while the second term gives the **convective rate of change** of the density in the neighborhood of a particle as it moves to a place having a different density. The first term vanishes in a steady flow, while the second term vanishes in a uniform flow.

19 Upon substitution in the last three equations, we obtain the continuity equation

$$\boxed{\frac{\mathrm{d}\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0 \quad \text{or} \quad \frac{\mathrm{d}\rho}{dt} + \rho\boldsymbol{\nabla}\cdot\mathbf{v} = 0} \tag{6.13}$$

The vector form is independent of any choice of coordinates. This equation shows that the divergence of the velocity vector field equals $(-1/\rho)(\mathrm{d}\rho/\mathrm{d}t)$ and measures the rate of flow of material away from the particle and is equal to the unit rate of decrease of density ρ in the neighborhood of the particle.

20 If the material is incompressible, so that the density in the neighborhood of each material particle remains constant as it moves, then the continuity equation takes the simpler form

$$\boxed{\frac{\partial v_i}{\partial x_i} = 0 \quad \text{or} \quad \boldsymbol{\nabla}\cdot\mathbf{v} = 0} \tag{6.14}$$

this is the **condition of incompressibility**

6.3 Linear Momentum Principle; Equation of Motion

6.3.1 Momentum Principle

21 The momentum principle states that *the time rate of change of the total momentum of a given set of particles equals the vector sum of all external forces acting on the particles of the set, provided Newton's Third Law applies*. The continuum form of this principle is a basic **postulate** of continuum mechanics.

$$\frac{d}{\mathrm{d}t}\int_V \rho\mathbf{v}\mathrm{d}V = \int_S \mathbf{t}\mathrm{d}S + \int_V \rho\mathbf{b}\mathrm{d}V \tag{6.15}$$

22 Then we substitute $t_i = T_{ij}n_j$ and apply the divergence theorem to obtain

$$\int_V \left(\frac{\partial T_{ij}}{\partial x_j} + \rho b_i\right)\mathrm{d}V = \int_V \rho\frac{\mathrm{d}v_i}{\mathrm{d}t}\mathrm{d}V \tag{6.16-a}$$

$$\int_V \left[\frac{\partial T_{ij}}{\partial x_j} + \rho b_i - \rho\frac{\mathrm{d}v_i}{\mathrm{d}t}\right]\mathrm{d}V = 0 \tag{6.16-b}$$

or for an arbitrary volume

$$\boxed{\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{\mathrm{d}v_i}{\mathrm{d}t} \quad \text{or} \quad \boldsymbol{\nabla}\cdot\mathbf{T} + \rho\mathbf{b} = \rho\frac{\mathrm{d}v}{\mathrm{d}t}} \tag{6.17}$$

which is **Cauchy's (first) equation of motion**, or **the linear momentum principle**, or more simply **equilibrium equation**.

23 When expanded in 3D, this equation yields:

$$\begin{aligned}\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho b_1 &= 0\\ \frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho b_2 &= 0\\ \frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}+\rho b_3 &= 0\end{aligned} \tag{6.18}$$

24 We note that these equations could also have been derived from the free body diagram shown in Fig. 6.2 with the assumption of **equilibrium** (via Newton's second law) considering an infinitesimal element of dimensions $dx_1 \times dx_2 \times dx_3$. Writing the summation of forces, will yield

$$\boxed{T_{ij,j}+\rho b_i = 0} \tag{6.19}$$

where ρ is the density, b_i is the body force (including inertia).

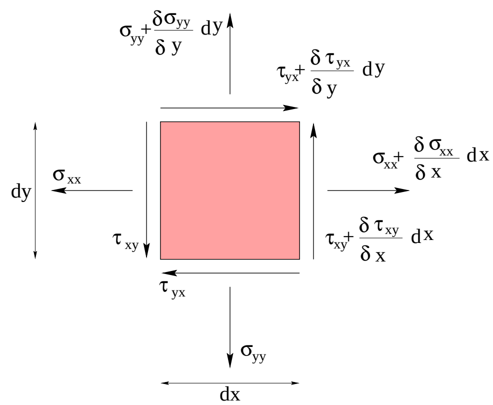


Figure 6.2: Equilibrium of Stresses, Cartesian Coordinates

■ Example 6-1: Equilibrium Equation

In the absence of body forces, does the following stress distribution

$$\begin{bmatrix} x_2^2+\nu(x_1^2-x_x^2) & -2\nu x_1 x_2 & 0\\ -2\nu x_1 x_2 & x_1^2+\nu(x_2^2-x_1^2) & 0\\ 0 & 0 & \nu(x_1^2+x_2^2)\end{bmatrix} \tag{6.20}$$

where ν is a constant, satisfy equilibrium?

Solution:

$$\frac{\partial T_{1j}}{\partial x_j} = \frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3} = 2\nu x_1 - 2\nu x_1 = 0\surd \tag{6.21-a}$$

$$\frac{\partial T_{2j}}{\partial x_j} = \frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3} = -2\nu x_2 + 2\nu x_2 = 0\surd \tag{6.21-b}$$

$$\frac{\partial T_{3j}}{\partial x_j} = \frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3} = 0\surd \tag{6.21-c}$$

Therefore, equilibrium is satisfied. ■

6.3.2 †Moment of Momentum Principle

25 The moment of momentum principle states that *the time rate of change of the total moment of momentum of a given set of particles equals the vector sum of the moments of all external forces acting on the particles of the set.*

26 Thus, in the absence of **distributed couples** (this theory of Cosserat will not be covered in this course) we postulate the same principle for a continuum as

$$\boxed{\int_S (\mathbf{r}\times\mathbf{t})\mathrm{d}S + \int_V (\mathbf{r}\times\rho\mathbf{b})\mathrm{d}V = \frac{d}{\mathrm{d}t}\int_V (\mathbf{r}\times\rho\mathbf{v})\mathrm{d}V} \tag{6.22}$$

6.4 Conservation of Energy; First Principle of Thermodynamics

27 The first principle of thermodynamics relates the work done on a (closed) system and the heat transfer into the system to the change in energy of the system. We shall assume that the only energy transfers to the system are by mechanical work done on the system by surface traction and body forces, by heat transfer through the boundary.

6.4.1 Global Form

28 If mechanical quantities only are considered, the **principle of conservation of energy** for the continuum may be derived directly from the equation of motion given by Eq. 6.17. This is accomplished by taking the integral over the volume V of the scalar product between Eq. 6.17 and the velocity v_i.

$$\int_V \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathrm{d}V = \int_V v_i T_{ji,j}\mathrm{d}V + \int_V \rho b_i v_i \mathrm{d}V \tag{6.23}$$

29 If we consider the left hand side

$$\int_V \rho v_i \frac{\mathrm{d}v_i}{\mathrm{d}t}\mathrm{d}V = \frac{d}{\mathrm{d}t}\int_V \frac{1}{2}\rho v_i v_i \mathrm{d}V = \frac{d}{\mathrm{d}t}\int_V \frac{1}{2}\rho v^2 \mathrm{d}V = \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \tag{6.24}$$

which represents the time rate of change of the **kinetic energy** $\mathcal{K}$ in the continuum.

30 †Also we have

$$v_i T_{ji,j} = (v_i T_{ji})_{,j} - \underbrace{v_{i,j}}_{L_{ij}} T_{ji} \tag{6.25}$$

and from Eq. 6.5-b we have $v_{i,j} = D_{ij} + W_{ij}$. It can be shown that since W_{ij} is skew-symmetric, and $\mathbf{T}$ is symmetric, that $T_{ij}W_{ij} = 0$, and thus $T_{ij}L_{ij} = T_{ij}D_{ij}$. $\mathbf{T}\dot{\mathbf{D}}$ is called the **stress power**.

31 If we consider thermal processes, the rate of increase of total heat into the continuum is given by

$$Q = -\int_S q_i n_i \mathrm{d}S + \int_V \rho r \mathrm{d}V \tag{6.26}$$

Q has the dimension[2] of power, that is ML^2T^{-3}, and the SI unit is the Watt (W). $\mathbf{q}$ is the **heat flux** per unit area by conduction, its dimension is MT^{-3} and the corresponding SI unit is Wm^{-2}. Finally, r is the **radiant heat constant** per unit mass, its dimension is $MT^{-3}L^{-4}$ and the corresponding SI unit is Wm^{-6}.

[2] Work=$FL = ML^2T^{-2}$; Power=Work/time

32 We thus have

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} + \int_V D_{ij} T_{ij} \mathrm{d}V = \int_V (v_i T_{ji})_{,j} \mathrm{d}V + \int_V \rho v_i b_i \mathrm{d}V + Q \tag{6.27}$$

33 We next convert the first integral on the right hand side to a surface integral by the divergence theorem ($\int_V \boldsymbol{\nabla}\cdot\mathbf{T} dV = \int_S \mathbf{T}.\mathbf{n} \mathrm{d}S$) and since $t_i = T_{ij} n_j$ we obtain

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} + \int_V D_{ij} T_{ij} \mathrm{d}V = \int_S v_i t_i \mathrm{d}S + \int_V \rho v_i b_i \mathrm{d}V + Q \tag{6.28}$$

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} + Q \tag{6.29}$$

this equation relates the time rate of change of total mechanical energy of the continuum on the left side to the rate of work done by the surface and body forces on the right hand side.

34 If both mechanical and non mechanical energies are to be considered, the first principle states that *the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to, or removed from the continuum per unit time (heat, chemical, electromagnetic, etc.).*

35 For a thermomechanical continuum, it is customary to express the time rate of change of internal energy by the integral expression

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = \frac{d}{\mathrm{d}t} \int_V \rho u \mathrm{d}V \tag{6.30}$$

where u is the internal energy per unit mass or **specific internal energy**. We note that $\mathcal{U}$ appears only as a differential in the first principle, hence if we really need to evaluate this quantity, we need to have a reference value for which $\mathcal{U}$ will be null. The dimension of $\mathcal{U}$ is one of energy dim $\mathcal{U} = ML^2T^{-2}$, and the SI unit is the Joule, similarly dim $u = L^2T^{-2}$ with the SI unit of Joule/Kg.

36 In terms of energy integrals, the first principle can be rewritten as

$$\overbrace{\underbrace{\frac{d}{\mathrm{d}t}\int_V \frac{1}{2}\rho v_i v_i \mathrm{d}V}_{\frac{d\mathcal{K}}{dt}=\dot{\mathcal{K}}} + \underbrace{\frac{d}{\mathrm{d}t}\int_V \rho u \mathrm{d}V}_{\frac{d\mathcal{U}}{dt}=\dot{\mathcal{U}}}}^{\text{Rate of increase}} = \underbrace{\overbrace{\int_S t_i v_i \mathrm{d}S}^{\text{Exchange}} + \overbrace{\int_V \rho v_i b_i \mathrm{d}V}^{\text{Source}}}_{\frac{d\mathcal{W}}{dt}(\mathcal{P}_{ext})} + \underbrace{\overbrace{\int_V \rho r \mathrm{d}V}^{\text{Source}} - \overbrace{\int_S q_i n_i \mathrm{d}S}^{\text{Exchange}}}_{Q(\mathcal{P}_{cal})} \tag{6.31}$$

37 † If we apply Gauss theorem and convert the surface integral, collect terms and use the fact that dV is arbitrary we obtain

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{T}{:}\mathbf{D} + \rho r - \boldsymbol{\nabla}\cdot\mathbf{q} \tag{6.32}$$

$$\text{or} \tag{6.33}$$

$$\rho \frac{\mathrm{d}u}{\mathrm{d}t} = T_{ij} D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j} \tag{6.34}$$

This equation expresses the rate of change of **internal energy** as the sum of the **stress power** plus the **heat** added to the continuum.

38 In ideal elasticity, heat transfer is considered insignificant, and all of the input work is assumed converted into internal energy in the form of recoverable stored elastic strain energy, which can be recovered as work when the body is unloaded.

39 In general, however, the major part of the input work into a deforming material is not recoverably stored, but dissipated by the deformation process causing an increase in the body's temperature and eventually being conducted away as heat.

6.4.2 Local Form

40 Examining the third term in Eq. 6.31

$$\int_S t_i v_i \mathrm{d}S = \int_S v_i T_{ij} n_j \mathrm{d}S = \int_V \frac{\partial (v_i T_{ij})}{\partial x_j} \mathrm{d}V \tag{6.35-a}$$

$$= \int_V T_{ij} \frac{\partial v_i}{\partial x_j} \mathrm{d}V + \int_V v_i \frac{\partial T_{ij}}{\partial x_j} \mathrm{d}V = \int_V \mathbf{T}:\dot{\varepsilon} \mathrm{d}V + \int_V \mathbf{v} \cdot (\nabla \cdot \mathbf{T}) \mathrm{d}V \tag{6.35-b}$$

41 We now evaluate $\mathcal{P}_{ext}$ in Eq. 6.31

$$\mathcal{P}_{ext} = \int_S t_i v_i \mathrm{d}S + \int_V \rho v_i b_i \mathrm{d}V \tag{6.36-a}$$

$$= \int_V \mathbf{v} \cdot (\rho \mathbf{b} + \nabla \cdot \mathbf{T}) \mathrm{d}V + \int_V \mathbf{T}:\dot{\varepsilon} \mathrm{d}V \tag{6.36-b}$$

Using Eq. 6.17 ($T_{ij,j} + \rho b_i = \rho \dot{v}_i$), this reduces to

$$\mathcal{P}_{ext} = \underbrace{\int_V \mathbf{v} \cdot (\rho \dot{\mathbf{v}}) \mathrm{d}V}_{d\mathcal{K}} + \underbrace{\int_V \mathbf{T}:\dot{\varepsilon} \mathrm{d}V}_{\mathcal{P}_{int}} \tag{6.37}$$

(note that $\mathcal{P}_{int}$ corresponds to the stress power).

42 Hence, we can rewrite Eq. 6.31 as

$$\dot{\mathcal{U}} = \mathcal{P}_{int} + \mathcal{P}_{cal} = \int_V \mathbf{T}:\dot{\varepsilon} \mathrm{d}V + \int_V (\rho r - \nabla \cdot \mathbf{q}) \mathrm{d}V \tag{6.38}$$

Introducing the specific internal energy u (taken per unit mass), we can express the internal energy of the finite body as $\mathcal{U} = \int_V \rho u \mathrm{d}V$, and rewrite the previous equation as

$$\int_V (\rho \dot{u} - \mathbf{T}:\dot{\varepsilon} - \rho r + \nabla \cdot \mathbf{q}) \mathrm{d}V = 0 \tag{6.39}$$

Since this equation must hold for any arbitrary partial volume V, we obtain the **local form** of the First Law

$$\boxed{\rho \dot{u} = \mathbf{T}:\dot{\varepsilon} + \rho r - \nabla \cdot \mathbf{q}} \tag{6.40}$$

or the rate of increase of internal energy in an elementary material volume is equal to the sum of 1) the power of stress $\mathbf{T}$ working on the strain rate $\dot{\varepsilon}$, 2) the heat supplied by an internal source of intensity r, and 3) the negative divergence of the heat flux which represents the net rate of heat entering the elementary volume through its boundary.

6.5 Second Principle of Thermodynamics

6.5.1 Equation of State

43 The complete characterization of a thermodynamic system is said to describe the **state** of a system (here a continuum). This description is specified, in general, by several thermodynamic and kinematic **state variables**. A change in time of those state variables constitutes a **thermodynamic process**. Usually state variables are not all independent, and functional relationships exist among them through

equations of state. Any state variable which may be expressed as a single valued function of a set of other state variables is known as a **state function**.

44 The first principle of thermodynamics can be regarded as an expression of the interconvertibility of heat and work, maintaining an energy balance. It places no restriction on the direction of the process. In classical mechanics, kinetic and potential energy can be easily transformed from one to the other in the absence of friction or other dissipative mechanism.

45 The first principle leaves unanswered the question of the extent to which conversion process is **reversible** or **irreversible**. If thermal processes are involved (friction) dissipative processes are irreversible processes, and it will be up to the second principle of thermodynamics to put limits on the direction of such processes.

6.5.2 Entropy

46 The basic criterion for irreversibility is given by the **second principle of thermodynamics** through the statement on the limitation of **entropy production**. This law postulates the existence of two distinct state functions: θ the **absolute temperature** and $\mathcal{S}$ the **entropy** with the following properties:

1. θ is a positive quantity.
2. Entropy is an extensive property, i.e. the total entropy in a system is the sum of the entropies of its parts.

47 Thus we can write

$$ds = ds^{(e)} + ds^{(i)} \tag{6.41}$$

where $ds^{(e)}$ is the increase due to interaction with the exterior, and $ds^{(i)}$ is the internal increase, and

$$ds^{(e)} > 0 \quad \text{irreversible process} \tag{6.42-a}$$
$$ds^{(i)} = 0 \quad \text{reversible process} \tag{6.42-b}$$

48 Entropy expresses a variation of energy associated with a variation in the temperature.

6.5.2.1 †Statistical Mechanics

49 In statistical mechanics, entropy is related to the probability of the occurrence of that state among all the possible states that could occur. It is found that changes of states are more likely to occur in the direction of greater **disorder** when a system is left to itself. Thus *increased entropy means increased disorder.*

50 Hence Boltzman's principle postulates that entropy of a state is proportional to the logarithm of its probability, and for a gas this would give

$$\mathcal{S} = kN[\ln V + \frac{3}{2}\ln\theta] + C \tag{6.43}$$

where $\mathcal{S}$ is the total entropy, V is volume, θ is absolute temperature, k is Boltzman's constant, and C is a constant and N is the number of molecules.

6.5.2.2 Classical Thermodynamics

51 In a reversible process (more about that later), the change in **specific entropy** s is given by

$$ds = \left(\frac{dq}{\theta}\right)_{rev} \tag{6.44}$$

52 †If we consider an ideal gas governed by

$$pv = R\theta \tag{6.45}$$

where R is the gas constant, and assuming that the specific energy u is only a function of temperature θ, then the first principle takes the form

$$\mathrm{d}u = \mathrm{d}q - p\mathrm{d}v \tag{6.46}$$

and for constant volume this gives

$$\mathrm{d}u = \mathrm{d}q = c_v \mathrm{d}\theta \tag{6.47}$$

wher c_v is the specific heat at constant volume. The assumption that $u = u(\theta)$ implies that c_v is a function of θ only and that

$$\mathrm{d}u = c_v(\theta)\mathrm{d}\theta \tag{6.48}$$

53 †Hence we rewrite the first principle as

$$\mathrm{d}q = c_v(\theta)\mathrm{d}\theta + R\theta\frac{\mathrm{d}v}{v} \tag{6.49}$$

or division by θ yields

$$\boxed{s - s_0 = \int_{p_0,v_0}^{p,v} \frac{\mathrm{d}q}{\theta} = \int_{\theta_0}^{\theta} c_v(\theta)\frac{\mathrm{d}\theta}{\theta} + R\ln\frac{v}{v_0}} \tag{6.50}$$

which gives the change in entropy for any reversible process in an ideal gas. In this case, entropy is a state function which returns to its initial value whenever the temperature returns to its initial value that is p and v return to their initial values.

54 The Clausius-Duhem inequality, an important relation associated with the second principle, will be separately examined in Sect. 18.2.

6.6 Balance of Equations and Unknowns

55 In the preceding sections several equations and unknowns were introduced. Let us count them. for both the coupled and uncoupled cases.

		Coupled	Uncoupled
$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$	Continuity Equation	1	1
$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt}$	Equation of motion	3	3
$\rho\frac{du}{dt} = T_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j}$	Energy equation	1	
Total number of equations		**5**	**4**

56 Assuming that the body forces b_i and distributed heat sources r are prescribed, then we have the following unknowns:

		Coupled	Uncoupled
Density	ρ	1	1
Velocity (or displacement)	v_i (u_i)	3	3
Stress components	T_{ij}	6	6
Heat flux components	q_i	3	-
Specific internal energy	u	1	-
Entropy density	s	1	-
Absolute temperature	θ	1	-
Total number of unknowns		**16**	**10**

and in addition the Clausius-Duhem inequality $\frac{\mathrm{d}s}{\mathrm{d}t} \geq \frac{r}{\theta} - \frac{1}{\rho}\mathrm{div}\ \frac{\mathbf{q}}{\theta}$ which governs entropy production must hold.

57 We thus need an additional $16 - 5 = 11$ additional equations to make the system determinate. These will be later on supplied by:

6	constitutive equations
3	temperature heat conduction
2	thermodynamic equations of state
11	**Total number of additional equations**

58 The next chapter will thus discuss constitutive relations, and a subsequent one will separately discuss thermodynamic equations of state.

59 We note that for the uncoupled case

1. The energy equation is essentially the integral of the equation of motion.
2. The 6 missing equations will be entirely supplied by the constitutive equations.
3. The temperature field is regarded as known, or at most, the heat-conduction problem must be solved separately and independently from the mechanical problem.

Chapter 7

CONSTITUTIVE EQUATIONS; Part I Engineering Approach

ceiinosssttuu
Hooke, 1676
Ut tensio sic vis
Hooke, 1678

7.1 Experimental Observations

1 We shall discuss two experiments which will yield the elastic **Young's modulus**, and then the **bulk modulus**. In the former, the simplicity of the experiment is surrounded by the intriguing character of Hooke, and in the later, the bulk modulus is mathematically related to the Green deformation tensor **C**, the deformation gradient **F** and the Lagrangian strain tensor **E**.

7.1.1 Hooke's Law

2 Hooke's Law is determined on the basis of a very simple experiment in which a uniaxial force is applied on a specimen which has one dimension much greater than the other two (such as a rod). The elongation is measured, and then the stress is plotted in terms of the strain (elongation/length). The slope of the line is called **Young's modulus**.

3 Hooke anticipated some of the most important discoveries and inventions of his time but failed to carry many of them through to completion. He formulated the theory of planetary motion as a problem in mechanics, and grasped, but did not develop mathematically, the fundamental theory on which Newton formulated the law of gravitation.

His most important contribution was published in 1678 in the paper *De Potentia Restitutiva*. It contained results of his experiments with elastic bodies, and was the first paper in which the elastic properties of material was discussed.

> "*Take a wire string of 20, or 30, or 40 ft long, and fasten the upper part thereof to a nail, and to the other end fasten a Scale to receive the weights: Then with a pair of compasses take the distance of the bottom of the scale from the ground or floor underneath, and set down the said distance, then put inweights into the said scale and measure the several stretchings of the said string, and set them down. Then compare the several stretchings of the said string, and you will find that they will always bear the same proportions one to the other that the weights do that made them*".

This became **Hooke's Law**

$$\boxed{\sigma = E\varepsilon} \tag{7.1}$$

4 Because he was concerned about patent rights to his invention, he did not publish his law when first discovered it in 1660. Instead he published it in the form of an anagram "*ceiinosssttuu*" in 1676 and the solution was given in 1678. *Ut tensio sic vis* (at the time the two symbols u and v were employed interchangeably to denote either the vowel u or the consonant v), i.e. *extension varies directly with force.*

7.1.2 Bulk Modulus

5 If, instead of subjecting a material to a uniaxial state of stress, we now subject it to a hydrostatic pressure p and measure the change in volume ΔV.

6 From the summary of Table 4.1 we know that:

$$V = (\det \mathbf{F})V_0 \tag{7.2-a}$$
$$\det \mathbf{F} = \sqrt{\det \mathbf{C}} = \sqrt{\det[\mathbf{I}+2\mathbf{E}]} \tag{7.2-b}$$

therefore,

$$\frac{V+\Delta V}{V} = \sqrt{\det[\mathbf{I}+2\mathbf{E}]} \tag{7.3}$$

we can expand the determinant of the tensor $\det[\mathbf{I}+2\mathbf{E}]$ to find

$$\det[\mathbf{I}+2\mathbf{E}] = 1 + 2I_E + 4II_E + 8III_E \tag{7.4}$$

but for small strains, $I_E \gg II_E \gg III_E$ since the first term is linear in $\mathbf{E}$, the second is quadratic, and the third is cubic. Therefore, we can approximate $\det[\mathbf{I}+2\mathbf{E}] \approx 1+2I_E$, hence we define the **volumetric dilatation** as

$$\boxed{\frac{\Delta V}{V} \equiv e \approx I_E = \text{tr } \mathbf{E}} \tag{7.5}$$

this quantity is readily measurable in an experiment.

7.2 Stress-Strain Relations in Generalized Elasticity

7.2.1 Anisotropic

7 From Eq. 18.31 and 18.32 we obtain the stress-strain relation for homogeneous anisotropic material

$$\boxed{\underbrace{\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31}\end{Bmatrix}}_{T_{ij}} = \underbrace{\begin{bmatrix} c_{1111} & c_{1112} & c_{1133} & c_{1112} & c_{1123} & c_{1131}\\ & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2231}\\ & & c_{3333} & c_{3312} & c_{3323} & c_{3331}\\ & & & c_{1212} & c_{1223} & c_{1231}\\ & \text{SYM.} & & & c_{2323} & c_{2331}\\ & & & & & c_{3131}\end{bmatrix}}_{c_{ijkm}} \underbrace{\begin{Bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ 2E_{12}(\gamma_{12})\\ 2E_{23}(\gamma_{23})\\ 2E_{31}(\gamma_{31})\end{Bmatrix}}_{E_{km}}} \tag{7.6}$$

which is **Hooke's law** for small strain in linear elasticity.

8 †We also observe that for symmetric c_{ij} we retrieve **Clapeyron formula**

$$\boxed{W = \frac{1}{2}T_{ij}E_{ij}} \tag{7.7}$$

9 In general the elastic moduli c_{ij} relating the cartesian components of stress and strain depend on the orientation of the coordinate system with respect to the body. If the form of elastic potential function W and the values c_{ij} are independent of the orientation, the material is said to be **isotropic**, if not it is **anisotropic**.

10 c_{ijkm} is a fourth order tensor resulting with $3^4 = 81$ terms.

$$\begin{bmatrix}
\begin{pmatrix} c_{1,1,1,1} & c_{1,1,1,2} & c_{1,1,1,3} \\ c_{1,1,2,1} & c_{1,1,2,2} & c_{1,1,2,3} \\ c_{1,1,3,1} & c_{1,1,3,2} & c_{1,1,3,3} \end{pmatrix} &
\begin{pmatrix} c_{1,2,1,1} & c_{1,2,1,2} & c_{1,2,1,3} \\ c_{1,2,2,1} & c_{1,2,2,2} & c_{1,2,2,3} \\ c_{1,2,3,1} & c_{1,2,3,2} & c_{1,2,3,3} \end{pmatrix} &
\begin{pmatrix} c_{1,3,1,1} & c_{1,3,1,2} & c_{1,3,1,3} \\ c_{1,3,2,1} & c_{1,3,2,2} & c_{1,3,2,3} \\ c_{1,3,3,1} & c_{1,3,3,2} & c_{1,3,3,3} \end{pmatrix} \\
\begin{pmatrix} c_{2,1,1,1} & c_{2,1,1,2} & c_{2,1,1,3} \\ c_{2,1,2,1} & c_{2,1,2,2} & c_{2,1,2,3} \\ c_{2,1,3,1} & c_{2,1,3,2} & c_{2,1,3,3} \end{pmatrix} &
\begin{pmatrix} c_{2,2,1,1} & c_{2,2,1,2} & c_{2,2,1,3} \\ c_{2,2,2,1} & c_{2,2,2,2} & c_{2,2,2,3} \\ c_{2,2,3,1} & c_{2,2,3,2} & c_{2,2,3,3} \end{pmatrix} &
\begin{pmatrix} c_{2,3,1,1} & c_{2,3,1,2} & c_{2,3,1,3} \\ c_{2,3,2,1} & c_{2,3,2,2} & c_{2,3,2,3} \\ c_{2,3,3,1} & c_{2,3,3,2} & c_{2,3,3,3} \end{pmatrix} \\
\begin{pmatrix} c_{3,1,1,1} & c_{3,1,1,2} & c_{3,1,1,3} \\ c_{3,1,2,1} & c_{3,1,2,2} & c_{3,1,2,3} \\ c_{3,1,3,1} & c_{3,1,3,2} & c_{3,1,3,3} \end{pmatrix} &
\begin{pmatrix} c_{3,2,1,1} & c_{3,2,1,2} & c_{3,2,1,3} \\ c_{3,2,2,1} & c_{3,2,2,2} & c_{3,2,2,3} \\ c_{3,2,3,1} & c_{3,2,3,2} & c_{3,2,3,3} \end{pmatrix} &
\begin{pmatrix} c_{3,3,1,1} & c_{3,3,1,2} & c_{3,3,1,3} \\ c_{3,3,2,1} & c_{3,3,2,2} & c_{3,3,2,3} \\ c_{3,3,3,1} & c_{3,3,3,2} & c_{3,3,3,3} \end{pmatrix}
\end{bmatrix} \tag{7.8}$$

But the matrix must be symmetric thanks to Cauchy's second law of motion (i.e symmetry of both the stress and the strain), and thus for **anisotropic** material we will have a symmetric 6 by 6 matrix with $\frac{(6)(6+1)}{2} =$ **21 independent coefficients**.

11 †By means of coordinate transformation we can relate the material properties in one coordinate system (old) x_i, to a new one $\overline{x}_i$, thus from Eq. 1.39 ($\overline{v}_j = a_j^p v_p$) we can rewrite

$$W = \frac{1}{2} c_{rstu} E_{rs} E_{tu} = \frac{1}{2} c_{rstu} a_i^r a_j^s a_k^t a_m^u \overline{E}_{ij} \overline{E}_{km} = \frac{1}{2} c_{ijkm} \overline{E}_{ij} \overline{E}_{km} \tag{7.9}$$

thus we deduce

$$c_{ijkm} = a_i^r a_j^s a_k^t a_m^u c_{rstu} \tag{7.10}$$

that is the fourth order tensor of material constants in old coordinates may be transformed into a new coordinate system through an eighth-order tensor $a_i^r a_j^s a_k^t a_m^u$

7.2.2 †Monotropic Material

12 A **plane of elastic symmetry** exists at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane. The axes of such coordinate systems are referred to as "equivalent elastic directions".

13 If we assume $\overline{x}_1 = x_1$, $\overline{x}_2 = x_2$ and $\overline{x}_3 = -x_3$, then the transformation $\overline{x}_i = a_i^j x_j$ is defined through

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7.11}$$

where the negative sign reflects the symmetry of the mirror image with respect to the x_3 plane.

14 We next substitute in Eq.7.10, and as an example we consider $c_{1123} = a_1^r a_1^s a_2^t a_3^u c_{rstu} = a_1^1 a_1^1 a_2^2 a_3^3 c_{1123} = (1)(1)(1)(-1)c_{1123} = -c_{1123}$, obviously, this is not possible, and the only way the relation can remanin valid is if $c_{1123} = 0$. We note that all terms in c_{ijkl} with the index 3 occurring an odd number of times will be equal to zero. Upon substitution, we obtain

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & 0 & 0 \\ & c_{2222} & c_{2233} & c_{2212} & 0 & 0 \\ & & c_{3333} & c_{3312} & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{bmatrix} \tag{7.12}$$

we now have **13 nonzero coefficients**.

7.2.3 † Orthotropic Material

15 If the material possesses three mutually perpendicular planes of elastic symmetry, (that is symmetric with respect to two planes x_2 and x_3), then the transformation $x_i = a_i^j x_j$ is defined through

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{7.13}$$

where the negative sign reflects the symmetry of the mirror image with respect to the x_3 plane. Upon substitution in Eq.7.10 we now would have

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & 0 \\ & & & & & c_{3131} \end{bmatrix} \tag{7.14}$$

We note that in here all terms of c_{ijkl} with the indices 3 and 2 occuring an odd number of times are again set to zero.

16 Wood is usually considered an orthotropic material and will have **9 nonzero coefficients**.

7.2.4 †Transversely Isotropic Material

17 A material is transversely isotropic if there is a preferential direction normal to all but one of the three axes. If this axis is x_3, then rotation about it will require that

$$a_i^j = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.15}$$

substituting Eq. 7.10 into Eq. 7.18, using the above transformation matrix, we obtain

$$\begin{aligned}
c_{1111} &= (\cos^4\theta)c_{1111} + (\cos^2\theta\sin^2\theta)(2c_{1122} + 4c_{1212}) + (\sin^4\theta)c_{2222} && \text{(7.16-a)} \\
c_{1122} &= (\cos^2\theta\sin^2\theta)c_{1111} + (\cos^4\theta)c_{1122} - 4(\cos^2\theta\sin^2\theta)c_{1212} + (\sin^4\theta)c_{2211} && \text{(7.16-b)} \\
& \quad +(\sin^2\theta\cos^2\theta)c_{2222} && \text{(7.16-c)} \\
c_{1133} &= (\cos^2\theta)c_{1133} + (\sin^2\theta)c_{2233} && \text{(7.16-d)} \\
c_{2222} &= (\sin^4\theta)c_{1111} + (\cos^2\theta\sin^2\theta)(2c_{1122} + 4c_{1212}) + (\cos^4\theta)c_{2222} && \text{(7.16-e)} \\
c_{1212} &= (\cos^2\theta\sin^2\theta)c_{1111} - 2(\cos^2\theta\sin^2\theta)c_{1122} - 2(\cos^2\theta\sin^2\theta)c_{1212} + (\cos^4\theta)c_{1212} && \text{(7.16-f)} \\
& \quad +(\sin^2\theta\cos^2\theta)c_{2222} + \sin^4\theta c_{1212} && \text{(7.16-g)} \\
&\vdots
\end{aligned}$$

But in order to respect our initial assumption about symmetry, these results require that

$$\begin{aligned}
c_{1111} &= c_{2222} && \text{(7.17-a)} \\
c_{1133} &= c_{2233} && \text{(7.17-b)} \\
c_{2323} &= c_{3131} && \text{(7.17-c)} \\
c_{1212} &= \frac{1}{2}(c_{1111} - c_{1122}) && \text{(7.17-d)}
\end{aligned}$$

yielding

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & \frac{1}{2}(c_{1111} - c_{1122}) & 0 & 0 \\ & \text{SYM.} & & & c_{2323} & 0 \\ & & & & & c_{3131} \end{bmatrix} \tag{7.18}$$

we now have **5 nonzero coefficients**.

18 It should be noted that very few natural or man-made materials are truly orthotropic (certain crystals as topaz are), but a number are transversely isotropic (laminates, shist, quartz, roller compacted concrete, etc...).

7.2.5 Isotropic Material

19 An isotropic material is symmetric with respect to every plane and every axis, that is the elastic properties are identical in all directions.

20 To mathematically characterize an isotropic material, we require coordinate transformation with rotation about x_2 and x_1 axes in addition to all previous coordinate transformations. This process will enforce symmetry about all planes and all axes.

21 The rotation about the x_2 axis is obtained through

$$a_i^j = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{7.19}$$

we follow a similar procedure to the case of transversely isotropic material to obtain

$$c_{1111} = c_{3333} \tag{7.20-a}$$
$$c_{3131} = \frac{1}{2}(c_{1111} - c_{1133}) \tag{7.20-b}$$

22 next we perform a rotation about the x_1 axis

$$a_i^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{7.21}$$

it follows that

$$c_{1122} = c_{1133} \tag{7.22-a}$$
$$c_{3131} = \frac{1}{2}(c_{3333} - c_{1133}) \tag{7.22-b}$$
$$c_{2323} = \frac{1}{2}(c_{2222} - c_{2233}) \tag{7.22-c}$$

which will finally give

$$c_{ijkm} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & a & 0 & 0 \\ & \text{SYM.} & & & b & 0 \\ & & & & & c \end{bmatrix} \tag{7.23}$$

with $a = \frac{1}{2}(c_{1111} - c_{1122})$, $b = \frac{1}{2}(c_{2222} - c_{2233})$, and $c = \frac{1}{2}(c_{3333} - c_{1133})$.

23 If we denote $c_{1122} = c_{1133} = c_{2233} = \lambda$ and $c_{1212} = c_{2323} = c_{3131} = \mu$ then from the previous relations we determine that $c_{1111} = c_{2222} = c_{3333} = \lambda + 2\mu$, or

$$
\begin{aligned}
c_{ijkm} &= \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & \text{SYM.} & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} && (7.24) \\
&= \lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{kj}) && (7.25)
\end{aligned}
$$

and we are thus left with only two independent non zero coefficients λ and μ which are called **Lame's constants**.

24 Substituting the last equation into Eq. 7.6,

$$T_{ij} = [\lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{kj})]E_{km} \quad (7.26)$$

Or in terms of λ and μ, **Hooke's Law** for an isotropic body is written as

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij} \quad \text{or} \quad \mathbf{T} = \lambda \mathbf{I}_E + 2\mu\mathbf{E} \quad (7.27)$$

$$E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}T_{kk}\right) \quad \text{or} \quad \mathbf{E} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\mathbf{I}_T + \frac{1}{2\mu}\mathbf{T} \quad (7.28)$$

25 It should be emphasized that Eq. 7.24 is written in terms of the **Engineering strains** (Eq. 7.6) that is $\gamma_{ij} = 2E_{ij}$ for $i \neq j$. On the other hand the preceding equations are written in terms of the **tensorial strains** E_{ij}

7.2.5.1 Engineering Constants

26 The stress-strain relations were expressed in terms of Lame's parameters which can not be readily measured experimentally. As such, in the following sections we will reformulate those relations in terms of "engineering constants" (Young's and the bulk's modulus). This will be done for both the isotropic and transversely isotropic cases.

7.2.5.1.1 Isotropic Case

7.2.5.1.1.1 Young's Modulus

27 In order to avoid certain confusion between the strain $\mathbf{E}$ and the elastic constant E, we adopt the usual engineering notation $T_{ij} \rightarrow \sigma_{ij}$ and $E_{ij} \rightarrow \varepsilon_{ij}$

28 If we consider a simple uniaxial state of stress in the x_1 direction ($\sigma_{11} = \sigma, \sigma_{22} = \sigma_{33} = 0$), then from Eq. 7.28

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma && (7.29\text{-a}) \\
\varepsilon_{22} = \varepsilon_{33} &= \frac{-\lambda}{2\mu(3\lambda+2\mu)}\sigma && (7.29\text{-b}) \\
0 &= \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{13} && (7.29\text{-c})
\end{aligned}
$$

29 Yet we have the elementary relations in terms engineering constants E **Young's modulus** and ν **Poisson's ratio**

$$\varepsilon_{11} = \frac{\sigma}{E} \tag{7.30-a}$$
$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \tag{7.30-b}$$

then it follows that

$$\boxed{\begin{aligned} \frac{1}{E} &= \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}; \nu = \frac{\lambda}{2(\lambda+\mu)} && (7.31)\\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)}; \mu = G = \frac{E}{2(1+\nu)} && (7.32)\end{aligned}}$$

30 Similarly in the case of pure shear in the x_1x_3 and x_2x_3 planes, we have

$$\sigma_{21} = \sigma_{12} = \tau \quad \text{all other} \quad \sigma_{ij} = 0 \tag{7.33-a}$$
$$2\varepsilon_{12} = \frac{\tau}{G} \tag{7.33-b}$$

and the μ is equal to the **shear modulus** G.

31 Hooke's law for isotropic material in terms of engineering constants becomes

$$\boxed{\begin{aligned} \sigma_{ij} &= \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \text{ or } \boldsymbol{\sigma} = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\mathbf{I}_\varepsilon\right) && (7.34)\\ \varepsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \text{ or } \boldsymbol{\varepsilon} = \frac{1+\nu}{E}\boldsymbol{\sigma} - \frac{\nu}{E}\mathbf{I}_\sigma && (7.35)\end{aligned}}$$

32 When the strain equation is expanded in 3D cartesian coordinates it would yield:

$$\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} = \frac{1}{E}\left[\begin{array}{cccccc}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{array}\right]\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\} \tag{7.36}$$

33 If we invert this equation, we obtain

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\} = \left[\begin{array}{cc}\frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{ccc}1-\nu & \nu & \nu\\ \nu & 1-\nu & \nu\\ \nu & \nu & 1-\nu\end{array}\right] & 0\\ 0 & G\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right]\end{array}\right]\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \gamma_{xy}(2\varepsilon_{xy})\\ \gamma_{yz}(2\varepsilon_{yz})\\ \gamma_{zx}(2\varepsilon_{zx})\end{array}\right\} \tag{7.37}$$

7.2.5.1.1.2 Bulk's Modulus; Volumetric and Deviatoric Strains

34 We can express the trace of the stress I_σ in terms of the **volumetric** strain I_ε From Eq. 7.27

$$\sigma_{ii} = \lambda\delta_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ii} = (3\lambda+2\mu)\varepsilon_{ii} \equiv 3K\varepsilon_{ii} \tag{7.38}$$

or

$$\boxed{K = \lambda + \frac{2}{3}\mu} \tag{7.39}$$

35 We can provide a complement to the volumetric part of the constitutive equations by substracting the trace of the stress from the stress tensor, hence we define the **deviatoric** stress and strains as as

$$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{1}{3}(\text{tr }\boldsymbol{\sigma})\mathbf{I} \tag{7.40}$$

$$\boldsymbol{\varepsilon}' \equiv \boldsymbol{\varepsilon} - \frac{1}{3}(\text{tr }\boldsymbol{\varepsilon})\mathbf{I} \tag{7.41}$$

and the corresponding constitutive relation will be

$$\boldsymbol{\sigma} = Ke\mathbf{I} + 2\mu\boldsymbol{\varepsilon}' \tag{7.42}$$

$$\boldsymbol{\varepsilon} = \frac{p}{3K}\mathbf{I} + \frac{1}{2\mu}\boldsymbol{\sigma}' \tag{7.43}$$

where $p \equiv \frac{1}{3}\text{tr }(\boldsymbol{\sigma})$ is the pressure, and $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - p\mathbf{I}$ is the stress deviator.

7.2.5.1.1.3 †Restriction Imposed on the Isotropic Elastic Moduli

36 We can rewrite Eq. 18.29 as

$$dW = T_{ij}dE_{ij} \tag{7.44}$$

but since dW is a scalar invariant (energy), it can be expressed in terms of volumetric (hydrostatic) and deviatoric components as

$$dW = -pde + \sigma'_{ij}dE'_{ij} \tag{7.45}$$

substituting $p = -Ke$ and $\sigma'_{ij} = 2GE'_{ij}$, and integrating, we obtain the following expression for the **isotropic strain energy**

$$W = \frac{1}{2}Ke^2 + GE'_{ij}E'_{ij} \tag{7.46}$$

and since positive work is required to cause any deformation $W > 0$ thus

$$\lambda + \frac{2}{3}G \equiv K > 0 \tag{7.47-a}$$

$$G > 0 \tag{7.47-b}$$

ruling out $K = G = 0$, we are left with

$$\boxed{E > 0; \quad -1 < \nu < \frac{1}{2}} \tag{7.48}$$

37 The isotropic strain energy function can be alternatively expressed as

$$W = \frac{1}{2}\lambda e^2 + GE_{ij}E_{ij} \tag{7.49}$$

38 From Table 7.1, we observe that $\nu = \frac{1}{2}$ implies $G = \frac{E}{3}$, and $\frac{1}{K} = 0$ or elastic **incompressibility**.

39 The elastic properties of selected materials is shown in Table 7.2.

	λ, μ	E, ν	μ, ν	E, μ	K, ν
λ	λ	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	$\frac{3K\nu}{1+\nu}$
μ	μ	$\frac{E}{2(1+\nu)}$	μ	μ	$\frac{3K(1-2\nu)}{2(1+\nu)}$
K	$\lambda + \frac{2}{3}\mu$	$\frac{E}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E}{3(3\mu-E)}$	K
E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E	$2\mu(1+\nu)$	E	$3K(1-2\nu)$
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	$\frac{E}{2\mu} - 1$	ν

Table 7.1: Conversion of Constants for an Isotropic Elastic Material

Material	E (MPa)	ν
A316 Stainless Steel	196,000	0.3
A5 Aluminum	68,000	0.33
Bronze	61,000	0.34
Plexiglass	2,900	0.4
Rubber	2	→0.5
Concrete	60,000	0.2
Granite	60,000	0.27

Table 7.2: Elastic Properties of Selected Materials at 20^{0c}

7.2.5.1.2 †Transversly Isotropic Case

40 For transversely isotropic, we can express the stress-strain relation in tems of

$$\begin{aligned}
\varepsilon_{xx} &= a_{11}\sigma_{xx} + a_{12}\sigma_{yy} + a_{13}\sigma_{zz} \\
\varepsilon_{yy} &= a_{12}\sigma_{xx} + a_{11}\sigma_{yy} + a_{13}\sigma_{zz} \\
\varepsilon_{zz} &= a_{13}(\sigma_{xx} + \sigma_{yy}) + a_{33}\sigma_{zz} \\
\gamma_{xy} &= 2(a_{11} - a_{12})\tau_{xy} \\
\gamma_{yz} &= a_{44}\tau_{xy} \\
\gamma_{xz} &= a_{44}\tau_{xz}
\end{aligned} \tag{7.50}$$

and

$$a_{11} = \frac{1}{E}; \quad a_{12} = -\frac{\nu}{E}; \quad a_{13} = -\frac{\nu'}{E'}; \quad a_{33} = -\frac{1}{E'}; \quad a_{44} = -\frac{1}{\mu'} \tag{7.51}$$

where E is the Young's modulus in the plane of isotropy and E' the one in the plane normal to it. ν corresponds to the transverse contraction in the plane of isotropy when tension is applied in the plane; ν' corresponding to the transverse contraction in the plane of isotropy when tension is applied normal to the plane; μ' corresponding to the shear moduli for the plane of isotropy and any plane normal to it, and μ is shear moduli for the plane of isotropy.

7.2.5.2 Special 2D Cases

41 Often times one can make simplifying assumptions to reduce a 3D problem into a 2D one.

7.2.5.2.1 Plane Strain

42 For problems involving a long body in the z direction with no variation in load or geometry, then

$\varepsilon_{zz} = \gamma_{yz} = \gamma_{xz} = \tau_{xz} = \tau_{yz} = 0$. Thus, replacing into Eq. 7.37 we obtain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{7.52}$$

7.2.5.2.2 Axisymmetry

43 In solids of revolution, we can use a polar coordinate sytem and

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \tag{7.53-a}$$
$$\varepsilon_{\theta\theta} = \frac{u}{r} \tag{7.53-b}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{7.53-c}$$
$$\varepsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{7.53-d}$$

44 The constitutive relation is again analogous to 3D/plane strain

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{Bmatrix} \tag{7.54}$$

7.2.5.2.3 Plane Stress

45 If the longitudinal dimension in z direction is much smaller than in the x and y directions, then $\tau_{yz} = \tau_{xz} = \sigma_{zz} = \gamma_{xz} = \gamma_{yz} = 0$ throughout the thickness. Again, substituting into Eq. 7.37 we obtain:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{7.55-a}$$
$$\varepsilon_{zz} = -\frac{1}{1-\nu}\nu(\varepsilon_{xx} + \varepsilon_{yy}) \tag{7.55-b}$$

7.3 †Linear Thermoelasticity

46 If thermal effects are accounted for, the components of the linear strain tensor E_{ij} may be considered as the sum of

$$E_{ij} = E_{ij}^{(T)} + E_{ij}^{(\Theta)} \tag{7.56}$$

where $E_{ij}^{(T)}$ is the contribution from the stress field, and $E_{ij}^{(\Theta)}$ the contribution from the temperature field.

47 When a body is subjected to a temperature change $\Theta - \Theta_0$ with respect to the reference state temperature, the strain componenet of an elementary volume of an unconstrained isotropic body are given by

$$E_{ij}^{(\Theta)} = \alpha(\Theta - \Theta_0)\delta_{ij} \tag{7.57}$$

where α is the **linear coefficient of thermal expansion**.

48 Inserting the preceding two equation into Hooke's law (Eq. 7.28) yields

$$\boxed{E_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}T_{kk}\right) + \alpha(\Theta - \Theta_0)\delta_{ij}} \tag{7.58}$$

which is known as **Duhamel-Neumann** relations.

49 If we invert this equation, we obtain the **thermoelastic constitutive equation**:

$$\boxed{T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij} - (3\lambda + 2\mu)\alpha\delta_{ij}(\Theta - \Theta_0)} \tag{7.59}$$

50 Alternatively, if we were to consider the derivation of the Green-elastic hyperelastic equations, (Sect. 18.5.1), we required the constants c_1 to c_6 in Eq. 18.31 to be zero in order that the stress vanish in the unstrained state. If we accounted for the temperature change $\Theta - \Theta_0$ with respect to the reference state temperature, we would have $c_k = -\beta_k(\Theta - \Theta_0)$ for $k = 1$ to 6 and would have to add like terms to Eq. 18.31, leading to

$$T_{ij} = -\beta_{ij}(\Theta - \Theta_0) + c_{ijrs}E_{rs} \tag{7.60}$$

for linear theory, we suppose that β_{ij} is independent from the strain and c_{ijrs} independent of temperature change with respect to the natural state. Finally, for isotropic cases we obtain

$$T_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij} - \beta_{ij}(\Theta - \Theta_0)\delta_{ij} \tag{7.61}$$

which is identical to Eq. 7.59 with $\beta = \frac{E\alpha}{1-2\nu}$. Hence

$$\boxed{T_{ij}^{\Theta} = \frac{E\alpha}{1 - 2\nu}} \tag{7.62}$$

51 In terms of deviatoric stresses and strains we have

$$\boxed{T'_{ij} = 2\mu E'_{ij} \quad \text{and} \quad E'_{ij} = \frac{T'_{ij}}{2\mu}} \tag{7.63}$$

and in terms of volumetric stress/strain:

$$\boxed{p = -Ke + \beta(\Theta - \Theta_0) \quad \text{and} \quad e = \frac{p}{K} + 3\alpha(\Theta - \Theta_0)} \tag{7.64}$$

7.4 Fourrier Law

52 Consider a solid through which there is a *flow* $\mathbf{q}$ of heat (or some other quantity such as mass, chemical, etc...)

53 The rate of transfer per unit area is $\mathbf{q}$

54 The direction of flow is in the direction of maximum "potential" (temperature in this case, but could be, piezometric head, or ion concentration) decreases (Fourrier, Darcy, Fick...).

$$\mathbf{q} = \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix} = -\mathbf{D}\begin{Bmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{Bmatrix} = -\mathbf{D}\boldsymbol{\nabla}\phi \tag{7.65}$$

D is a three by three (symmetric) **constitutive/conductivity** matrix
The conductivity can be either

Isotropic

$$\mathbf{D} = k \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.66}$$

Anisotropic

$$\mathbf{D} = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \tag{7.67}$$

Orthotropic

$$\mathbf{D} = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \tag{7.68}$$

Note that for flow through porous media, Darcy's equation is only valid for laminar flow.

7.5 Updated Balance of Equations and Unknowns

55 In light of the new equations introduced in this chapter, it would be appropriate to revisit our balance of equations and unknowns.

		Coupled	Uncoupled
$\frac{d\rho}{dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$	Continuity Equation	1	1
$\frac{\partial T_{ij}}{\partial x_j} + \rho b_i = \rho\frac{dv_i}{dt}$	Equation of motion	3	3
$\rho\frac{du}{dt} = T_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j}$	Energy equation	1	
$\mathbf{T} = \lambda\mathbf{I}_E + 2\mu\mathbf{E}$	Hooke's Law	6	6
$\mathbf{q} = -\mathbf{D}\boldsymbol{\nabla}\phi$	Heat Equation (Fourrier)	3	
$\Theta = \Theta(s, \boldsymbol{\nu});\quad \tau_j = \tau_j(s, \boldsymbol{\nu})$	Equations of state	2	
Total number of equations		**16**	**10**

and we repeat our list of unknowns

		Coupled	Uncoupled
Density	ρ	1	1
Velocity (or displacement)	v_i (u_i)	3	3
Stress components	T_{ij}	6	6
Heat flux components	q_i	3	-
Specific internal energy	u	1	-
Entropy density	s	1	-
Absolute temperature	Θ	1	-
Total number of unknowns		**16**	**10**

and in addition the Clausius-Duhem inequality $\frac{ds}{dt} \geq \frac{r}{\Theta} - \frac{1}{\rho}\text{div}\,\frac{\mathbf{q}}{\Theta}$ which governs entropy production must hold.

56 Hence we now have as many equations as unknowns and are (almost) ready to pose and solve problems in continuum mechanics.

Part II

ELASTICITY/SOLID MECHANICS

Chapter 8

BOUNDARY VALUE PROBLEMS in ELASTICITY

8.1 Preliminary Considerations

1 All problems in elasticity require three basic components:

3 Equations of Motion (Equilibrium): i.e. Equations relating the applied tractions and body forces to the stresses (3)

$$\frac{\partial T_{ij}}{\partial X_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{8.1}$$

6 Stress-Strain relations: (Hooke's Law)

$$\mathbf{T} = \lambda \mathbf{I}_E + 2\mu \mathbf{E} \tag{8.2}$$

6 Geometric (kinematic) equations: i.e. Equations of geometry of deformation relating displacement to strain (6)

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_{\mathbf{x}} + \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u}) \tag{8.3}$$

2 Those 15 equations are written in terms of 15 unknowns: 3 displacement u_i, 6 stress components T_{ij}, and 6 strain components E_{ij}.

3 In addition to these equations which describe what is happening inside the body, we must describe what is happening on the surface or boundary of the body, just like for the solution of a differential equation. These extra conditions are called **boundary conditions**.

8.2 Boundary Conditions

4 In describing the boundary conditions (B.C.), we must note that:

1. Either we know the displacement but not the traction, or we know the traction and not the corresponding displacement. We can never know both *a priori*.

2. Not all boundary conditions specifications are acceptable. For example we can not apply tractions to the entire surface of the body. Unless those tractions are specially prescribed, they may not necessarily satisfy equilibrium.

5 Properly specified boundary conditions result in **well-posed** boundary value problems, while improperly specified boundary conditions will result in **ill-posed** boundary value problem. Only the former can be solved.

6 Thus we have two types of boundary conditions in terms of *known* quantitites, Fig. 8.1:

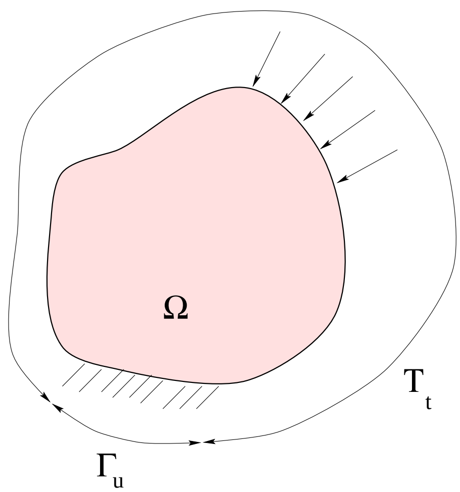


Figure 8.1: Boundary Conditions in Elasticity Problems

Displacement boundary conditions along Γ_u with the three components of u_i prescribed on the boundary. The displacement is decomposed into its cartesian (or curvilinear) components, i.e. u_x, u_y

Traction boundary conditions along Γ_t with the three traction components $t_i = n_j T_{ij}$ prescribed at a boundary where the unit normal is **n**. The traction is decomposed into its normal and shear(s) components, i.e t_n, t_s.

Mixed boundary conditions where displacement boundary conditions are prescribed on a part of the bounding surface, while traction boundary conditions are prescribed on the remainder.

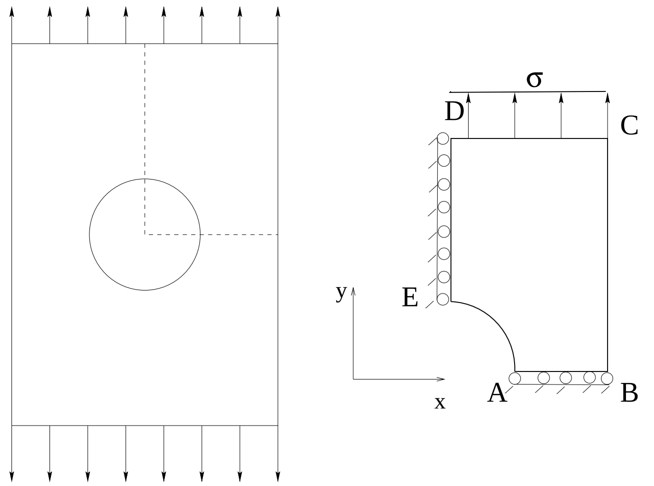
We note that at some points, traction may be specified in one direction, and displacement at another. Displacement and tractions can never be specified at the same point in the same direction.

7 Various terms have been associated with those boundary conditions in the litterature, those are suumarized in Table 8.1.

u, Γ_u	**t**, Γ_t
Dirichlet	Neuman
Field Variable	Derivative(s) of Field Variable
Essential	Non-essential
Forced	**Natural**
Geometric	Static

Table 8.1: Boundary Conditions in Elasticity

8 Often time we take advantage of symmetry not only to simplify the problem, but also to properly define the appropriate boundary conditions, Fig. 8.2.



	Γ_u		Γ_t	
	u_x	u_y	t_n	t_s
AB	?	0	?	0
BC	?	?	0	0
CD	?	?	σ	0
DE	0	?	?	0
EA	?	?	0	0

Note: Unknown tractions=Reactions

Figure 8.2: Boundary Conditions in Elasticity Problems

8.3 Boundary Value Problem Formulation

9 Hence, the boundary value formulation is suumarized by

$$\frac{\partial T_{ij}}{\partial X_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad \text{in} \quad \Omega \tag{8.4}$$

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{u}\boldsymbol{\nabla}_\mathbf{x} + \boldsymbol{\nabla}_\mathbf{x}\mathbf{u}) \tag{8.5}$$

$$\mathbf{T} = \lambda \mathbf{I}_E + 2\mu\mathbf{E} \quad \text{in} \quad \Omega \tag{8.6}$$

$$\mathbf{u} = \overline{\mathbf{u}} \quad \text{in} \quad \Gamma_u \tag{8.7}$$

$$\mathbf{t} = \overline{\mathbf{t}} \quad \text{in} \quad \Gamma_t \tag{8.8}$$

and is illustrated by Fig. 8.3. This is now a **well posed problem**.

8.4 †Compact Forms

10 Solving a boundary value problem with 15 unknowns through 15 equations is a formidable task. Hence, there are numerous methods to reformulate the problem in terms of fewer unknows.

8.4.1 Navier-Cauchy Equations

11 One such approach is to substitute the displacement-strain relation into Hooke's law (resulting in stresses in terms of the gradient of the displacement), and the resulting equation into the equation of motion to obtain three second-order partial differential equations for the three displacement components known as **Navier's Equation**

$$(\lambda+\mu)\frac{\partial^2 u_k}{\partial X_i \partial X_k} + \mu \frac{\partial^2 u_i}{\partial X_k \partial X_k} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{8.9}$$

or

$$(\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mu\boldsymbol{\nabla}^2\mathbf{u} + \rho\mathbf{b} = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{8.10}$$

(8.11)

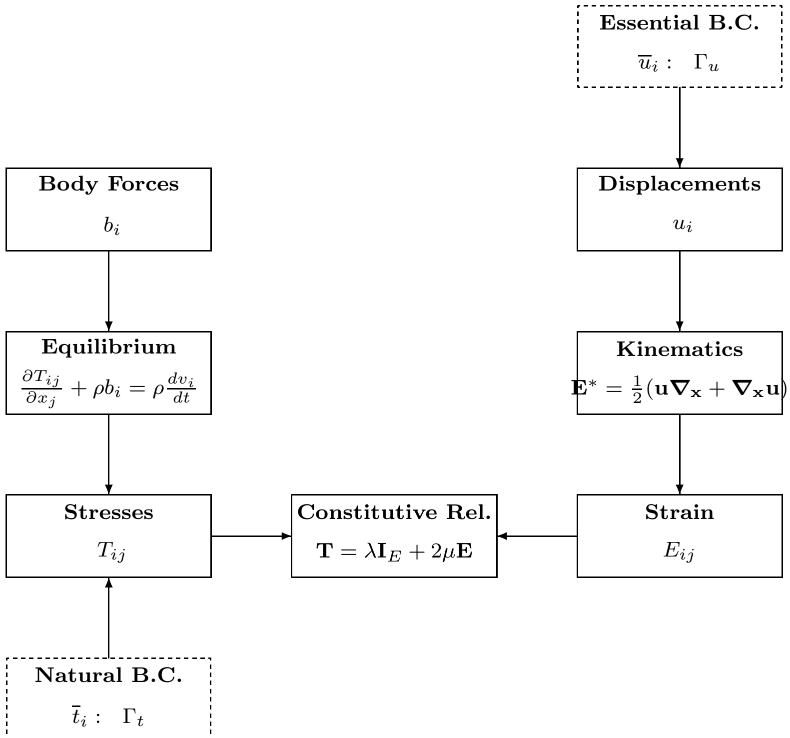


Figure 8.3: Fundamental Equations in Solid Mechanics

8.4.2 Beltrami-Mitchell Equations

12 Whereas Navier-Cauchy equation was expressed in terms of the gradient of the displacement, we can follow a similar approach and write a single equation in term of the gradient of the tractions.

$$\boldsymbol{\nabla}^2 T_{ij} + \frac{1}{1+\nu} T_{pp,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\boldsymbol{\nabla}\cdot(\rho\mathbf{b}) - \rho(b_{i,j} + b_{j,i}) \quad (8.12)$$

or

$$T_{ij,pp} + \frac{1}{1+\nu} T_{pp,ij} = -\frac{\nu}{1-\nu}\delta_{ij}\rho b_{p,p} - \rho(b_{i,j} + b_{j,i}) \quad (8.13)$$

8.4.3 Airy Stress Function

Airy stress function, for plane strain problems will be separately covered in Sect. 9.2.

8.4.4 Ellipticity of Elasticity Problems

8.5 †Strain Energy and Extenal Work

13 For the isotropic Hooke's law, we saw that there always exist a strain energy function W which is positive-definite, homogeneous quadratic function of the strains such that, Eq. 18.29

$$T_{ij} = \frac{\partial W}{\partial E_{ij}} \quad (8.14)$$

hence it follows that

$$W = \frac{1}{2} T_{ij} E_{ij} \quad (8.15)$$

14 The external work done by a body in equilibrium under body forces b_i and surface traction t_i is equal to $\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma$. Substituting $t_i = T_{ij} n_j$ and applying Gauss theorem, the second term becomes

$$\int_\Gamma T_{ij} n_j u_i d\Gamma = \int_\Omega (T_{ij} u_i)_{,j} d\Omega = \int_\Omega (T_{ij,j} u_i + T_{ij} u_{i,j}) d\Omega \tag{8.16}$$

but $T_{ij} u_{i,j} = T_{ij}(E_{ij} + \Omega_{ij}) = T_{ij} E_{ij}$ and from equilibrium $T_{ij,j} = -\rho b_i$, thus

$$\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma = \int_\Omega \rho b_i u_i d\Omega + \int_\Omega (T_{ij} E_{ij} - \rho b_i u_i) d\Omega \tag{8.17}$$

or

$$\boxed{\underbrace{\int_\Omega \rho b_i u_i d\Omega + \int_\Gamma t_i u_i d\Gamma}_{\text{External Work}} = 2 \underbrace{\int_\Omega \frac{T_{ij} E_{ij}}{2} d\Omega}_{\text{Internal Strain Energy}}} \tag{8.18}$$

that is *For an elastic system, the total strain energy is one half the work done by the external forces acting through their displacements u_i.*

8.6 †Uniqueness of the Elastostatic Stress and Strain Field

15 Because the equations of linear elasticity are linear equations, the principles of superposition may be used to obtain additional solutions from those established. Hence, given two sets of solution $T_{ij}^{(1)}$, $u_i^{(1)}$, and $T_{ij}^{(2)}$, $u_i^{(2)}$, then $T_{ij} = T_{ij}^{(2)} - T_{ij}^{(1)}$, and $u_i = u_i^{(2)} - u_i^{(1)}$ with $b_i = b_i^{(2)} - b_i^{(1)} = 0$ must also be a solution.

16 Hence for this "difference" solution, Eq. 8.18 would yield $\int_\Gamma t_i u_i d\Gamma = 2\int_\Omega u^* d\Omega$ but the left hand side is zero because $t_i = t_i^{(2)} - t_i^{(1)} = 0$ on Γ_u, and $u_i = u_i^{(2)} - u_i^{(1)} = 0$ on Γ_t, thus $\int_\Omega u^* d\Omega = 0$.

17 But u^* is positive-definite and continuous, thus the integral can vanish if and only if $u^* = 0$ everywhere, and this is only possible if $E_{ij} = 0$ everywhere so that

$$\boxed{E_{ij}^{(2)} = E_{ij}^{(1)} \Rightarrow T_{ij}^{(2)} = Tij^{(1)}} \tag{8.19}$$

hence, there can not be two different stress and strain fields corresponding to the same externally imposed body forces and boundary conditions[1] and satisfying the linearized elastostatic Eqs 8.1, 8.14 and 8.3.

8.7 Saint Venant's Principle

18 This famous **principle** of Saint Venant was enunciated in 1855 and is of great importance in applied elasticity where it is often invoked to justify certain "simplified" solutions to complex problem.

> In elastostatics, if the boundary tractions on a part Γ_1 of the boundary Γ are replaced by a statically equivalent traction distribution, the effects on the stress distribution in the body are negligible at points whose distance from Γ_1 is large compared to the maximum distance between points of Γ_1.

19 For instance the analysis of the problem in Fig. 8.4 can be greatly simplified if the tractions on Γ_1 are replaced by a concentrated statically equivalent force.

[1] This theorem is attributed to Kirchoff (1858).

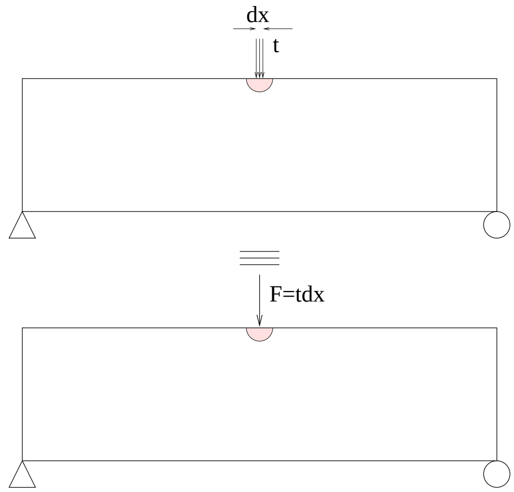


Figure 8.4: St-Venant's Principle

8.8 Cylindrical Coordinates

20 So far all equations have been written in either vector, indicial, or engineering notation. The last two were so far restricted to an othonormal cartesian coordinate system.

21 We now rewrite some of the fundamental relations in **cylindrical** coordinate system, Fig. 8.5, as this would enable us to analytically solve some simple problems of great practical usefulness (torsion, pressurized cylinders, ...). This is most often achieved by reducing the dimensionality of the problem from 3 to 2 or even to 1.

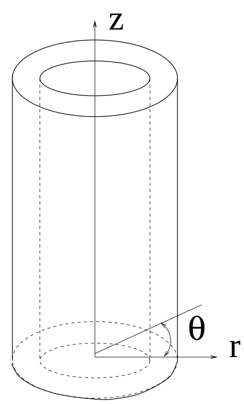


Figure 8.5: Cylindrical Coordinates

8.8.1 Strains

22 With reference to Fig. 8.6, we consider the displacement of point P to P^*. the displacements can be expressed in cartesian coordinates as u_x, u_y, or in polar coordinates as u_r, u_θ. Hence,

$$u_x = u_r \cos\theta - u_\theta \sin\theta \tag{8.20-a}$$
$$u_y = u_r \sin\theta + u_\theta \cos\theta \tag{8.20-b}$$

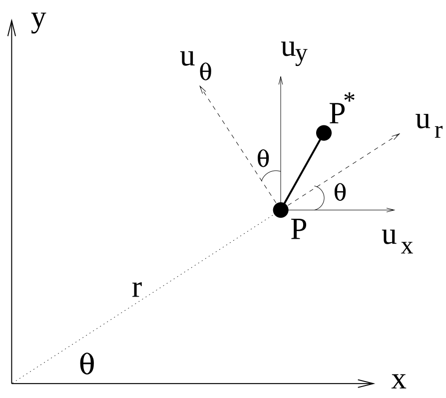


Figure 8.6: Polar Strains

substituting into the strain definition for ε_{xx} (for small displacements) we obtain

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u_x}{\partial r}\frac{\partial r}{\partial x} \quad \text{(8.21-a)}$$

$$\frac{\partial u_x}{\partial \theta} = \frac{\partial u_r}{\partial \theta}\cos\theta - u_r\sin\theta - \frac{\partial u_\theta}{\partial \theta}\sin\theta - u_\theta\cos\theta \quad \text{(8.21-b)}$$

$$\frac{\partial u_x}{\partial r} = \frac{\partial u_r}{\partial r}\cos\theta - \frac{\partial u_\theta}{\partial r}\sin\theta \quad \text{(8.21-c)}$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad \text{(8.21-d)}$$

$$\frac{\partial r}{\partial x} = \cos\theta \quad \text{(8.21-e)}$$

$$\varepsilon_{xx} = \left(-\frac{\partial u_r}{\partial \theta}\cos\theta + u_r\sin\theta + \frac{\partial u_\theta}{\partial \theta}\sin\theta + u_\theta\cos\theta\right)\frac{\sin\theta}{r} + \left(\frac{\partial u_r}{\partial r}\cos\theta - \frac{\partial u_\theta}{\partial r}\sin\theta\right)\cos\theta \quad \text{(8.21-f)}$$

Noting that as $\theta \to 0$, $\varepsilon_{xx} \to \varepsilon_{rr}$, $\sin\theta \to 0$, and $\cos\theta \to 1$, we obtain

$$\varepsilon_{rr} = \varepsilon_{xx}|_{\theta\to 0} = \frac{\partial u_r}{\partial r} \quad \text{(8.22)}$$

23 Similarly, if $\theta \to \pi/2$, $\varepsilon_{xx} \to \varepsilon_{\theta\theta}$, $\sin\theta \to 1$, and $\cos\theta \to 0$. Hence,

$$\varepsilon_{\theta\theta} = \varepsilon_{xx}|_{\theta\to\pi/2} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad \text{(8.23)}$$

finally, we may express ε_{xy} as a function of u_r, u_θ and θ and noting that $\varepsilon_{xy} \to \varepsilon_{r\theta}$ as $\theta \to 0$, we obtain

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\varepsilon_{xy}|_{\theta\to 0} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \quad \text{(8.24)}$$

24 In summary, and with the addition of the z components (not explicitely derived), we obtain

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{8.25}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \tag{8.26}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{8.27}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right] \tag{8.28}$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] \tag{8.29}$$

$$\varepsilon_{rz} = \frac{1}{2}\left[\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right] \tag{8.30}$$

8.8.2 Equilibrium

25 Whereas the equilibrium equation as given In Eq. 6.17 was obtained from the linear momentum principle (without any reference to the notion of equilibrium of forces), its derivation (as mentioned) could have been obtained by equilibrium of forces considerations. This is the approach which we will follow for the polar coordinate system with respect to Fig. 8.7.

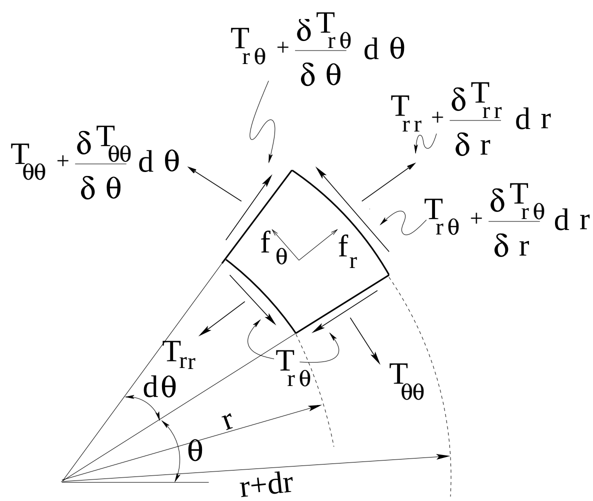


Figure 8.7: Stresses in Polar Coordinates

26 Summation of forces parallel to the radial direction through the center of the element with unit thickness in the z direction yields:

$$\begin{aligned}\left(T_{rr} + \tfrac{\partial T_{rr}}{\partial r}dr\right)(r+dr)d\theta - T_{rr}(rd\theta)& \\ - \left(T_{\theta\theta} + \tfrac{\partial T_{\theta\theta}}{\partial \theta} + T_{\theta\theta}\right) dr \sin\tfrac{d\theta}{2}& \\ + \left(T_{\theta r} + \tfrac{\partial T_{\theta r}}{\partial \theta}d\theta - T_{\theta r}\right) dr\cos\tfrac{d\theta}{2} + f_r r dr d\theta &= 0\end{aligned} \tag{8.31}$$

we approximate $\sin(d\theta/2)$ by $d\theta/2$ and $\cos(d\theta/2)$ by unity, divide through by $rdrd\theta$,

$$\frac{1}{r}T_{rr} + \frac{\partial T_{rr}}{\partial r}\left(1+\frac{dr}{r}\right) - \frac{T_{\theta\theta}}{r} - \frac{\partial T_{\theta\theta}}{\partial \theta}\frac{d\theta}{dr} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + f_r = 0 \tag{8.32}$$

27 Similarly we can take the summation of forces in the θ direction. In both cases if we were to drop the dr/r and $d\theta/r$ in the limit, we obtain

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) + f_r = 0 \quad (8.33)$$
$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r}(T_{r\theta} - T_{\theta r}) + f_\theta = 0 \quad (8.34)$$

28 It is often necessary to express cartesian stresses in terms of polar stresses and vice versa. This can be done through the following relationships

$$\begin{bmatrix} T_{xx} & T_{xy} \\ T_{xy} & T_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \quad (8.35)$$

yielding

$$T_{xx} = T_{rr}\cos^2\theta + T_{\theta\theta}\sin^2\theta - T_{r\theta}\sin 2\theta \quad (8.36\text{-a})$$
$$T_{yy} = T_{rr}\sin^2\theta + T_{\theta\theta}\cos^2\theta + T_{r\theta}\sin 2\theta \quad (8.36\text{-b})$$
$$T_{xy} = (T_{rr} - T_{\theta\theta})\sin\theta\cos\theta + T_{r\theta}(\cos^2\theta - \sin^2\theta) \quad (8.36\text{-c})$$

(recalling that $\sin^2\theta = 1/2\sin 2\theta$, and $\cos^2\theta = 1/2(1+\cos 2\theta)$).

8.8.3 Stress-Strain Relations

29 In orthogonal curvilinear coordinates, the physical components of a tensor at a point are merely the Cartesian components in a local coordinate system at the point with its axes tangent to the coordinate curves. Hence,

$$T_{rr} = \lambda e + 2\mu\varepsilon_{rr} \quad (8.37)$$
$$T_{\theta\theta} = \lambda e + 2\mu\varepsilon_{\theta\theta} \quad (8.38)$$
$$T_{r\theta} = 2\mu\varepsilon_{r\theta} \quad (8.39)$$
$$T_{zz} = \nu(T_{rr} + T_{\theta\theta}) \quad (8.40)$$

with $e = \varepsilon_{rr} + \varepsilon_{\theta\theta}$. alternatively,

$$E_{rr} = \frac{1}{E}\left[(1-\nu^2)T_{rr} - \nu(1+\nu)T_{\theta\theta}\right] \quad (8.41)$$
$$E_{\theta\theta} = \frac{1}{E}\left[(1-\nu^2)T_{\theta\theta} - \nu(1+\nu)T_{rr}\right] \quad (8.42)$$
$$E_{r\theta} = \frac{1+\nu}{E}T_{r\theta} \quad (8.43)$$
$$E_{rz} = E_{\theta z} = E_{zz} = 0 \quad (8.44)$$

8.8.3.1 Plane Strain

30 For Plane strain problems, from Eq. 7.52:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{r\theta} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ \nu & \nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix} \quad (8.45)$$

and $\varepsilon_{zz} = \gamma_{rz} = \gamma_{\theta z} = \tau_{rz} = \tau_{\theta z} = 0$.

31 Inverting,

$$\begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & 1-\nu^2 & 0 \\ \nu & \nu & 0 \\ 0 & 0 & 2(1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{r\theta} \end{Bmatrix} \quad (8.46)$$

8.8.3.2 Plane Stress

32 For plane stress problems, from Eq. 7.55-a

$$\left\{ \begin{array}{c} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{array} \right\} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \end{array} \right\} \quad (8.47\text{-a})$$

$$\varepsilon_{zz} = -\frac{1}{1-\nu}\nu(\varepsilon_{rr} + \varepsilon_{\theta\theta}) \quad (8.47\text{-b})$$

and $\tau_{rz} = \tau_{\theta z} = \sigma_{zz} = \gamma_{rz} = \gamma_{\theta z} = 0$

33 Inverting

$$\left\{ \begin{array}{c} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \gamma_{r\theta} \end{array} \right\} = \frac{1}{E} \left[\begin{array}{ccc} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{array} \right] \left\{ \begin{array}{c} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \tau_{r\theta} \end{array} \right\} \quad (8.48\text{-a})$$

Chapter 9

SOME ELASTICITY PROBLEMS

1 Practical solutions of two-dimensional boundary-value problem in simply connected regions can be accomplished by numerous techniques. Those include: a) Finite-difference approximation of the differential equation, b) Complex function method of Muskhelisvili (most useful in problems with stress concentration), c) Variational methods, d) Semi-inverse methods, and e) Airy stress functions.

2 Only the last two methods will be discussed in this chapter.

9.1 Semi-Inverse Method

3 Often a solution to an elasticity problem may be obtained without seeking simulateneous solutions to the equations of motion, Hooke's Law and boundary conditions. One may attempt to seek solutions by making certain assumptions or guesses about the components of strain stress or displacement while leaving enough freedom in these assumptions so that the equations of elasticity be satisfied.

4 If the assumptions allow us to satisfy the elasticity equations, then by the uniqueness theorem, we have succeeded in obtaining the solution to the problem.

5 This method was employed by Saint-Venant in his treatment of the torsion problem, hence it is often referred to as the **Saint-Venant semi-inverse method**.

9.1.1 Example: Torsion of a Circular Cylinder

6 Let us consider the elastic deformation of a cylindrical bar with circular cross section of radius a and length L twisted by equal and opposite end moments M_1, Fig. 9.1.

7 From symmetry, it is reasonable to assume that the motion of each cross-sectional plane is a rigid body rotation about the x_1 axis. Hence, for a small rotation angle θ, the displacement field will be given by:

$$\mathbf{u} = (\theta\mathbf{e}_1)\times\mathbf{r} = (\theta\mathbf{e}_1)\times(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = \theta(x_2\mathbf{e}_3 - x_3\mathbf{e}_2) \tag{9.1}$$

or

$$u_1 = 0; \qquad u_2 = -\theta x_3; \qquad u_3 = \theta x_2 \tag{9.2}$$

where $\theta = \theta(x_1)$.

8 The corresponding strains are given by

$$E_{11} = E_{22} = E_{33} = 0 \tag{9.3-a}$$

$$E_{12} = -\frac{1}{2}x_3\frac{\partial\theta}{\partial x_1} \tag{9.3-b}$$

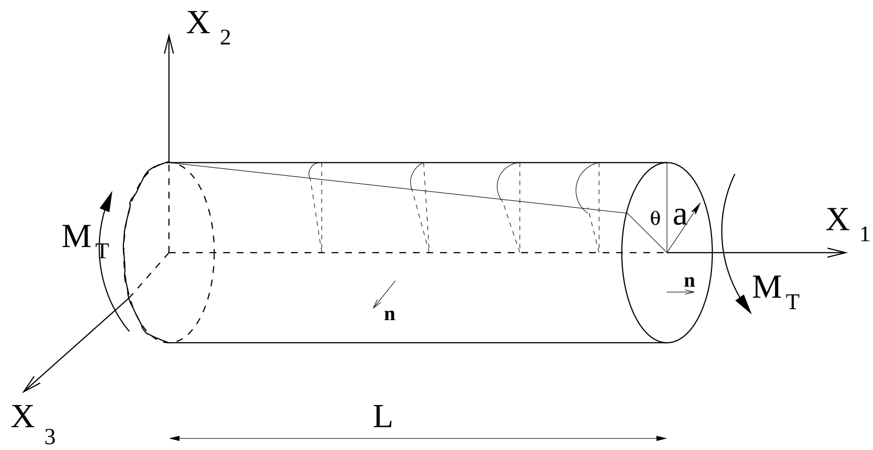


Figure 9.1: Torsion of a Circular Bar

$$E_{13} = \frac{1}{2}x_2\frac{\partial\theta}{\partial x_1} \quad (9.3\text{-c})$$

9 The non zero stress components are obtained from Hooke's law

$$T_{12} = -\mu x_3\frac{\partial\theta}{\partial x_1} \quad (9.4\text{-a})$$

$$T_{13} = \mu x_2\frac{\partial\theta}{\partial x_1} \quad (9.4\text{-b})$$

10 We need to check that this state of stress satisfies equilibrium $\partial T_{ij}/\partial x_j = 0$. The first one $j = 1$ is identically satisfied, whereas the other two yield

$$-\mu x_3\frac{\mathrm{d}^2\theta}{\mathrm{d}x_1^2} = 0 \quad (9.5\text{-a})$$

$$\mu x_2\frac{\mathrm{d}^2\theta}{\mathrm{d}x_1^2} = 0 \quad (9.5\text{-b})$$

thus,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x_1} \equiv \theta' = \text{constant} \quad (9.6)$$

Physically, this means that equilibrium is only satisfied if the increment in angular rotation (twist per unit length) is a constant.

11 We next determine the corresponding surface tractions. On the lateral surface we have a unit normal vector $\mathbf{n} = \frac{1}{a}(x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$, therefore the surface traction on the lateral surface is given by

$$\{\mathbf{t}\} = [\mathbf{T}]\{\mathbf{n}\} = \frac{1}{a}\begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ T_{31} & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{a}\begin{Bmatrix} x_2T_{12} \\ 0 \\ 0 \end{Bmatrix} \quad (9.7)$$

12 Substituting,

$$\mathbf{t} = \frac{\mu}{a}(-x_2x_3\theta' + x_2x_3\theta')\mathbf{e}_1 = \mathbf{0} \quad (9.8)$$

which is in agreement with the fact that the bar is twisted by end moments only, the lateral surface is traction free.

13 On the face $x_1 = L$, we have a unit normal $\mathbf{n} = \mathbf{e}_1$ and a surface traction

$$\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \tag{9.9}$$

this distribution of surface traction on the end face gives rise to the following resultants

$$R_1 = \int T_{11} dA = 0 \tag{9.10-a}$$

$$R_2 = \int T_{21} dA = \mu\theta' \int x_3 dA = 0 \tag{9.10-b}$$

$$R_3 = \int T_{31} dA = \mu\theta' \int x_2 dA = 0 \tag{9.10-c}$$

$$M_1 = \int (x_2 T_{31} - x_3 T_{21}) dA = \mu\theta' \int (x_2^2 + x_3^2) dA = \mu\theta' J \tag{9.10-d}$$

$$M_2 = M_3 = 0 \tag{9.10-e}$$

We note that $\int (x_2^2 + x_3^3)^2 dA$ is the **polar moment of inertia** of the cross section and is equal to $J = \pi a^4/2$, and we also note that $\int x_2 dA = \int x_3 dA = 0$ because the area is symmetric with respect to the axes.

14 From the last equation we note that

$$\theta' = \frac{M}{\mu J} \tag{9.11}$$

which implies that the shear modulus μ can be determined froma simple torsion experiment.

15 Finally, in terms of the twisting couple M, the stress tensor becomes

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\frac{Mx_3}{J} & \frac{Mx_2}{J} \\ -\frac{Mx_3}{J} & 0 & 0 \\ \frac{Mx_2}{J} & 0 & 0 \end{bmatrix} \tag{9.12}$$

9.2 Airy Stress Functions; Plane Strain

16 If the deformation of a cylindrical body is such that there is no axial components of the displacement and that the other components do not depend on the axial coordinate, then the body is said to be in a state of plane strain. If $\mathbf{e}_3$ is the direction corresponding to the cylindrical axis, then we have

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0 \tag{9.13}$$

and the strain components corresponding to those displacements are

$$E_{11} = \frac{\partial u_1}{\partial x_1} \tag{9.14-a}$$

$$E_{22} = \frac{\partial u_2}{\partial x_2} \tag{9.14-b}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{9.14-c}$$

$$E_{13} = E_{23} = E_{33} = 0 \tag{9.14-d}$$

and the non-zero stress components are $T_{11}, T_{12}, T_{22}, T_{33}$ where

$$T_{33} = \nu(T_{11} + T_{22}) \tag{9.15}$$

17 Considering a static stress field with no body forces, the equilibrium equations reduce to:

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = 0 \quad (9.16\text{-a})$$

$$\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} = 0 \quad (9.16\text{-b})$$

$$\frac{\partial T_{33}}{\partial x_1} = 0 \quad (9.16\text{-c})$$

we note that since $T_{33} = T_{33}(x_1, x_2)$, the last equation is always satisfied.

18 Hence, it can be easily verified that for any arbitrary scalar variable Φ, if we compute the stress components from

$$T_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} \quad (9.17)$$

$$T_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} \quad (9.18)$$

$$T_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \quad (9.19)$$

then the first two equations of equilibrium are automatically satisfied. This function Φ is called **Airy stress function**.

19 However, if stress components determined this way are **statically admissible** (i.e. they satisfy equilibrium), they are not necessarily **kinematically admissible** (i.e. satisfy compatibility equations).

20 To ensure compatibility of the strain components, we express the strains components in terms of Φ from Hooke's law, Eq. 7.36 and Eq. 9.15.

$$E_{11} = \frac{1}{E}\left[(1-\nu^2)T_{11} - \nu(1+\nu)T_{22}\right] = \frac{1}{E}\left[(1-\nu^2)\frac{\partial^2 \Phi}{\partial x_2^2} - \nu(1+\nu)\frac{\partial^2 \Phi}{\partial x_1^2}\right] \quad (9.20\text{-a})$$

$$E_{22} = \frac{1}{E}\left[(1-\nu^2)T_{22} - \nu(1+\nu)T_{11}\right] = \frac{1}{E}\left[(1-\nu^2)\frac{\partial^2 \Phi}{\partial x_1^2} - \nu(1+\nu)\frac{\partial^2 \Phi}{\partial x_2^2}\right] \quad (9.20\text{-b})$$

$$E_{12} = \frac{1}{E}(1+\nu)T_{12} = -\frac{1}{E}(1+\nu)\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} \quad (9.20\text{-c})$$

For plane strain problems, the only compatibility equation, 4.140, that is not automatically satisfied is

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} \quad (9.21)$$

substituting,

$$(1-\nu)\left(\frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_1^4}\right) = 0 \quad (9.22)$$

or

$$\frac{\partial^4 \Phi}{\partial x_1^4} + 2\frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0 \quad \text{or} \quad \nabla^4 \Phi = 0 \quad (9.23)$$

Hence, any function which satisfies the preceding equation will satisfy **both** equilibrium, kinematic, stress-strain (albeit plane strain) and is thus an acceptable elasticity solution.

21 †We can also obtain from the Hooke's law, the compatibility equation 9.21, and the equilibrium equations the following

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(T_{11} + T_{22}) = 0 \quad \text{or} \quad \nabla^2(T_{11} + T_{22}) = 0 \quad (9.24)$$

22 †Any polynomial of degree three or less in x and y satisfies the biharmonic equation (Eq. 9.23). A systematic way of selecting coefficients begins with

$$\Phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}x^m y^n \tag{9.25}$$

23 †The stresses will be given by

$$T_{xx} = \sum_{m=0}^{\infty}\sum_{n=2}^{\infty} n(n-1)C_{mn}x^m y^{n-2} \tag{9.26-a}$$

$$T_{yy} = \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} m(m-1)C_{mn}x^{m-1} y^{n} \tag{9.26-b}$$

$$T_{xy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnC_{mn}x^{m-1} y^{n-1} \tag{9.26-c}$$

24 †Substituting into Eq. 9.23 and regrouping we obtain

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}[(m+2)(m+1)m(m-1)C_{m+2,n-2}+2m(m-1)n(n-1)C_{mn}+(n+2)(n+1)n(n-1)C_{m-2,n+2}]x^{m-2}y^{n-2} = 0 \tag{9.27}$$

but since the equation must be identically satisfied for all x and y, the term in bracket must be equal to zero.

$$(m+2)(m+1)m(m-1)C_{m+2,n-2}+2m(m-1)n(n-1)C_{mn}+(n+2)(n+1)n(n-1)C_{m-2,n+2} = 0 \tag{9.28}$$

Hence, the recursion relation establishes relationships among groups of three alternate coefficients which can be selected from

$$\begin{bmatrix} 0 & 0 & C_{02} & C_{03} & \boxed{C_{04}} & C_{05} & C_{06} & \cdots \\ 0 & C_{11} & C_{12} & C_{13} & C_{14} & \underline{C_{15}} & \cdots & \\ C_{20} & C_{21} & \boxed{C_{22}} & C_{23} & C_{24} & \cdots & & \\ C_{30} & C_{31} & C_{32} & \underline{C_{33}} & \cdots & & & \\ \boxed{C_{40}} & C_{41} & C_{42} & \cdots & & & & \\ C50 & \underline{C_{51}} & \cdots & & & & & \\ C_{60} & & & & & & & \cdots \end{bmatrix} \tag{9.29}$$

For example if we consider $m = n = 2$, then

$$(4)(3)(2)(1)C_{40} + (2)(2)(1)(2)(1)C_{22} + (4)(3)(2)(1)C_{04} = 0 \tag{9.30}$$

or $3C_{40} + C_{22} + 3C_{04} = 0$

9.2.1 Example: Cantilever Beam

25 We consider the homogeneous fourth-degree polynomial

$$\Phi_4 = C_{40}x^4 + C_{31}x^3y + C_{22}x^2y^2 + C_{13}xy^3 + C_{04}y^4 \tag{9.31}$$

with $3C_{40} + C_{22} + 3C_{04} = 0$,

26 The stresses are obtained from Eq. 9.26-a-9.26-c

$$T_{xx} = 2C_{22}x^2 + 6C_{13}xy + 12C_{04}y^2 \tag{9.32-a}$$

$$T_{yy} = 12C_{40}x^2 + 6C_{31}xy + 2C_{22}y^2 \tag{9.32-b}$$

$$T_{xy} = -3C_{31}x^2 - 4C_{22}xy - 3C_{13}y^2 \tag{9.32-c}$$

These can be used for the end-loaded cantilever beam with width b along the z axis, depth $2a$ and length L.

27 If all coefficients except C_{13} are taken to be zero, then

$$\begin{aligned} T_{xx} &= 6C_{13}xy & (9.33\text{-a}) \\ T_{yy} &= 0 & (9.33\text{-b}) \\ T_{xy} &= -3C_{13}y^2 & (9.33\text{-c}) \end{aligned}$$

28 This will give a parabolic shear traction on the loaded end (correct), but also a uniform shear traction $T_{xy} = -3C_{13}a^2$ on top and bottom. These can be removed by superposing uniform shear stress $T_{xy} = +3C_{13}a^2$ corresponding to $\Phi_2 = \underbrace{-3C_{13}a^2}_{C_{11}} xy$. Thus

$$T_{xy} = 3C_{13}(a^2 - y^2) \quad (9.34)$$

note that $C_{20} = C_{02} = 0$, and $C_{11} = -3C_{13}a^2$.

29 The constant C_{13} is determined by requiring that

$$P = b\int_{-a}^{a} -T_{xy}dy = -3bC_{13}\int_{-a}^{a}(a^2 - y^2)dy \quad (9.35)$$

hence

$$C_{13} = -\frac{P}{4a^3b} \quad (9.36)$$

and the solution is

$$\begin{aligned} \Phi &= \frac{3P}{4ab}xy - \frac{P}{4a^3b}xy^3 & (9.37\text{-a}) \\ T_{xx} &= -\frac{3P}{2a^3b}xy & (9.37\text{-b}) \\ T_{xy} &= -\frac{3P}{4a^3b}(a^2 - y^2) & (9.37\text{-c}) \\ T_{yy} &= 0 & (9.37\text{-d}) \end{aligned}$$

30 We observe that the second moment of area for the rectangular cross section is $I = b(2a)^3/12 = 2a^3b/3$, hence this solution agrees with the elementary beam theory solution

$$\begin{aligned} \Phi &= C_{11}xy + C_{13}xy^3 = \frac{3P}{4ab}xy - \frac{P}{4a^3b}xy^3 & (9.38\text{-a}) \\ T_{xx} &= -\frac{P}{I}xy = -M\frac{y}{I} = -\frac{M}{S} & (9.38\text{-b}) \\ T_{xy} &= -\frac{P}{2I}(a^2 - y^2) & (9.38\text{-c}) \\ T_{yy} &= 0 & (9.38\text{-d}) \end{aligned}$$

9.2.2 Polar Coordinates

9.2.2.1 Plane Strain Formulation

31 In polar coordinates, the strain components in plane strain are, Eq. 8.46

$$E_{rr} = \frac{1}{E}\left[(1-\nu^2)T_{rr} - \nu(1+\nu)T_{\theta\theta}\right] \quad (9.39\text{-a})$$

$$E_{\theta\theta} = \frac{1}{E}\left[(1-\nu^2)T_{\theta\theta} - \nu(1+\nu)T_{rr}\right] \quad (9.39\text{-b})$$

$$E_{r\theta} = \frac{1+\nu}{E}T_{r\theta} \quad (9.39\text{-c})$$

$$E_{rz} = E_{\theta z} = E_{zz} = 0 \quad (9.39\text{-d})$$

and the equations of equilibrium are

$$\frac{1}{r}\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} - \frac{T_{\theta\theta}}{r} = 0 \quad (9.40\text{-a})$$

$$\frac{1}{r^2}\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \quad (9.40\text{-b})$$

32 Again, it can be easily verified that the equations of equilibrium are identically satisfied if

$$T_{rr} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} \quad (9.41)$$

$$T_{\theta\theta} = \frac{\partial^2 \Phi}{\partial r^2} \quad (9.42)$$

$$T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) \quad (9.43)$$

33 In order to satisfy the compatibility conditions, the cartesian stress components must also satisfy Eq. 9.24. To derive the equivalent expression in cylindrical coordinates, we note that $T_{11} + T_{22}$ is the first scalar invariant of the stress tensor, therefore

$$T_{11} + T_{22} = T_{rr} + T_{\theta\theta} = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial r^2} \quad (9.44)$$

34 We also note that in cylindrical coordinates, the Laplacian operator takes the following form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad (9.45)$$

35 Thus, the function Φ must satisfy the biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) = 0 \quad \text{or} \quad \nabla^4 = 0 \quad (9.46)$$

9.2.2.2 Axially Symmetric Case

36 If Φ is a function of r only, we have

$$T_{rr} = \frac{1}{r}\frac{\mathrm{d}\Phi}{\mathrm{d}r}; \qquad T_{\theta\theta} = \frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2}; \qquad T_{r\theta} = 0 \quad (9.47)$$

and

$$\frac{\mathrm{d}^4\Phi}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3\Phi}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \quad (9.48)$$

37 The general solution to this problem; using Mathematica:

```
DSolve[phi''''[r]+2 phi'''[r]/r-phi''[r]/r^2+phi'[r]/r^3==0,phi[r],r]
```

$$\Phi = A \ln r + Br^2 \ln r + Cr^2 + D \tag{9.49}$$

38 The corresponding stress field is

$$T_{rr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C \tag{9.50}$$
$$T_{\theta\theta} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C \tag{9.51}$$
$$T_{r\theta} = 0 \tag{9.52}$$

and the strain components are (from Sect. 8.8.1)

$$E_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left[\frac{(1+\nu)A}{r^2} + (1 - 3\nu - 4\nu^2)B + 2(1-\nu-2\nu^2)B\ln r + 2(1-\nu-2\nu^2)C\right] \tag{9.53}$$
$$E_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{E}\left[-\frac{(1+\nu)A}{r^2} + (3 - \nu - 4\nu^2)B + 2(1-\nu-2\nu^2)B\ln r + 2(1-\nu-2\nu^2)C\right] \tag{9.54}$$
$$E_{r\theta} = 0 \tag{9.55}$$

39 Finally, the displacement components can be obtained by integrating the above equations

$$u_r = \frac{1}{E}\left[-\frac{(1+\nu)A}{r} - (1+\nu)Br + 2(1-\nu-2\nu^2)r\ln rB + 2(1-\nu-2\nu^2)rC\right] \tag{9.56}$$
$$u_\theta = \frac{4r\theta B}{E}(1-\nu^2) \tag{9.57}$$

9.2.2.3 Example: Thick-Walled Cylinder

40 If we consider a circular cylinder with internal and external radii a and b respectively, subjected to internal and external pressures p_i and p_o respectively, Fig. 9.2, then the boundary conditions for the plane strain problem are

$$T_{rr} = -p_i \quad \text{at} \quad r = a \tag{9.58-a}$$
$$T_{rr} = -p_o \quad \text{at} \quad r = b \tag{9.58-b}$$

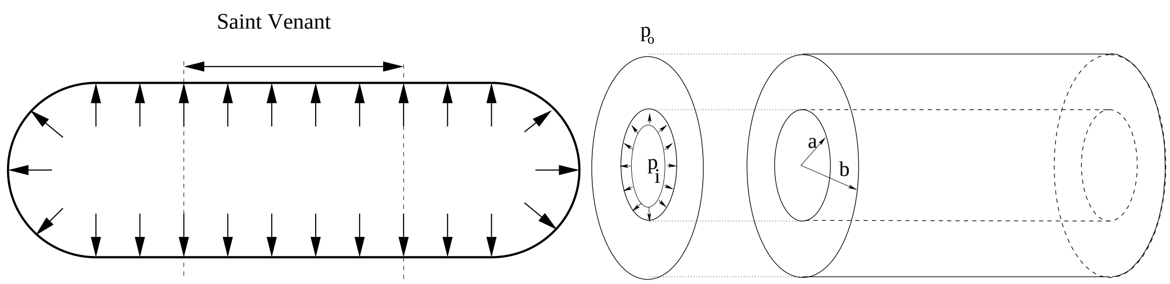


Figure 9.2: Pressurized Thick Tube

41 These Boundary conditions can be easily shown to be satisfied by the following stress field

$$T_{rr} = \frac{A}{r^2} + 2C \tag{9.59-a}$$
$$T_{\theta\theta} = -\frac{A}{r^2} + 2C \tag{9.59-b}$$
$$T_{r\theta} = 0 \tag{9.59-c}$$

These equations are taken from Eq. 9.50, 9.51 and 9.52 with $B = 0$ and therefore represent a possible state of stress for the plane strain problem.

42 We note that if we take $B \neq 0$, then $u_\theta = \frac{4r\theta B}{E}(1-\nu^2)$ and this is not acceptable because if we were to start at $\theta = 0$ and trace a curve around the origin and return to the same point, than $\theta = 2\pi$ and the displacement would then be different.

43 Applying the boundary condition we find that

$$T_{rr} = -p_i \frac{(b^2/r^2)-1}{(b^2/a^2)-1} - p_0 \frac{1-(a^2/r^2)}{1-(a^2/b^2)} \quad (9.60)$$

$$T_{\theta\theta} = p_i \frac{(b^2/r^2)+1}{(b^2/a^2)-1} - p_0 \frac{1+(a^2/r^2)}{1-(a^2/b^2)} \quad (9.61)$$

$$T_{r\theta} = 0 \quad (9.62)$$

44 We note that if only the internal pressure p_i is acting, then T_{rr} is always a compressive stress, and $T_{\theta\theta}$ is always positive.

45 If the cylinder is thick, then the strains are given by Eq. 9.53, 9.54 and 9.55. For a very thin cylinder in the axial direction, then the strains will be given by

$$E_{rr} = \frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{E}(T_{rr} - \nu T_{\theta\theta}) \quad (9.63\text{-a})$$

$$E_{\theta\theta} = \frac{u}{r} = \frac{1}{E}(T_{\theta\theta} - \nu T_{rr}) \quad (9.63\text{-b})$$

$$E_{zz} = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\nu}{E}(T_{rr} + T_{\theta\theta}) \quad (9.63\text{-c})$$

$$E_{r\theta} = \frac{(1+\nu)}{E} T_{r\theta} \quad (9.63\text{-d})$$

46 It should be noted that applying Saint-Venant's principle the above solution is only valid away from the ends of the cylinder.

9.2.2.4 Example: Hollow Sphere

47 We consider next a hollow sphere with internal and xternal radii a_i and a_o respectively, and subjected to internal and external pressures of p_i and p_o, Fig. 9.3.

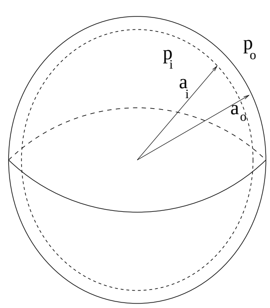


Figure 9.3: Pressurized Hollow Sphere

48 With respect to the spherical ccordinates (r, θ, ϕ), it is clear due to the spherical symmetry of the geometry and the loading that each particle of the elastic sphere will expereince only a radial displacement

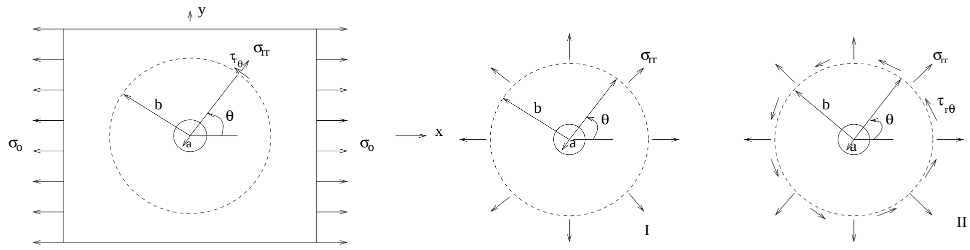


Figure 9.4: Circular Hole in an Infinite Plate

whose magnitude depends on r only, that is

$$u_r = u_r(r), \qquad u_\theta = u_\phi = 0 \tag{9.64}$$

9.3 Circular Hole, (Kirsch, 1898)

49 Analysing the infinite plate under uniform tension with a circular hole of diameter a, and subjected to a uniform stress σ_0, Fig. 9.4.

50 The peculiarity of this problem is that the far-field boundary conditions are better expressed in cartesian coordinates, whereas the ones around the hole should be written in polar coordinate system.

51 We will solve this problem by replacing the plate with a thick tube subjected to two different set of loads. The first one is a thick cylinder subjected to uniform radial pressure (solution of which is well known from *Strength of Materials*), the second one is a thick cylinder subjected to both radial and shear stresses which must be compatible with the traction applied on the rectangular plate.

52 First we select a stress function which satisfies the biharmonic Equation (Eq. **??**), and the far-field boundary conditions. From St Venant principle, away from the hole, the boundary conditions are given by:

$$\sigma_{xx} = \sigma_0; \qquad \sigma_{yy} = \tau_{xy} = 0 \tag{9.65}$$

53 Recalling (Eq. 9.19) that $\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2}$, this would would suggest a stress function Φ of the form $\Phi = \sigma_0 y^2$. Alternatively, the presence of the circular hole would suggest a polar representation of Φ. Thus, substituting $y = r\sin\theta$ would result in $\Phi = \sigma_0 r^2 \sin^2\theta$.

54 Since $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, we could simplify the stress function into

$$\Phi = f(r)\cos 2\theta \tag{9.66}$$

55 Substituting this function into the biharmonic equation (Eq. **??**) yields

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\right) = 0 \tag{9.67-a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{4}{r^2}\right)\left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{4f}{r^2}\right) = 0 \tag{9.67-b}$$

(note that the $\cos 2\theta$ term is dropped)

56 The general solution of this ordinary linear fourth order differential equation is

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D \tag{9.68}$$

thus the stress function becomes

$$\Phi = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D \right) \cos 2\theta \tag{9.69}$$

57 Next, we must determine the constants A, B, C, and D. Using Eq. **??**, the stresses are given by

$$\begin{array}{rclcl} \sigma_{rr} & = & \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} & = & -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta \\ \sigma_{\theta\theta} & = & \frac{\partial^2 \Phi}{\partial r^2} & = & \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} & = & -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right) & = & \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta \end{array} \tag{9.70}$$

58 Next we seek to solve for the four constants of integration by applying the boundary conditions. We will identify two sets of boundary conditions:

1. Outer boundaries: around an infinitely large circle of radius b inside a plate subjected to uniform stress σ_0, the stresses in polar coordinates are obtained from *Strength of Materials*

$$\left[\begin{array}{cc} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{array} \right] = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right] \left[\begin{array}{cc} \sigma_0 & 0 \\ 0 & 0 \end{array} \right] \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array} \right]^T \tag{9.71}$$

   yielding (recalling that $\sin^2\theta = 1/2 \sin 2\theta$, and $\cos^2\theta = 1/2(1+\cos 2\theta)$).

$$(\sigma_{rr})_{r=b} = \sigma_0 \cos^2\theta = \frac{1}{2}\sigma_0(1+\cos 2\theta) \tag{9.72-a}$$

$$(\sigma_{r\theta})_{r=b} = \frac{1}{2}\sigma_0 \sin 2\theta \tag{9.72-b}$$

$$(\sigma_{\theta\theta})_{r=b} = \frac{\sigma_0}{2}(1-\cos 2\theta) \tag{9.72-c}$$

   For reasons which will become apparent later, it is more convenient to decompose the state of stress given by Eq. 9.72-a and 9.72-b, into state I and II:

$$(\sigma_{rr})^I_{r=b} = \frac{1}{2}\sigma_0 \tag{9.73-a}$$

$$(\sigma_{r\theta})^I_{r=b} = 0 \tag{9.73-b}$$

$$(\sigma_{rr})^{II}_{r=b} = \frac{1}{2}\sigma_0 \cos 2\theta \leftarrow \tag{9.73-c}$$

$$(\sigma_{r\theta})^{II}_{r=b} = \frac{1}{2}\sigma_0 \sin 2\theta \leftarrow \tag{9.73-d}$$

   Where state I corresponds to a thick cylinder with external pressure applied on $r = b$ and of magnitude $\sigma_0/2$. Hence, only the last two equations will provide us with boundary conditions.

2. Around the hole: the stresses should be equal to zero:

$$(\sigma_{rr})_{r=a} = 0 \leftarrow \tag{9.74-a}$$

$$(\sigma_{r\theta})_{r=a} = 0 \leftarrow \tag{9.74-b}$$

59 Upon substitution in Eq. 9.70 the four boundary conditions (Eq. 9.73-c, 9.73-d, 9.74-a, and 9.74-b) become

$$-\left(2A + \frac{6C}{b^4} + \frac{4D}{b^2}\right) = \frac{1}{2}\sigma_0 \tag{9.75-a}$$

$$\left(2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2}\right) = \frac{1}{2}\sigma_0 \tag{9.75-b}$$

$$-\left(2A + \frac{6C}{a^4} + \frac{4D}{a^2}\right) = 0 \tag{9.75-c}$$

$$\left(2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2}\right) = 0 \tag{9.75-d}$$

60 Solving for the four unknowns, and taking $\frac{a}{b} = 0$ (i.e. an infinite plate), we obtain:

$$A = -\frac{\sigma_0}{4}; \qquad B = 0; \qquad C = -\frac{a^4}{4}\sigma_0; \qquad D = \frac{a^2}{2}\sigma_0 \tag{9.76}$$

61 To this solution, we must superimpose the one of a thick cylinder subjected to a uniform radial traction $\sigma_0/2$ on the outer surface, and with b much greater than a (Eq. 9.73-a and 9.73-b. These stresses are obtained from *Strength of Materials* yielding for this problem (carefull about the sign)

$$\sigma_{rr} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) \tag{9.77-a}$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) \tag{9.77-b}$$

62 Thus, substituting Eq. 9.75-a- into Eq. 9.70, we obtain

$$\sigma_{rr} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) + \left(1 + 3\frac{a^4}{r^4} - \frac{4a^2}{r^2}\right)\frac{1}{2}\sigma_0 \cos 2\theta \tag{9.78-a}$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\frac{1}{2}\sigma_0 \cos 2\theta \tag{9.78-b}$$

$$\sigma_{r\theta} = -\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\frac{1}{2}\sigma_0 \sin 2\theta \tag{9.78-c}$$

63 We observe that as $r \to \infty$, both σ_{rr} and $\sigma_{r\theta}$ are equal to the values given in Eq. 9.72-a and 9.72-b respectively.

64 Alternatively, at the edge of the hole when $r = a$ we obtain

$$\sigma_{rr} = 0 \tag{9.79}$$

$$\sigma_{r\theta} = 0 \tag{9.80}$$

$$\sigma_{\theta\theta}|_{r=a} = \sigma_0(1 - 2\cos 2\theta) \tag{9.81}$$

which for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ gives a stress concentration factor (SCF) of 3. For $\theta = 0$ and $\theta = \pi$, $\sigma_{\theta\theta} = -\sigma_0$.

Part III

FRACTURE MECHANICS

Chapter 10

ELASTICITY BASED SOLUTIONS FOR CRACK PROBLEMS

1 Following the solution for the stress concentration around a circular hole, and as a transition from elasticity to fracture mechanics, we now examine the stress field around a sharp crack. This problem, first addressed by Westergaard, is only one in a long series of similar ones, Table 10.1.

Problem	Coordinate System	Real/Complex	Solution	Date
Circular Hole	Polar	Real	Kirsh	1898
Elliptical Hole	Curvilinear	Complex	Inglis	1913
Crack	Cartesian	Complex	Westergaard	1939
V Notch	Polar	Complex	Willimas	1952
Dissimilar Materials	Polar	Complex	Williams	1959
Anisotropic Materials	Cartesian	Complex	Sih	1965

Table 10.1: Summary of Elasticity Based Problems Analysed

2 But first, we need to briefly review complex variables, and the formulation of the Airy stress functions in the complex space.

10.1 †Complex Variables

3 In the preceding chapter, we have used the Airy stress function with real variables to determine the stress field around a circular hole, however we need to extend Airy stress functions to complex variables in order to analyze stresses at the tip of a crack.

4 First we define the complex number z as:

$$\boxed{z = x_1 + ix_2 = re^{i\theta}} \tag{10.1}$$

where $i = \sqrt{-1}$, x_1 and x_2 are the cartesian coordinates, and r and θ are the polar coordinates.

5 We further define an **analytic function**, $f(z)$ one which derivatives depend only on z. Applying the chain rule

$$\frac{\partial}{\partial x_1} f(z) = \frac{\partial}{\partial z} f(z) \frac{\partial z}{\partial x_1} = f'(z) \frac{\partial z}{\partial x_1} = f'(z) \tag{10.2-a}$$

$$\frac{\partial}{\partial x_2} f(z) = \frac{\partial}{\partial z} f(z) \frac{\partial z}{\partial x_2} = f'(z) \frac{\partial z}{\partial x_2} = i f'(z) \tag{10.2-b}$$

6 If $f(z) = \alpha + i\beta$ where α and β are real functions of x_1 and x_2, and $f(z)$ is analytic, then from Eq. 10.2-a and 10.2-b we have:

$$\left.\begin{array}{l} \frac{\partial f(z)}{\partial x_1} = \frac{\partial \alpha}{\partial x_1} + i\frac{\partial \beta}{\partial x_1} = f'(z) \\ \frac{\partial f(z)}{\partial x_2} = \frac{\partial \alpha}{\partial x_2} + i\frac{\partial \beta}{\partial x_2} = i f'(z) \end{array}\right\} i\left(\frac{\partial \alpha}{\partial x_1} + i\frac{\partial \beta}{\partial x_1}\right) = \frac{\partial \alpha}{\partial x_2} + i\frac{\partial \beta}{\partial x_2} \tag{10.3}$$

7 Equating the real and imaginary parts yields the **Cauchy-Riemann** equations:

$$\boxed{\frac{\partial \alpha}{\partial x_1} = \frac{\partial \beta}{\partial x_2}; \quad \frac{\partial \alpha}{\partial x_2} = -\frac{\partial \beta}{\partial x_1}} \tag{10.4}$$

8 If we differentiate those two equation, first with respect to x_1, then with respect to x_2, and then add them up we obtain

$$\boxed{\frac{\partial^2 \alpha}{\partial x_1^2} + \frac{\partial^2 \alpha}{\partial x_2^2} = 0 \qquad \text{or} \qquad \nabla^2(\alpha) = 0} \tag{10.5}$$

which is Laplace's equation.

9 Similarly we can have:

$$\nabla^2(\beta) = 0 \tag{10.6}$$

Hence both the real (α) and the immaginary part (β) of an analytic function will separately provide solution to Laplace's equation, and α and β are **conjugate harmonic functions**.

10.2 †Complex Airy Stress Functions

10 It can be shown that any stress function can be expressed as

$$\Phi = \text{Re}[(x_1 - ix_2)\psi(z) + \chi(z)] \tag{10.7}$$

provided that both $\psi(z)$ (psi) and $\chi(z)$ (chi) are harmonic (i.e $\nabla^2(\psi) = \nabla^2(\chi) = 0$) analytic functions of x_1 and x_2. ψ and χ are often refered to as the Kolonov-Muskhelishvili complex potentials.

11 If $f(z) = \alpha + i\beta$ and both α and β are real, then its conjugate function is defined as:

$$\boxed{\bar{f}(\bar{z}) = \alpha - i\beta} \tag{10.8}$$

12 Note that conjugate functions should not be confused with the conjugate harmonic functions. Hence we can rewrite Eq. 10.7 as:

$$\boxed{\Phi = \text{Re}[\bar{z}\psi(z) + \chi(z)]} \tag{10.9}$$

13 Substituting Eq. 10.9 into Eq. 9.19, we can determine the stresses

$$\sigma_{11} + \sigma_{22} = 4\text{Re}\psi'(z) \tag{10.10}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\psi''(z) + \chi''(z)] \tag{10.11}$$

and by separation of real and imaginary parts we can then solve for $\sigma_{22} - \sigma_{11}$ & σ_{12}.

14 Displacements can be similarly obtained.

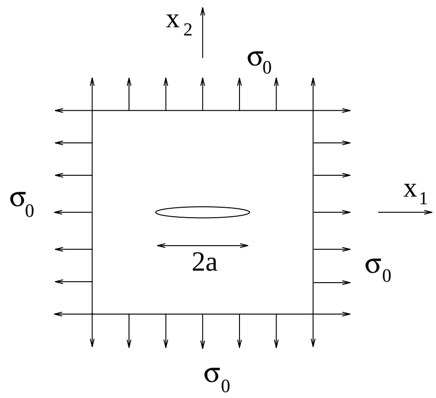


Figure 10.1: Crack in an Infinite Plate

10.3 Crack in an Infinite Plate, (Westergaard, 1939)

15 Just as both Kolosoff (1910) and Inglis (1913) independently solved the problem of an elliptical hole, there are two classical solutions for the crack problem. The first one was proposed by Westergaard, and the later by Williams. Whereas the first one is simpler to follow, the second has the advantage of being extended to cracks at the interface of two different homogeneous isotropic materials and be applicable for V notches.

16 Let us consider an infinite plate subjected to uniform biaxial stress σ_0 with a central crack of length $2a$, Fig. 10.1. From Inglis solution, we know that there would be a theoretically infinite stress at the tip of the crack, however neither the nature of the singularity nor the stress field can be derived from it.

17 Westergaard's solution, (Westergaard 1939) starts by assuming $\Phi(z)$ as a harmonic function (thus satisfying Laplace's equation $\nabla^2(\Phi) = 0$). Denoting by $\phi'(z)$ and $\phi''(z)$ the first and second derivatives respectively, and $\bar{\phi}(z)$ and $\bar{\bar{\phi}}(z)$ its first and second integrals respectively of the function $\phi(z)$.

18 Westergaard has postulated that

$$\Phi = \text{Re}\bar{\bar{\phi}}(z) + x_2\text{Im}\bar{\phi}(z) \tag{10.12}$$

is a solution to the crack problem[1].

19 †Let us verify that Φ satisfies the biharmonic equation. Taking the first derivatives, and recalling from from Eq. 10.2-a that $\frac{\partial}{\partial x_1} f(z) = f'(z)$, we have

$$\frac{\partial\Phi}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\text{Re}\,\bar{\bar{\phi}}\right) + \left[x_2\frac{\partial}{\partial x_1}\text{Im}\bar{\phi}(z) + \text{Im}\bar{\phi}(z)\underbrace{\frac{\partial x_2}{\partial x_1}}_{0}\right] \tag{10.13-a}$$

$$= \text{Re}\bar{\phi}(z) + x_2\text{Im}\phi(z) \tag{10.13-b}$$

$$\sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(\text{Re}\bar{\phi}(z)\right) + \left[x_2\frac{\partial}{\partial x_1}\text{Im}\phi(z) + \text{Im}\phi(z)\frac{\partial x_2}{\partial x_1}\right] \tag{10.13-c}$$

$$= \text{Re}\phi(z) + x_2\text{Im}\phi'(z) \tag{10.13-d}$$

[1]Note that we should not confuse the Airy stress function Φ with the complex function $\phi(z)$.

20 †Similarly, differentiating with respect to x_2, and recalling from Eq. 10.2-b that $\frac{\partial}{\partial x_2} f(z) = if'(z)$, we obtain

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_2} &= \frac{\partial}{\partial x_2}(\text{Re}\bar{\bar{\phi}}(z) + [x_2 \frac{\partial}{\partial x_2}\left(\text{Im}\bar{\phi}(z)\right) + \frac{\partial x_2}{\partial x_2}\text{Im}\bar{\phi}(z)] && (10.14\text{-a}) \\
&= -\text{Im}\bar{\phi}(z) + x_2\text{Re}\phi(z) + \text{Im}\bar{\phi}(z) && (10.14\text{-b}) \\
\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} &= x_2 \frac{\partial}{\partial x_2}\text{Re}\phi(z) + \text{Re}\phi(z)\frac{\partial x_2}{\partial x_2} && (10.14\text{-c}) \\
&= -x_2\text{Im}\phi'(z) + \text{Re}\phi(z) && (10.14\text{-d})
\end{aligned}$$

21 †Similarly, it can be shown that

$$\sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = -x_2\text{Re}\phi'(z) \quad (10.15)$$

22 †Having derived expressions for the stresses and the second partial derivatives of Φ, substituting into Eq. **??**, it can be shown that the biharmonic equation is satisfied, thus Φ is a valid solution.

23 †If we want to convince ourselves that the stresses indeed satisfy both the equilibrium and compatibility equations (which they do by virtue of Φ satisfiying the bi-harmonic equation), we have from Eq. 6.18 in 2D:

1. Equilibrium:

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} &= 0 && (10.16\text{-a}) \\
\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1} &= 0 && (10.16\text{-b})
\end{aligned}$$

   Let us consider the first equation

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \Phi}{\partial x_2^2}\right) &= \frac{\partial}{\partial x_1}[\text{Re}\phi(z) - x_2\text{Im}\phi'(z)] && (10.17\text{-a}) \\
&= \text{Re}\phi'(z) - x_2\text{Im}\phi''(z) && (10.17\text{-b}) \\
\frac{\partial \sigma_{12}}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}\right) &= \frac{\partial}{\partial x_2}[-x_2\text{Re}\phi'(z)] && (10.17\text{-c}) \\
&= -\text{Re}\phi'(z) + x_2\frac{\partial}{\partial x_1}\text{Im}\phi'(z) && (10.17\text{-d}) \\
&= -\text{Re}\phi'(z) + x_2\text{Im}\phi''(z) && (10.17\text{-e})
\end{aligned}$$

   If we substitute those two equations into Eq. 10.16-a then we do obtain zero. Similarly, it can be shown that Eq. 10.16-b is satisfied.

2. Compatibility: In plane strain, displacements are given by

$$\begin{aligned}
2\mu u_1 &= (1-2\nu)\text{Re}\,\bar{\phi}(z) - x_2\text{Im}\phi(z) && (10.18\text{-a}) \\
2\mu u_2 &= 2(1-\nu)\text{Im}\bar{\phi}(z) - x_2\text{Re}\,\phi(z) && (10.18\text{-b})
\end{aligned}$$

   and are obtained by integration of the strains which were in turn obtained from the stresses. As a check we compute

$$\begin{aligned}
2\mu\varepsilon_{11} &= \frac{\partial u_1}{\partial x_1} && (10.19\text{-a}) \\
&= (1-2\nu)\text{Re}\phi(z) - x_2\text{Im}\phi'(z) && (10.19\text{-b}) \\
&= (1-\nu)\underbrace{[\text{Re}\phi(z) - x_2\text{Im}\phi'(z)]}_{\sigma_{11}} - \nu\underbrace{[\text{Re}\phi(z) + x_2\text{Im}\phi'(z)]}_{\sigma_{22}} && (10.19\text{-c}) \\
&= (1-\nu)\sigma_{11} - \nu\sigma_{22} && (10.19\text{-d})
\end{aligned}$$

Recalling that

$$\mu = \frac{E}{2(1+\nu)} \tag{10.20}$$

then

$$E\varepsilon_{11} = (1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22} \tag{10.21}$$

this shows that $E\varepsilon_{11} = \sigma_{11} - \nu(\sigma_{22} - \sigma_{33})$, and for plane strain, $\varepsilon_{33} = 0 \Rightarrow \sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$ and $E\varepsilon_{11} = (1-\nu^2)\sigma_{11} - \nu(1+\nu)\sigma_{22}$

24 So far Φ was defined independently of the problem, and we simply determined the stresses in terms of it, and verified that the bi-harmonic equation was satisfied.

25 Next, we must determine ϕ such that the boundary conditions are satisfied. For reasons which will become apparent later, we generalize our problem to one in which we have a biaxial state of stress applied on the plate. Hence:

1. Along the crack: at $x_2 = 0$ and $-a < x_1 < a$ we have $\sigma_{22} = 0$ (traction free crack).
2. At infinity: at $x_2 = \pm\infty$, $\sigma_{22} = \sigma_0$

We note from Eq. 10.13-d that at $x_2 = 0$ σ_{22} reduces to

$$(\sigma_{22})_{x_2=0} = \text{Re}\phi(z) \tag{10.22}$$

26 Furthermore, we expect $\sigma_{22} \to \sigma_0$ as $x_1 \to \infty$, and σ_{22} to be greater than σ_0 when $| x_1 - a |> \epsilon$ (due to anticipated singularity predicted by Inglis), thus a possible choice for σ_{22} would be $\sigma_{22} = \frac{\sigma_0}{1-\frac{a}{x_1}}$, for symmetry, this is extended to $\sigma_{22} = \frac{\sigma_0}{\left(1-\frac{a^2}{x_1^2}\right)}$. However, we also need to have $\sigma_{22} = 0$ when $x_2 = 0$ and $-a < x_1 < a$, thus the function $\phi(z)$ should become imaginary along the crack, and

$$\sigma_{22} = \text{Re}\left(\frac{\sigma_0}{\sqrt{1-\frac{a^2}{x_1^2}}}\right) \tag{10.23}$$

27 Thus from Eq. 10.22 we have (note the transition from x_1 to z).

$$\phi(z) = \frac{\sigma_0}{\sqrt{1-\frac{a^2}{z^2}}} \tag{10.24}$$

28 If we perform a change of variable and define $\eta = z - a = re^{i\theta}$ and assuming $\frac{\eta}{a} \ll 1$, and recalling that $e^{i\theta} = \cos\theta + i\sin\theta$, then the first term of Eq. 10.13-d can be rewritten as

$$\begin{aligned}\text{Re}\phi(z) &= \text{Re}\frac{\sigma_0}{\sqrt{\frac{\eta^2+2a\eta}{\eta^2+a^2+2a\eta}}} \approx \text{Re}\frac{\sigma_0}{\sqrt{\frac{2a\eta}{a^2}}} \\ &\approx \text{Re}\sigma_0\sqrt{\frac{a}{2\eta}} \approx \text{Re}\sigma_0\sqrt{\frac{a}{2re^{i\theta}}} \approx \sigma_0\sqrt{\frac{a}{2r}}e^{-i\frac{\theta}{2}} \approx \sigma_0\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\end{aligned} \tag{10.25-a}$$

29 Recalling that $\sin 2\theta = 2\sin\theta\cos\theta$ and that $e^{-i\theta} = \cos\theta - i\sin\theta$, we substituting $x_2 = r\sin\theta$ into the second term

$$x_2\text{Im}\phi' = r\sin\theta\text{Im}\frac{\sigma_0}{2}\sqrt{\frac{a}{2(re^{i\theta})^3}} = \sigma_0\sqrt{\frac{a}{2r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{3\theta}{2} \tag{10.26}$$

30 Combining the above equations, with Eq. 10.13-d, 10.14-d, and 10.15 we obtain

$$\boxed{\begin{aligned}
\sigma_{22} &= \sigma_0\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\ \sin\frac{3\theta}{2}\right)+\cdots && (10.27)\\
\sigma_{11} &= \sigma_0\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\ \sin\frac{3\theta}{2}\right)+\cdots && (10.28)\\
\sigma_{12} &= \sigma_0\sqrt{\frac{a}{2r}}\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\ \cos\frac{3\theta}{2}+\cdots && (10.29)
\end{aligned}}$$

31 Recall that this was the biaxial case, the uniaxial case may be reproduced by superimposing a pressure in the x_1 direction equal to $-\sigma_0$, however this should not affect the stress field close to the crack tip.

32 Using a similar approach, we can derive expressions for the stress field around a crack tip in a plate subjected to far field shear stresses (mode II as defined later) using the following expression of ϕ

$$\Phi_{II}(z) = -x_2\text{Re}\bar{\phi}_{II}(z) \Rightarrow \phi_{II} = \frac{\tau}{\sqrt{1-\frac{a^2}{z^2}}} \tag{10.30}$$

and for the same crack but subjected to antiplane shear stresses (mode III)

$$\Phi'_{III}(z) = \frac{\sigma_{13}}{\sqrt{1-\frac{a^2}{z^2}}} \tag{10.31}$$

10.4 Stress Intensity Factors (Irwin)

33 Irwin[2] (Irwin 1957) introduced the concept of *stress intensity factor* defined as:

$$\boxed{\left\{\begin{array}{c} K_{\rm I} \\ K_{\rm II} \\ K_{\rm III}\end{array}\right\} = \lim_{r\rightarrow 0,\theta=0}\sqrt{2\pi r}\left\{\begin{array}{c}\sigma_{22}\\ \sigma_{12}\\ \sigma_{23}\end{array}\right\}} \tag{10.32}$$

where σ_{ij} are the near crack tip stresses, and K_i are associated with three independent kinematic movements of the upper and lower crack surfaces with respect to each other, as shown in Fig. 10.2:

- *Opening Mode, I*: The two crack surfaces are pulled apart in the y direction, but the deformations are symmetric about the $x-z$ and $x-y$ planes.
- *Shearing Mode, II*: The two crack surfaces slide over each other in the x-direction, but the deformations are symmetric about the $x-y$ plane and skew symmetric about the $x-z$ plane.
- *Tearing Mode, III*: The crack surfaces slide over each other in the z-direction, but the deformations are skew symmetric about the $x-y$ and $x-z$ planes.

34 From Eq. 10.27, 10.28 and 10.29 with $\theta = 0$, we have

$$\begin{aligned}
K_I &= \sqrt{2\pi r}\sigma_{22}\\
&= \sqrt{2\pi r}\sigma_0\sqrt{\frac{a}{2r}}\\
&= \sigma_0\sqrt{\pi a} && (10.33\text{-a})
\end{aligned}$$

where r is the length of a small vector extending directly forward from the crack tip.

[2] Irwin was asked by the Office of Naval Research (ONR) to investigate the Liberty ships failure during World War II, just as thirty years earlier Inglis was investigating the failure of British ships.

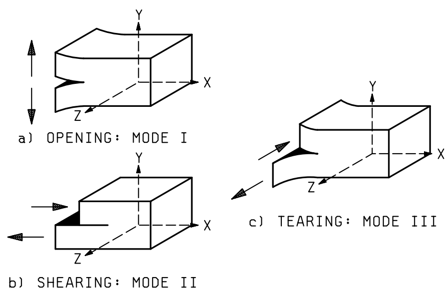


Figure 10.2: Independent Modes of Crack Displacements

35 Thus stresses and displacements can all be rewritten in terms of the SIF

$$\left\{ \begin{array}{c} \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \end{array} \right\} = \frac{1}{\sqrt{2\pi r}} \left[\begin{array}{ccc} f_{11}^{I}(\theta) & f_{11}^{II}(\theta) & f_{11}^{III}(\theta) \\ f_{22}^{I}(\theta) & f_{22}^{II}(\theta) & f_{22}^{III}(\theta) \\ f_{12}^{I}(\theta) & f_{12}^{II}(\theta) & f_{12}^{III}(\theta) \end{array} \right] \left\{ \begin{array}{c} K_{\rm I} \\ K_{\rm II} \\ K_{\rm III} \end{array} \right\} \tag{10.34-a}$$

i.e.

$$\sigma_{12} = \frac{K_{\rm II}}{\sqrt{2\pi r}} \underbrace{\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}}_{f_{22}^{II}} \tag{10.35}$$

1. Since higher order terms in r were neglected, previous equations are exact in the limit as $r \to 0$
2. Distribution of elastic stress field at tip can be described by K_I, K_{II} and K_{III}. Note that this polar distribution is identical for all cases. As we shall see later, for anisotropic cases, the spatial distribution is a function of elastic constants.
3. SIF are additives, i.e.
4. The SIF is the measure of the strength of the singularity (analogous to SCF)
5. $K = f(g)\sigma\sqrt{\pi a}$ where $f(g)$ is a parameter[3] that depends on the specimen, crack geometry, and loading.
6. Tada "Stress Analysis of Cracks", (Tada, Paris and Irwin 1973); and Cartwright & Rooke, "Compendium of Stress Intensity Factors" (Rooke and Cartwright 1976).
7. One of the underlying principles of FM is that unstable fracture occurs when the SIF reaches a critical value K_{Ic}. K_{Ic} or fracture toughness represents the inherent ability of a material to withstand a given stress field intensity at the tip of a crack and to resist progressive tensile crack extensions.

10.5 Near Crack Tip Stresses and Displacements in Isotropic Cracked Solids

36 Using Irwin's concept of the stress intensity factors, which characterize the strength of the singularity at a crack tip, the near crack tip ($r \ll a$) stresses and displacements are always expressed as:

[3]Note that in certain literature, (specially the one of Lehigh University), instead of $K = f(g)\sigma\sqrt{\pi a}$, $k = f(g)\sigma\sqrt{a}$ is used.

Pure mode I loading:

$$\sigma_{xx} = \frac{K_{\rm I}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \quad (10.36\text{-a})$$

$$\sigma_{yy} = \frac{K_{\rm I}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \quad (10.36\text{-b})$$

$$\tau_{xy} = \frac{K_{\rm I}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (10.36\text{-c})$$

$$\sigma_{zz} = \nu(\sigma_x+\sigma_y)\tau_{xz} = \tau_{yz} = 0 \quad (10.36\text{-d})$$

$$u = \frac{K_{\rm I}}{2\mu}\left[\frac{r}{2\pi}\right]^{\frac{1}{2}}\cos\frac{\theta}{2}\left[\kappa-1+2\sin^2\frac{\theta}{2}\right] \quad (10.36\text{-e})$$

$$v = \frac{K_{\rm I}}{2\mu}\left[\frac{r}{2\pi}\right]^{\frac{1}{2}}\sin\frac{\theta}{2}\left[\kappa+1-2\cos^2\frac{\theta}{2}\right] \quad (10.36\text{-f})$$

$$w = 0 \quad (10.36\text{-g})$$

Pure mode II loading:

$$\sigma_{xx} = -\frac{K_{\rm II}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\left[2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right] \quad (10.37\text{-a})$$

$$\sigma_{yy} = \frac{K_{\rm II}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (10.37\text{-b})$$

$$\tau_{xy} = \frac{K_{\rm II}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \quad (10.37\text{-c})$$

$$\sigma_{zz} = \nu(\sigma_x+\sigma_y) \quad (10.37\text{-d})$$

$$\tau_{xz} = \tau_{yz} = 0 \quad (10.37\text{-e})$$

$$u = \frac{K_{\rm II}}{2\mu}\left[\frac{r}{2\pi}\right]^{\frac{1}{2}}\sin\frac{\theta}{2}\left[\kappa+1+2\cos^2\frac{\theta}{2}\right] \quad (10.37\text{-f})$$

$$v = -\frac{K_{\rm II}}{2\mu}\left[\frac{r}{2\pi}\right]^{\frac{1}{2}}\cos\frac{\theta}{2}\left[\kappa-1-2\sin^2\frac{\theta}{2}\right] \quad (10.37\text{-g})$$

$$w = 0 \quad (10.37\text{-h})$$

Pure mode III loading:

$$\tau_{xz} = -\frac{K_{\rm III}}{(2\pi r)^{\frac{1}{2}}}\sin\frac{\theta}{2} \quad (10.38\text{-a})$$

$$\tau_{yz} = \frac{K_{\rm III}}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2} \quad (10.38\text{-b})$$

$$\sigma_{xx} = \sigma_y = \sigma_z = \tau_{xy} = 0 \quad (10.38\text{-c})$$

$$w = \frac{K_{\rm III}}{\mu}\left[\frac{2r}{\pi}\right]^{\frac{1}{2}}\sin\frac{\theta}{2} \quad (10.38\text{-d})$$

$$u = v = 0 \quad (10.38\text{-e})$$

where $\kappa = 3-4\nu$ for plane strain, and $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress.

37 Using Eq. **??**, **??**, and **??** we can write the stresses in polar coordinates

Pure mode I loading:

$$\sigma_r = \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right) \tag{10.39-a}$$

$$\sigma_\theta = \frac{K_{\rm I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right) \tag{10.39-b}$$

$$\tau_{r\theta} = \frac{K_{\rm I}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \tag{10.39-c}$$

Pure mode II loading:

$$\sigma_r = \frac{K_{\rm II}}{\sqrt{2\pi r}} \left(-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{10.40-a}$$

$$\sigma_\theta = \frac{K_{\rm II}}{\sqrt{2\pi r}} \left(-\frac{3}{4}\sin\frac{\theta}{2}-\frac{3}{4}\sin\frac{3\theta}{2}\right) \tag{10.40-b}$$

$$\tau_{r\theta} = \frac{K_{\rm II}}{\sqrt{2\pi r}} \left(\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right) \tag{10.40-c}$$

Chapter 11

LEFM DESIGN EXAMPLES

1 Following the detailed coverage of the derivation of the linear elastic stress field around a crack tip, and the introduction of the concept of a stress intensity factor in the preceding chapter, we now seek to apply those equations to some (pure mode I) practical design problems.

2 First we shall examine how is linear elastic fracture mechanics (LEFM) effectively used in design examples, then we shall give analytical solutions to some simple commonly used test geometries, followed by a tabulation of fracture toughness of commonly used engineering materials. Finally, this chapter will conclude with some simple design/analysis examples.

11.1 Design Philosophy Based on Linear Elastic Fracture Mechanics

3 One of the underlying principles of fracture mechanics is that *unstable* fracture occurs when the stress intensity factor (SIF) reaches a critical value $K_{\rm Ic}$, also called *fracture toughness*. $K_{\rm Ic}$ represents the inherent ability of a material to withstand a given stress field intensity at the tip of a crack and to resist progressive tensile crack extension.

4 Thus a crack will propagate (under pure mode I), whenever the stress intensity factor K_I (which characterizes the strength of the singularity for a given problem) reaches a material constant K_{Ic}. Hence, under the assumptions of linear elastic fracture mechanics (LEFM), at the point of incipient crack growth:

$$K_{Ic} = \beta\sigma\sqrt{\pi a} \tag{11.1}$$

5 Thus for the design of a cracked, or potentially cracked, structure, the engineer would have to decide what design variables can be selected, as only, two of these variables can be fixed, and the third must be determined. The design variables are:

Material properties: (such as special steel to resist corrosive liquid) $\Rightarrow K_c$ is fixed.

Design stress level: (which may be governed by weight considerations) $\Rightarrow \sigma$ is fixed.

Flaw size: [1], a.

> 6 In assessing the safety of a cracked body, it is essential that the crack length a be properly known. In most cases it is not. Thus assumptions must be made for its value, and those assumptions are dependent upon the crack detection methodology adopted. The presence of a crack, equal to the smallest one that can be detected, must be assumed.

[1] In most cases, a refers to half the total crack length.

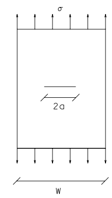


Figure 11.1: Middle Tension Panel

7 Thus, a simpler inspection procedure would result in a larger minimum crack size than one detected by expensive nondestructive techniques. In return, simplified inspection would result in larger crack size assumptions.

Once two parameters are specified, the third one is fixed. Finally, it should be mentioned that whereas in most cases the geometry is fixed (hence β), occasionally, there is the possibility to alter it in such a way to reduce (or maximize) β.

11.2 Stress Intensity Factors

6 As shown in the preceding chapter, analytic derivation of the stress intensity factors of even the simplest problem can be quite challenging. This explain the interest developed by some mathematician in solving fracture related problems. Fortunately, a number of simple problems have been solved and their analytic solution is found in stress intensity factor handbooks. The most commonly referenced ones are Tada, Paris and Irwin's (Tada et al. 1973), and Roorke and Cartwright, (Rooke and Cartwright 1976), and Murakami (Murakami 1987)

7 In addition, increasingly computer software with pre-programmed analytical solutions are becoming available, specially in conjunction with fatigue life predictions.

8 Because of their importance, expressions of SIF of commonly encountered geometries will be listed below:

Middle Tension Panel (MT), Fig. 11.1

$$K_I = \underbrace{\sqrt{\sec\frac{\pi a}{W}}}_{\beta}\sigma\sqrt{\pi a} \tag{11.2}$$

$$= \underbrace{\left[1 + 0.256\left(\frac{a}{W}\right) - 1.152\left(\frac{a}{W}\right)^2 + 12.2\left(\frac{a}{W}\right)^3\right]}_{\beta}\sigma\sqrt{\pi a} \tag{11.3}$$

We note that for W very large with respect to a, $\sqrt{\pi \sec\frac{\pi a}{W}} = 1$ as anticipated.

Single Edge Notch Tension Panel (SENT) for $\frac{L}{W} = 2$, Fig. 11.2

$$K_I = \underbrace{\left[1.12 - 0.23\left(\frac{a}{W}\right) + 10.56\left(\frac{a}{W}\right)^2 - 21.74\left(\frac{a}{W}\right)^3 + 30.42\left(\frac{a}{W}\right)^4\right]}_{\beta}\sigma\sqrt{\pi a} \tag{11.4}$$

We observe that here the β factor for small crack ($\frac{a}{W} \ll 1$) is grater than one and is approximately 1.12.

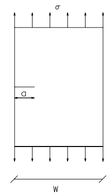


Figure 11.2: Single Edge Notch Tension Panel

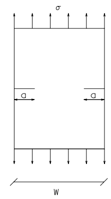


Figure 11.3: Double Edge Notch Tension Panel

Double Edge Notch Tension Panel (DENT), Fig. 11.3

$$K_I = \underbrace{\left[1.12 + 0.43\left(\frac{a}{W}\right) - 4.79\left(\frac{a}{W}\right)^2 + 15.46\left(\frac{a}{W}\right)^3\right]}_{\beta} \sigma\sqrt{\pi a} \tag{11.5}$$

Three Point Bend (TPB), Fig. 11.4

$$K_I = \frac{3\sqrt{\frac{a}{W}}\left[1.99 - \left(\frac{a}{W}\right)\left(1 - \frac{a}{W}\right)\left(2.15 - 3.93\frac{a}{W} + 2.7\left(\frac{a}{W}\right)^2\right)\right]}{2\left(1 + 2\frac{a}{W}\right)\left(1 - \frac{a}{W}\right)^{\frac{3}{2}}} \frac{PS}{BW^{\frac{3}{2}}} \tag{11.6}$$

Compact Tension Specimen (CTS), Fig. 11.5 used in ASTM E-399 (399 n.d.) Standard Test Method for Plane-Strain Fracture Toughness of Metallic Materials

$$K_I = \underbrace{\left[16.7 - 104.6\left(\frac{a}{W}\right) + 370\left(\frac{a}{W}\right)^2 - 574\left(\frac{a}{W}\right)^3 + 361\left(\frac{a}{W}\right)^4\right]}_{\beta} \underbrace{\frac{P}{BW}}_{\sigma} \sqrt{\pi a} \tag{11.7}$$

We note that this is not exactly the equation found in the ASTM standard, but rather an equivalent one written in the standard form.

Circular Holes: First let us consider the approximate solution of this problem, Fig. 11.6, then we will present the exact one:

Approximate: For a plate with a far field uniform stress σ, we know that there is a stress concentration factor of 3. for a crack radiating from this hole, we consider two cases

Short Crack: $\frac{a}{D} \to 0$, and thus we have an approximate far field stress of 3σ, and for an edge crack $\beta = 1.12$, Fig. 11.6 thus

$$\begin{aligned} K_I &= 1.12(3\sigma)\sqrt{\pi a} \\ &= 3.36\sigma\sqrt{\pi a} \end{aligned} \tag{11.8}$$

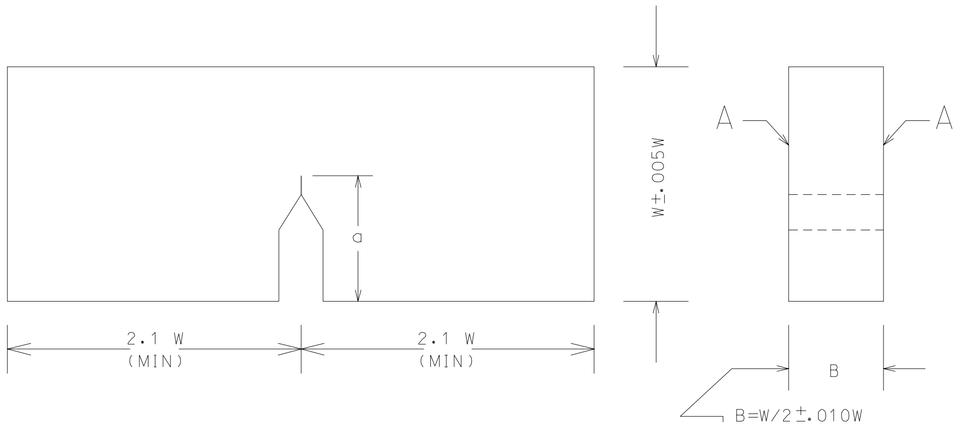


Figure 11.4: Three Point Bend Beam

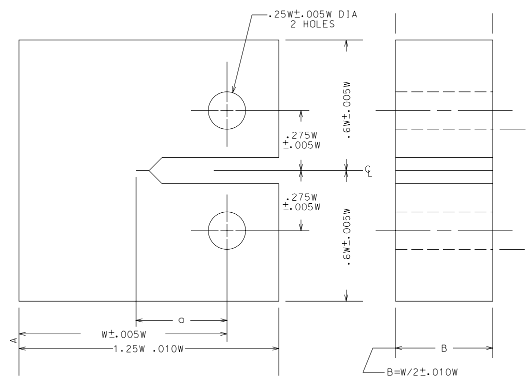


Figure 11.5: Compact Tension Specimen

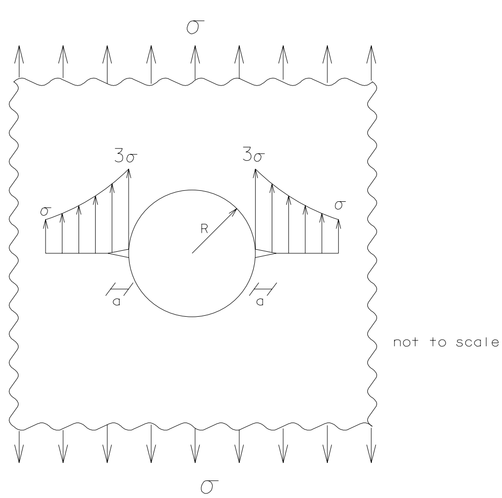


Figure 11.6: Approximate Solutions for Two Opposite Short Cracks Radiating from a Circular Hole in an Infinite Plate under Tension

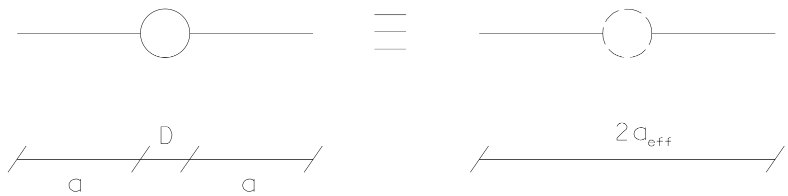


Figure 11.7: Approximate Solutions for Long Cracks Radiating from a Circular Hole in an Infinite Plate under Tension

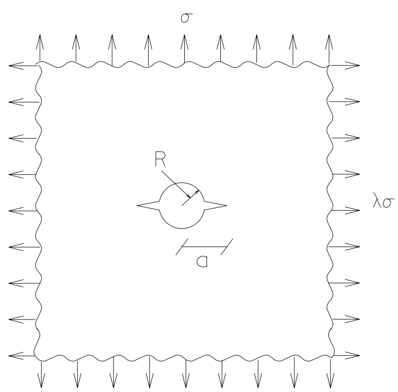


Figure 11.8: Radiating Cracks from a Circular Hole in an Infinite Plate under Biaxial Stress

Long Crack $D \ll 2a + D$, in this case, we can for all practical purposes ignore the presence of the hole, and assume that we have a central crack with an effective length $a_{eff} = \frac{2a+D}{2}$, thus

$$K_I = \sigma\sqrt{\pi\frac{2a+D}{2}} = \underbrace{\sqrt{1+\frac{D}{2a}}}_{\beta}\sigma\sqrt{\pi a} \tag{11.9}$$

Similarly, if we had only one single crack radiating from a hole, for short crack, β remains equal to 3.36, whereas for long crack, Fig. 11.7 we obtain:

$$K_I = \sigma\sqrt{\pi\frac{a+D}{2}} = \underbrace{\sqrt{\frac{1}{2}+\frac{D}{2a}}}_{\beta}\sigma\sqrt{\pi a} \tag{11.10}$$

Exact: Whereas the preceding equations give accurate results for both very short and very large cracks, in the intermediary stage an exact numerical solution was derived by Newman (Newman 1971), Fig. 11.8

$$K_I = \beta\sigma\sqrt{\pi a} \tag{11.11}$$

where, using Newman's solution β is given in Table 11.1

$\frac{a}{R}$	β Biaxial Stress			β Pressurized Hole	
	$\lambda = -1$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1$	$\lambda = 0$
1.01	0.4325	0.3256	0.2188	.2188	.1725
1.02	.5971	.4514	.3058	.3058	.2319
1.04	.7981	.6082	.4183	.4183	.3334
1.06	.9250	.7104	.4958	.4958	.3979
1.08	1.0135	.7843	.5551	.5551	.4485
1.10	1.0775	.8400	.6025	.6025	.4897
1.15	1.1746	.9322	.6898	.6898	.5688
1.20	1.2208	.9851	.7494	.7494	.6262
1.25	1.2405	1.0168	.7929	.7929	.6701
1.30	1.2457	1.0358	.8259	.8259	.7053
1.40	1.2350	1.0536	.8723	.8723	.7585
1.50	1.2134	1.0582	.9029	.9029	.7971
1.60	1.1899	1.0571	.9242	.9242	.8264
1.80	1.1476	1.0495	.9513	.9513	.8677
2.00	1.1149	1.0409	.9670	.9670	.8957
2.20	1.0904	1.0336	.9768	.9768	.9154
2.50	1.0649	1.0252	.9855	.9855	.9358
3.00	1.0395	1.0161	.99267	.99267	.9566
4.00	1.0178	1.0077	.9976	.9976	.9764

Table 11.1: Newman's Solution for Circular Hole in an Infinite Plate subjected to Biaxial Loading, and Internal Pressure

Pressurized Hole with Radiating Cracks: Again we will use Newman's solution for this problem, and distinguish two cases:

Pressurized Hole Only: or $\lambda = 0$, Fig. 11.9

$$K_I = \beta \frac{2pR}{\sqrt{\pi a}} \tag{11.12}$$

Pressurized Hole and Crack: or $\lambda = 1$

$$K_I = \beta p \sqrt{\pi a} \tag{11.13}$$

For both cases, β is given in Table 11.1. We note that for the pressurized hole only, K_I decreases with crack length, hence we would have a stable crack growth. We also note that K_I would be the same for a pressurized crack and borehole, as it would have been for an unpressurized hole but an identical far field stress. (WHY?)

Point Load Acting on Crack Surfaces of an Embedded Crack: The solution of this problem, Fig. 11.10 and the subsequent one, is of great practical importance, as it provides the Green's function for numerous other ones.

$$K_I^A = \frac{P}{\pi a}\sqrt{\frac{a+x}{a-x}} \tag{11.14}$$

$$K_I^B = \frac{P}{\pi a}\sqrt{\frac{a-x}{a+x}} \tag{11.15}$$

Point Load Acting on Crack Surfaces of an Edge Crack: The solution of this problem, Fig. 11.11 is

$$K_I = \frac{2P}{\pi a}\frac{C}{\sqrt{1+\left(\frac{x}{a}\right)^2}}\left[-0.4\left(\frac{x}{a}\right)^2 + 1.3\right] \tag{11.16}$$

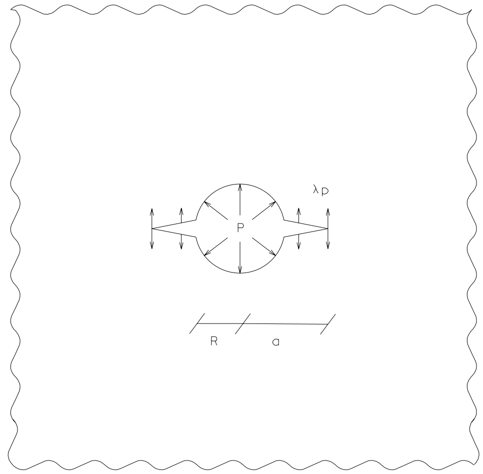


Figure 11.9: Pressurized Hole with Radiating Cracks

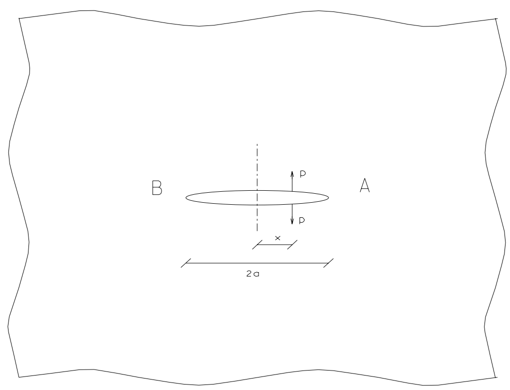


Figure 11.10: Two Opposite Point Loads acting on the Surface of an Embedded Crack

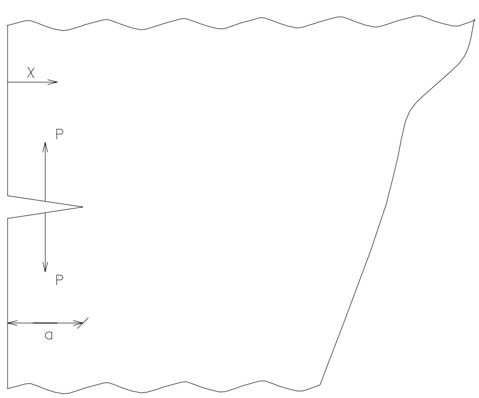


Figure 11.11: Two Opposite Point Loads acting on the Surface of an Edge Crack

$\frac{x}{a}$	C
< 0.6	1
0.6-0.7	1.01
0.7-0.8	1.03
0.8-0.9	1.07
> 0.9	1.11

Table 11.2: C Factors for Point Load on Edge Crack

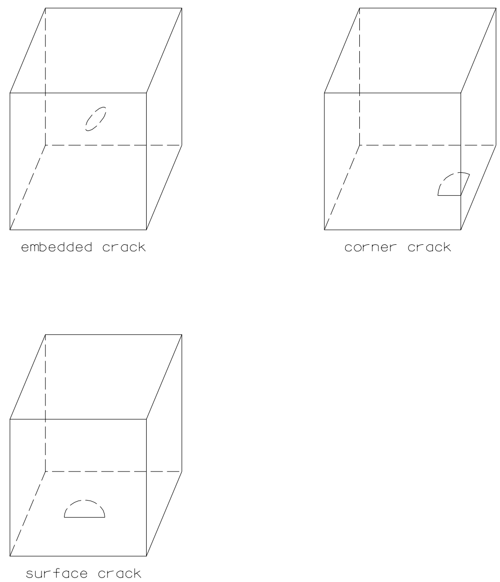


Figure 11.12: Embedded, Corner, and Surface Cracks

where C is tabulated in Table 11.2

Embedded Elliptical Crack A large number of naturally occurring defects are present as embedded, surface or corner cracks (such as fillet welding) Irwin, (Irwin 1962) proposed the following solution for the elliptical crack, with $x = a\cos\theta$ and $y = b\sin\theta$:

$$K_I(\theta) = \frac{1}{\Phi_0}\left(\sin^2\theta + \frac{b^2}{a^2}\cos^2\theta\right)^{\frac{1}{4}}\sigma\sqrt{\pi b} \tag{11.17}$$

where Φ_0 is a complete elliptical integral of the second kind

$$\Phi_0 = \int_0^{\frac{\pi}{2}}\sqrt{1 - \frac{a^2-b^2}{a^2}\sin^2\theta}\,d\theta \tag{11.18}$$

$$= \sqrt{Q} \tag{11.19}$$

An approximation to Eq. 11.17 was given by Cherepanov (Cherepanov 1979)

$$K_I = \left[\sin^2\theta + \left(\frac{b}{a}\right)^2\cos^2\theta\right]^{\frac{1}{4}}\sigma\sqrt{\pi b} \tag{11.20}$$

for $0 \le \frac{b}{a} \le 1$.

This solution calls for the following observations:

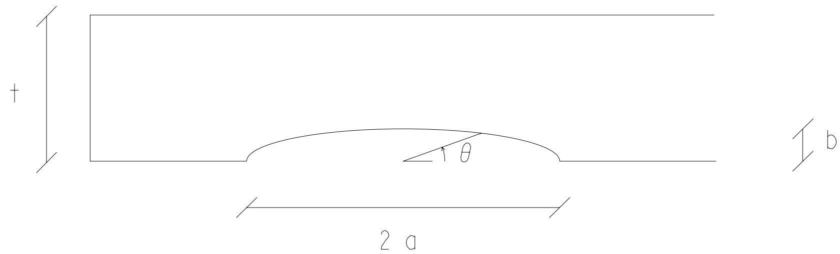


Figure 11.13: Elliptical Crack, and Newman's Solution

1. If $a = b$ then we have a "penny-shape' circular crack and Eq. 11.17 reduces to

$$K_I = \frac{2}{\pi}\sigma\sqrt{\pi a} \tag{11.21}$$

2. If $a = \infty$ & $\theta = \frac{\pi}{2}$ then we retrieve the solution $K_I = \sigma\sqrt{\pi a}$ of a through crack.
3. At the end of the minor axes, $\theta = \frac{\pi}{2}$ the stress intensity factor is maximum:

$$(K_I)_{\theta=\frac{\pi}{2}} = \frac{\sigma\sqrt{\pi b}}{\Phi_0} = \sigma\sqrt{\frac{\pi b}{Q}} \tag{11.22}$$

4. At the end of the major axes, $\theta = 0$ the stress intensity factor is minimum

$$(K_I)_{\theta=0} = \frac{\sigma\sqrt{\pi\frac{b^2}{a}}}{\Phi_0} \tag{11.23}$$

Thus an embedded elliptical crack will propagate into a circular one "penny-shaped".

Surface Cracks Irwin's original solution has been extended to semi-elliptical surface flaws, quarter elliptical corner cracks, and to surface cracks emanating from circular holes. Using the results of three dimensional finite element analysis, Newman and Raju (Newman and Raju 1981) developed an empirical SIF equation for semi-elliptical surface cracks, Fig. 11.13. The equation applies for cracks of arbitrary shape factor in finite size plates for both tension and bending loads. This is perhaps the most accurate solution and is almost universally used:

$$\begin{aligned} K &= \sigma\sqrt{\pi b}\left[M_1 + M_2\left(\frac{b}{t}\right)^2 + M_3\left(\frac{b}{t}\right)^4\right]\left[1 + 1.464\left(\frac{b}{a}\right)^{1.65}\right]^{-\frac{1}{2}} \\ &\left[\left(\frac{b}{a}\right)^2\cos^2\theta + \sin^2\theta\right]^{\frac{1}{4}}\left\{1 + \left[0.1 + 0.35\left(\frac{b}{t}\right)^2\right](1-\sin\theta)^2\right\} \end{aligned} \tag{11.24}$$

$$M_1 = 1.13 - 0.09\left(\frac{b}{a}\right) \tag{11.25}$$

$$M_2 = 0.89\left[0.2 + \left(\frac{b}{a}\right)\right]^{-1} - 0.54 \tag{11.26}$$

$$M_3 = 0.5 - \left[0.65 + \left(\frac{b}{a}\right)\right]^{-1} + 14\left[1. - \left(\frac{b}{a}\right)\right]^{24} \tag{11.27}$$

Material	K_{Ic} ksi$\sqrt{\text{in}}$
Steel, Medium Carbon	49
Steel, Pressure Vessel	190
Hardened Steel	20
Aluminum	20-30
Titanium	70
Copper	100
Lead	18
Glass	0.7
Westerly Granite	16
Cement Paste	0.5
Concrete	1
Nylon	3

Table 11.3: Approximate Fracture Toughness of Common Engineering Materials

Newman and Raju report that this equation is accurate within ±5 percent, provided $0 < \frac{b}{a} \leq 1.0$ and $\frac{b}{t} \leq 0.8$. For $\frac{b}{a}$ approximately equal to 0.25, K is roughly independent of θ. For shallow cracks $\frac{b}{t} \ll 1$, Equation 11.24 reduces to

$$K = 1.13\sigma\sqrt{\pi b}\left[1 - .08\left(\frac{b}{a}\right)\right]\left[1 + 1.464\left(\frac{b}{a}\right)^{1.65}\right]^{-\frac{1}{2}} \tag{11.28}$$

For very long cracks $\frac{b}{a} \ll 1$, Equation 11.24 reduces to

$$K = 1.13\sigma\sqrt{\pi b}\left[1 + 3.46\left(\frac{b}{t}\right)^2 + 11.5\left(\frac{b}{t}\right)^4\right] \tag{11.29}$$

11.3 Fracture Properties of Materials

9 Whereas fracture toughness testing will be the object of a separate chapter, we shall briefly mention the appropriate references from where fracture toughness values can be determined.

Metallic Alloys: 10 Testing procedures for fracture toughness of metallic alloys are standardized by various codes (see (399 n.d.) and (*British Standards Institution, BS 5447, London* 1977)). An exhaustive tabulation of fracture toughnesses of numerous alloys can be found in (Hudson and Seward 1978) and (Hudson and Seward 1982).

Concrete: Fracture mechanics evolved primarily from mechanical and metallurgical applications, but there has been much recent interest in its applicability to both concrete and rocks. Although there is not yet a standard for fracture toughness of concrete, Hillerborg (Anon. 1985) has proposed a standard procedure for determining G_F. Furthermore, a subcommittee of ASTM E399 is currently looking into a proposed testing procedure for concrete.

Rock: Ouchterlony has a comprehensive review of fracture toughnesses of numerous rocks in an appendix of (Ouchterlony 1986), and a proposed fracture toughness testing procedure can be found in (Ouchterlony 1982).

Table 11.3 provides an indication of the fracture toughness of common engineering materials. Note that stress intensity factors in metric units are commonly expressed in Mpa$\sqrt{\text{m}}$, and that

$$1\text{ksi}\sqrt{\text{in}} = 1.099\text{Mpa}\sqrt{\text{m}} \tag{11.30}$$

Yield Stress Ksi	K_{Ic} ksi$\sqrt{\text{in}}$
210	65
220	60
230	40
240	38
290	35
300	30

Table 11.4: Fracture Toughness vs Yield Stress for $.45C - N_i - C_r - M_o$ Steel

11.4 Examples

11.4.1 Example 1

Assume that a component in the shape of a large sheet is to be fabricated from $.45C - N_i - C_r - M_o$ steel, with a decreasing fracture toughness with increase in yield stress, Table 11.4. The smallest crack size ($2a$) which can be detected is approximately .12 in. The specified design stress is $\frac{\sigma_y}{2}$. To save weight, an increase of tensile strength from 220 ksi to 300 ksi is suggested. Is this reasonable?

At 220 ksi $K_{Ic} = 60$ ksi$\sqrt{\text{in}}$, and at 300 ksi $K_{Ic} = 30$ ksi$\sqrt{\text{in}}$. Thus, the design stress will be given by $\sigma_d = \frac{\sigma_y}{2}$ and from $K_{Ic} = \sigma_d\sqrt{\pi a_{cr}} \Rightarrow a_{cr} = \frac{1}{\pi}\left(\frac{K_{Ic}}{\frac{\sigma_y}{2}}\right)^2$

Thus,

Yield Stress σ_y	Design Stress σ_d	Fracture Toughness K_{Ic}	Critical Crack a_{cr}	Total Crack $2a_{cr}$
220	110	60	.0947	.189
300	150	30	.0127	.0255

We observe that for the first case, the total crack length is larger than the smallest one which can be detected (which is O.K.); Alternatively, for the second case the total critical crack size is approximately five times smaller than the minimum flaw size required and approximately eight times smaller than the flaw size tolerated at the 220 ksi level.

Hence, σ_y should not be raised to 300 ksi.

Finally, if we wanted to use the flaw size found with the 300 ksi alloy, we should have a decrease in design stress (since K_{Ic} and a_{cr} are now set) $K_{Ic} = \sigma_d\sqrt{\pi a_{vis}} \Rightarrow \sigma_d = \frac{K_{Ic}}{\sqrt{\pi a_{vis}}} = \frac{30\text{ksi}\sqrt{\text{in}}}{\sqrt{0.06\pi}} = 69$ ksi, with a potential factor of safety of one against cracking (we can not be sure 100% that there is no crack of that size or smaller as we can not detect it). We observe that since the design stress level is approximately half of that of the weaker alloy, there will be a two fold increase in weight.

11.4.2 Example 2

A small beer barrel of diameter 15" and wall thickness of .126" made of aluminum alloy exploded when a pressure reduction valve malfunctioned and the barrel experienced the 610 psi full pressure of the CO_2 cylinder supplying it with gas. Afterwards, cracks approximately 4.0 inch long by (probably) .07 inch deep were discovered on the inside of the salvaged pieces of the barrel (it was impossible to measure their depth). Independent tests gave 40. ksi$\sqrt{\text{in}}$ for K_{Ic} of the aluminum alloy. The question is whether the cracks were critical for the 610 psi pressure?

For a cylinder under internal pressure, the hoop stress is $\sigma = \frac{pD}{2t} = \frac{610}{}\frac{\text{lb}}{\text{in}^2}\frac{15}{2(.126)}\frac{\text{in}}{\text{in}} = 36,310$ psi $=$ 36.3 ksi. This can be used as the far field stress (neglecting curvature).

First we use the exact solution as given in Eq. 11.24, with $a = 2.$ in, $b = .07$ in, and $t = .126$ in. upon substitution we obtain:

$$
\begin{aligned}
M_1 &= 1.13 - 0.09\left(\frac{.07}{2.}\right) \\
&= 1.127 \\
M_2 &= 0.89\left[0.2 + \left(\frac{.07}{2.}\right)\right]^{-1} - 0.54 \\
&= 3.247 \\
M_3 &= 0.5 - \left[0.65 + \left(\frac{.07}{2}\right)\right]^{-1} + 14\left[1. - \left(\frac{.07}{2}\right)\right]^{24} \\
&= 4.994
\end{aligned}
$$

Substituting

$$
\begin{aligned}
K &= 36.3\sqrt{\pi .07}\left[1.127 + 3.247\left(\frac{.07}{.126}\right)^2 + 4.994\left(\frac{.07}{.126}\right)^4\right]\left[1 + 1.464\left(\frac{.07}{2}\right)^{1.65}\right]^{-\frac{1}{2}} \\
&\left[\left(\frac{.07}{2.}\right)^2 0 + 1\right]^{\frac{1}{4}}\left\{1 + \left[0.1 + 0.35\left(\frac{.07}{.126}\right)^2\right](1-1)^2\right\} \\
&= 44.2\text{ksi}\sqrt{\text{in}}
\end{aligned}
$$

This is about equal to the fracture toughness.

Note that if we were to use the approximate equation, for long cracks we would have obtained:

$$
\begin{aligned}
K &= (1.13)(36.3)\sqrt{\pi(.07)}\left[1 + 3.46\left(\frac{.07}{.126}\right)^2 + 11.5\left(\frac{.07}{.126}\right)^4\right] \\
&= 60.85 \\
&> K_{Ic}
\end{aligned}
$$

11.5 Additional Design Considerations

11.5.1 Leak Before Fail

10 As observed from the preceding example, many pressurized vessels are subject to crack growth if internal flaws are present. Two scenarios may happen, Fig. 11.14

Break-through: In this case critical crack configuration is reached before the crack has "daylighted", and there is a sudden and unstable crack growth.

Leak Before Fail: In this case, crack growth occur, and the crack "pierces" through the thickness of the vessel before unstable crack growth occurs. This in turn will result in a sudden depressurization, and this will stop any further crack gowth.

11 Hence, pressurized vessels should be designed to ensure a *leak before fail* failure scenario, as this would usually be immediately noticed and corrrected (assuming that there is no leak of flamable gas!).

12 Finally, it should be noted that leak before break assessment should be made on the basis of a complete *residual strength diagram* for both the part through and the through crack. Various ratios should be considered

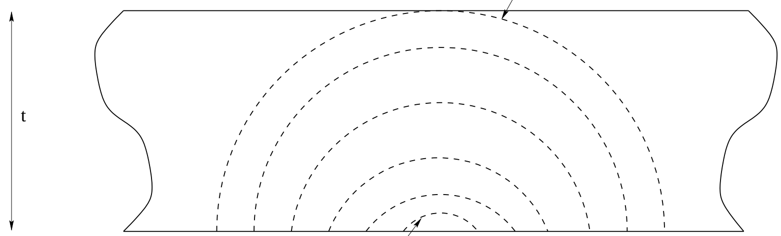


Figure 11.14: Growth of Semielliptical surface Flaw into Semicircular Configuration

11.5.2 Damage Tolerance Assessment

13 Fracture mechanics is not limited to determining the critical crack size, load, or stress combination. It can also be applied to establish a fracture control plan, or damage tolerance analysis with the following objectives:

1. Determine the effect of cracks on strength. This will result in a plot of crack size versus residual strength, or *Residual Strength Diagram*
2. Determine crack growth with time, resulting in *Crack Growth Curve*.

Chapter 12

THEORETICAL STRENGTH of SOLIDS; (Griffith I)

1 We recall that Griffith's involvement with fracture mechanics started as he was exploring the disparity in strength between glass rods of different sizes, (Griffith 1921). As such, he had postulated that this can be explained by the presence of internal flaws (idealized as elliptical) and then used Inglis solution to explain this discrepancy.

2 In this section, we shall develop an expression for the theoretical strength of perfect crystals (theoretically the strongest form of solid). This derivation, (Kelly 1974) is fundamentally different than the one of Griffith as it starts at the atomic level.

12.1 Derivation

3 We start by exploring the energy of interaction between two adjacent atoms at equilibrium separated by a distance a_0, Fig. 12.1. The total energy which must be supplied to separate atom C from C' is

$$U_0 = 2\gamma \tag{12.1}$$

where γ is the surface energy[1], and the factor of 2 is due to the fact that upon separation, we have two distinct surfaces.

12.1.1 Tensile Strength

12.1.1.1 Ideal Strength in Terms of Physical Parameters

We shall first derive an expression for the ideal strength in terms of physical parameters, and in the next section the strength will be expressed in terms of engineering ones.

Solution I: Force being the derivative of energy, we have $F = \frac{dU}{da}$, thus $F = 0$ at $a = a_0$, Fig. 12.2, and is maximum at the inflection point of the $U_0 - a$ curve. Hence, the slope of the force displacement curve is the stiffness of the atomic spring and should be related to E. If we let $x = a - a_0$, then the strain would be equal to $\varepsilon = \frac{x}{a_0}$. Furthermore, if we define the stress as $\sigma = \frac{F}{a_0^2}$, then the $\sigma - \varepsilon$

[1] From watching raindrops and bubbles it is obvious that liquid water has surface tension. When the surface of a liquid is extended (soap bubble, insect walking on liquid) work is done against this tension, and energy is stored in the new surface. When insects walk on water it sinks until the surface energy just balances the decrease in its potential energy. For solids, the chemical bonds are stronger than for liquids, hence the surface energy is stronger. The reason why we do not notice it is that solids are too rigid to be distorted by it. Surface energy γ is expressed in J/m^2 and the surface energies of water, most solids, and diamonds are approximately .077, 1.0, and 5.14 respectively.

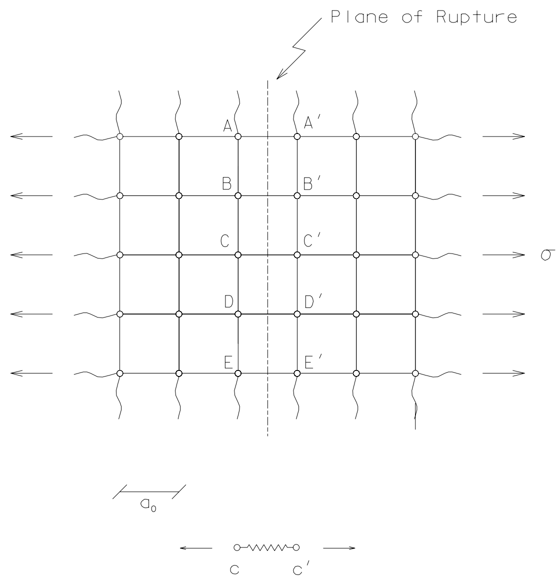


Figure 12.1: Uniformly Stressed Layer of Atoms Separated by a_0

curve will be as shown in Fig. 12.3. From this diagram, it would appear that the sine curve would be an adequate approximation to this relationship. Hence,

$$\sigma = \sigma_{max}^{theor} \sin 2\pi \frac{x}{\lambda} \tag{12.2}$$

and the maximum stress σ_{max}^{theor} would occur at $x = \frac{\lambda}{4}$. The energy required to separate two atoms is thus given by the area under the sine curve, and from Eq. 12.1, we would have

$$\begin{aligned} 2\gamma = U_0 &= \int_0^{\frac{\lambda}{2}} \sigma_{max}^{theor} \sin\left(2\pi\frac{x}{\lambda}\right) dx && (12.3) \\ &= \frac{\lambda}{2\pi}\sigma_{max}^{theor}[-\cos(\frac{2\pi x}{\lambda})]\,|_0^{\frac{\lambda}{2}} && (12.4) \\ &= \frac{\lambda}{2\pi}\sigma_{max}^{theor}[-\overbrace{\cos(\frac{2\pi\lambda}{2\lambda})}^{-1} + \overbrace{\cos(0)}^{1}] && (12.5) \\ \Rightarrow \lambda &= \frac{2\gamma\pi}{\sigma_{max}^{theor}} && (12.6) \end{aligned}$$

For very small displacements (small x) $\sin x \approx x$, Eq. 12.2 reduces to

$$\sigma \approx \sigma_{max}^{theor} \frac{2\pi x}{\lambda} \approx E\frac{x}{a_0} \tag{12.7}$$

elliminating x,

$$\sigma_{max}^{theor} \approx \frac{E}{a_0}\frac{\lambda}{2\pi} \tag{12.8}$$

Substituting for λ from Eq. 12.6, we get

$$\boxed{\sigma_{max}^{theor} \approx \sqrt{\frac{E\gamma}{a_0}}} \tag{12.9}$$

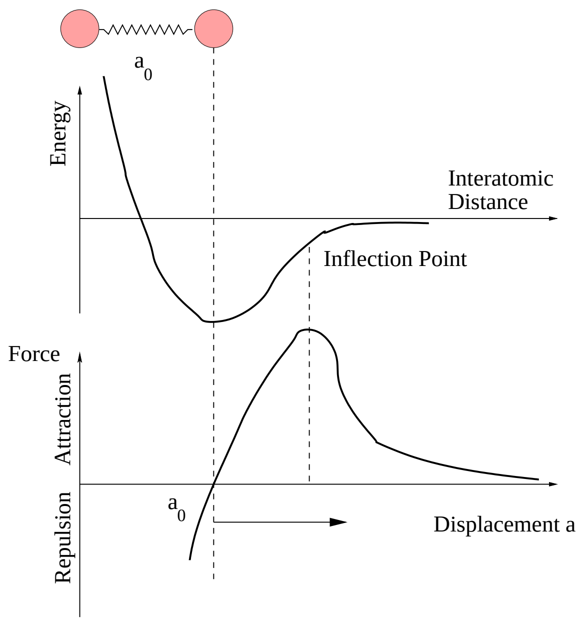


Figure 12.2: Energy and Force Binding Two Adjacent Atoms

Solution II: For two layers of atoms a_0 apart, the strain energy per unit area due to σ (for linear elastic systems) is

$$\left.\begin{array}{rcl} U &=& \frac{1}{2}\sigma\varepsilon a_o \\ \sigma &=& E\varepsilon \end{array}\right\} U = \frac{\sigma^2 a_o}{2E} \tag{12.10}$$

If γ is the surface energy of the solid per unit area, then the total surface energy of two new fracture surfaces is 2γ.

For our theoretical strength, $U = 2\gamma \Rightarrow \frac{(\sigma_{max}^{theor})^2 a_0}{2E} = 2\gamma$ or $\sigma_{max}^{theor} = 2\sqrt{\frac{\gamma E}{a_0}}$

Note that here we have assumed that the material obeys Hooke's Law up to failure, since this is seldom the case, we can simplify this approximation to:

$$\boxed{\sigma_{max}^{theor} = \sqrt{\frac{E\gamma}{a_0}}} \tag{12.11}$$

which is the same as Equation 12.9

Example: As an example, let us consider steel which has the following properties: $\gamma = 1\frac{J}{m^2}$; $E = 2 \times 10^{11}\frac{N}{m^2}$; and $a_0 \approx 2 \times 10^{-10}$ m. Thus from Eq. 12.9 we would have:

$$\sigma_{max}^{theor} \approx \sqrt{\frac{(2\times10^{11})(1)}{2\times10^{-10}}} \tag{12.12}$$

$$\approx 3.16\times10^{10}\frac{N}{m^2} \tag{12.13}$$

$$\approx \frac{E}{6} \tag{12.14}$$

Thus this would be the ideal theoretical strength of steel.

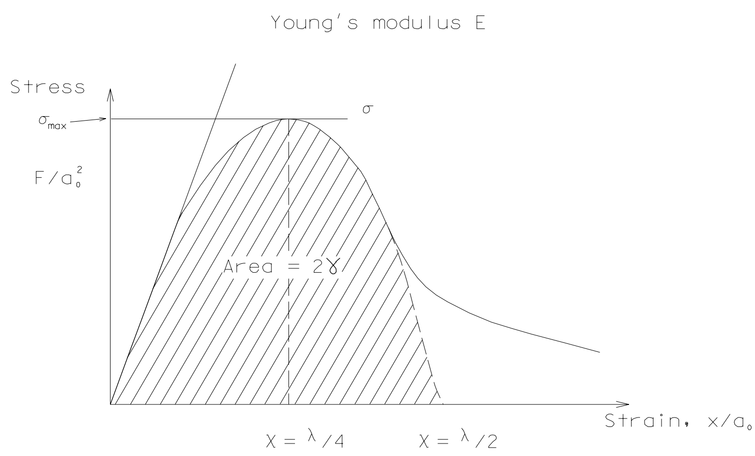


Figure 12.3: Stress Strain Relation at the Atomic Level

12.1.1.2 Ideal Strength in Terms of Engineering Parameter

We note that the force to separate two atoms drops to zero when the distance between them is $a_0 + a$ where a_0 corresponds to the origin and a to $\frac{\lambda}{2}$. Thus, if we take $a = \frac{\lambda}{2}$ or $\lambda = 2a$, combined with Eq. 12.8 would yield

$$\sigma_{max}^{theor} \approx \frac{E}{a_0}\frac{a}{\pi} \tag{12.15}$$

4 Alternatively combining Eq. 12.6 with $\lambda = 2a$ gives

$$a \approx \frac{\gamma\pi}{\sigma_{max}^{theor}} \tag{12.16}$$

5 Combining those two equations

$$\gamma \approx \frac{E}{a_0}\left(\frac{a}{\pi}\right)^2 \tag{12.17}$$

6 However, since as a first order approximation $a \approx a_0$ then the surface energy will be

$$\gamma \approx \frac{Ea_0}{10} \tag{12.18}$$

7 This equation, combined with Eq. 12.9 will finally give

$$\boxed{\sigma_{max}^{theor} \approx \frac{E}{\sqrt{10}}} \tag{12.19}$$

which is an approximate expression for the theoretical maximum strength in terms of E.

12.1.2 Shear Strength

8 Similar derivation can be done for shear. What happen if we slide the top row over the bottom one. Again, we can assume that the shear stress is

$$\tau = \tau_{max}^{theor}\sin 2\pi\frac{x}{\lambda} \tag{12.20}$$

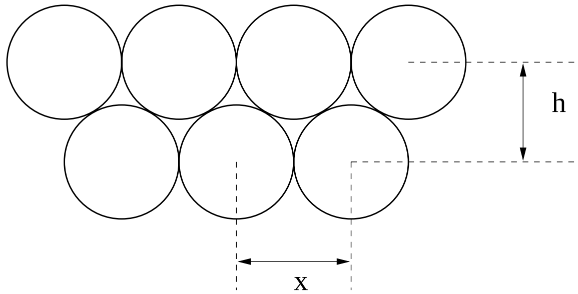


Figure 12.4: Influence of Atomic Misfit on Ideal Shear Strength

and from basic elasticity

$$\tau = G\gamma_{xy} \tag{12.21}$$

and, Fig. 12.4 $\gamma_{xy} = x/h$.

9 Because we do have very small displacement, we can elliminate x from

$$\begin{aligned} \tau_{max}^{theor} \sin 2\pi\frac{x}{\lambda} &\simeq \frac{2\pi x}{\lambda} = \gamma G = \frac{x}{h}G \\ &\Rightarrow \tau_{max}^{theor} = \frac{G\lambda}{2\pi h} \end{aligned} \tag{12.22-a}$$

10 If we do also assume that $\lambda = h$, and $G = E/2(1+\nu)$, then

$$\boxed{\tau_{max}^{theor} \simeq \frac{E}{12(1+\nu)} \simeq \frac{E}{18}} \tag{12.23}$$

12.2 Griffith Theory

11 Around 1920, Griffith was exploring the theoretical strength of solids by performing a series of experiments on glass rods of various diameters. He observed that the tensile strength (σ_t) of glass decreased with an increase in diameter, and that for a diameter $\phi \approx \frac{1}{10,000}$ in., $\sigma_t = 500,000$ psi; furthermore, by extrapolation to "zero" diameter he obtained a theoretical maximum strength of approximately 1,600,000 psi, and on the other hand for very large diameters the asymptotic values was around 25,000 psi.

12 Griffith had thus demonstrated that the theoretical strength could be experimentally approached, he now needed to show why the great majority of solids fell so far below it.

12.2.1 Derivation

13 In his quest for an explanation, he came across Inglis's paper, and his "strike of genius" was to assume that strength is reduced due to the presence of internal flaws. Griffith postulated that the theoretical strength can only be reached at the point of highest stress concentration, and accordingly the far-field applied stress will be much smaller.

14 Hence, assuming an elliptical imperfection, and from equation **??**

$$\sigma_{max}^{theor} = \sigma_{cr}^{act}\left(1 + 2\sqrt{\frac{a}{\rho}}\right) \tag{12.24}$$

σ is the stress at the tip of the ellipse which is caused by a (lower) far field stress σ_{cr}^{act}.

15 Asssuming $\rho \approx a_0$ and since $2\sqrt{\frac{a}{a_0}} \gg 1$, for an ideal plate under tension with only one single elliptical flaw the strength may be obtained from

$$\underbrace{\sigma_{max}^{theor}}_{\text{micro}} = \underbrace{2\sigma_{cr}^{act}\sqrt{\frac{a}{a_0}}}_{\text{macro}} \tag{12.25}$$

hence, equating with Eq. 12.9, we obtain

$$\boxed{\sigma_{max}^{theor} = \underbrace{2\sigma_{cr}^{act}\sqrt{\frac{a}{a_o}}}_{\text{Macro}} = \underbrace{\sqrt{\frac{E\gamma}{a_0}}}_{\text{Micro}}} \tag{12.26}$$

16 From this very important equation, we observe that

1. The left hand side is based on a linear elastic solution of a macroscopic problem solved by Inglis.
2. The right hand side is based on the theoretical strength derived from the sinusoidal stress-strain assumption of the interatomic forces, and finds its roots in micro-physics.

17 Finally, this equation would give (at fracture)

$$\boxed{\sigma_{cr}^{act} = \sqrt{\frac{E\gamma}{4a}}} \tag{12.27}$$

18 As an example, let us consider a flaw with a size of $2a = 5,000a_0$

$$\left.\begin{array}{lcl} \sigma_{cr}^{act} & = & \sqrt{\frac{E\gamma}{4a}} \\ \gamma & = & \frac{Ea_0}{10} \end{array}\right\} \begin{array}{lcl} \sigma_{cr}^{act} & = & \sqrt{\frac{E^2}{40}\frac{a_o}{a}} \\ \frac{a}{a_0} & = & 2,500 \end{array}\left.\vphantom{\begin{array}{l}a\\a\end{array}}\right\} \sigma_{cr}^{act} = \sqrt{\frac{E^2}{100{,}000}} = \frac{E}{100\sqrt{10}} \tag{12.28}$$

19 Thus if we set a flaw size of $2a = 5,000a_0$ in $\gamma \approx \frac{Ea_0}{10}$ this is enough to lower the theoretical fracture strength from $\frac{E}{\sqrt{10}}$ to a critical value of magnitude $\frac{E}{100\sqrt{10}}$, or a factor of 100.

20 Also

$$\left.\begin{array}{lcl} \sigma_{max}^{theor} & = & 2\sigma_{cr}^{act}\sqrt{\frac{a}{a_o}} \\ a & = & 10^{-6}m = 1\mu m \\ a_o & = & 1\mathring{A} = \rho = 10^{-10}m \end{array}\right\} \sigma_{max}^{theor} = 2\sigma_{cr}^{act}\sqrt{\frac{10^{-6}}{10^{-10}}} = 200\sigma_{cr}^{act} \tag{12.29}$$

21 Therefore at failure

$$\left.\begin{array}{lcl} \sigma_{cr}^{act} & = & \frac{\sigma_{max}^{theor}}{200} \\ \sigma_{max}^{theor} & = & \frac{E}{10} \end{array}\right\} \sigma_{cr}^{act} \approx \frac{E}{2,000} \tag{12.30}$$

which can be attained. For instance for steel $\frac{E}{2{,}000} = \frac{30{,}000}{2{,}000} = 15$ ksi

Chapter 13

ENERGY TRANSFER in CRACK GROWTH; (Griffith II)

1 In the preceding chapters, we have focused on the singular stress field around a crack tip. On this basis, a criteria for crack propagation, based on the strength of the singularity was first developed and then used in practical problems.

2 An alternative to this approach, is one based on energy transfer (or release), which occurs during crack propagation. This dual approach will be developed in this chapter.

3 Griffith's main achievement, in providing a basis for the fracture strengths of bodies containing cracks, was his realization that it was possible to derive a thermodynamic criterion for fracture by considering the total change in energy of a cracked body as the crack length increases, (Griffith 1921).

4 Hence, Griffith showed that material fail not because of a maximum stress, but rather because a certain energy criteria was met.

5 Thus, the Griffith model for elastic solids, and the subsequent one by Irwin and Orowan for elastic-plastic solids, show that crack propagation is caused by a transfer of energy transfer from external work and/or strain energy to surface energy.

6 It should be noted that this is a *global energy* approach, which was developed prior to the one of Westergaard which focused on the stress field surrounding a crack tip. It will be shown later that for linear elastic solids the two approaches are identical.

13.1 Thermodynamics of Crack Growth

13.1.1 General Derivation

7 If we consider a crack in a deformable continuum aubjected to arbitrary loading, then the first law of thermodynamics gives: The change in energy is proportional to the amount of work performed. Since only the change of energy is involved, any datum can be used as a basis for measure of energy. Hence energy is neither created nor consumed.

8 The first law of thermodynamics states *The time-rate of change of the total energy (i.e., sum of the kinetic energy and the internal energy) is equal to the sum of the rate of work done by the external forces and the change of heat content per unit time*:

$$\frac{d}{dt}(K + U + \Gamma) = W + Q \tag{13.1}$$

where K is the kinetic energy, U the total internal strain energy (elastic plus plastic), Γ the surface energy, W the external work, and Q the heat input to the system.

9 Since all changes with respect to time are caused by changes in crack size, we can write

$$\frac{\partial}{\partial t} = \frac{\partial A}{\partial t}\frac{\partial}{\partial A} \tag{13.2}$$

and for an adiabatic system (no heat exchange) and if loads are applied in a quasi static manner (no kinetic energy), then Q and K are equal to zero, and for a unit thickness we can replace A by a, then we can rewrite the first law as

$$\frac{\partial W}{\partial a} = \left(\frac{\partial U^e}{\partial a} + \frac{\partial U^p}{\partial a}\right) + \frac{\partial \Gamma}{\partial a} \tag{13.3}$$

10 This equation represents the energy balance during crack growth. It indicates that the work rate supplied to the continuum by the applied external loads is equal to the rate of strain energy (elastic and plastic) plus the energy dissipated during crack propagation.

11 Thus

$$\boxed{\begin{aligned} \Pi &= U^e - W && (13.4)\\ -\frac{\partial \Pi}{\partial a} &= \frac{\partial U^p}{\partial a} + \frac{\partial \Gamma}{\partial a} && (13.5)\end{aligned}}$$

that is the rate of potential energy decrease during crack growth is equal to the rate of energy dissipated in plastic deformation and crack growth.

12 It is very important to observe that the energy scales with a^2, whereas surface energy scales with a. It will be shown later that this can have serious implication on the stability of cracks, and on size effects.

13.1.2 Brittle Material, Griffith's Model

13 For a perfectly brittle material, we can rewrite Eq. 13.3 as

$$\boxed{G \stackrel{\text{def}}{=} -\frac{\partial \Pi}{\partial a} = \frac{\partial W}{\partial a} - \frac{\partial U^e}{\partial a} = \frac{\partial \Gamma}{\partial a} = 2\gamma} \tag{13.6}$$

the factor 2 appears because we have two material surfaces upon fracture. The left hand side represents the energy available for crack growth and is given the symbol G in honor of Griffith. Because G is derived from a potential function, it is often referred to as the crack driving force. The right hand side represents the resistance of the material that must be overcome for crack growth, and is a material constant (related to the toughness).

14 This equation represents the fracture criterion for crack growth, two limiting cases can be considered. They will be examined in conjunction with Fig. 13.1 in which we have a crack of length $2a$ located in an infinite plate subjected to load P. Griffith assumed that it was possible to produce a macroscopical load displacement $(P-u)$ curve for two different crack lengths a and $a+da$.

Two different boundary conditions will be considered, and in each one the change in potential energy as the crack extends from a to $a+da$ will be determined:

Fixed Grip: $(u_2 = u_1)$ loading, an increase in crack length from a to $a+da$ results in a decrease in stored elastic strain energy, ΔU,

$$\begin{aligned} \Delta U &= \frac{1}{2}P_2 u_1 - \frac{1}{2}P_1 u_1 && (13.7)\\ &= \frac{1}{2}(P_2 - P_1)\, u_1 && (13.8)\\ &< 0 && (13.9)\end{aligned}$$

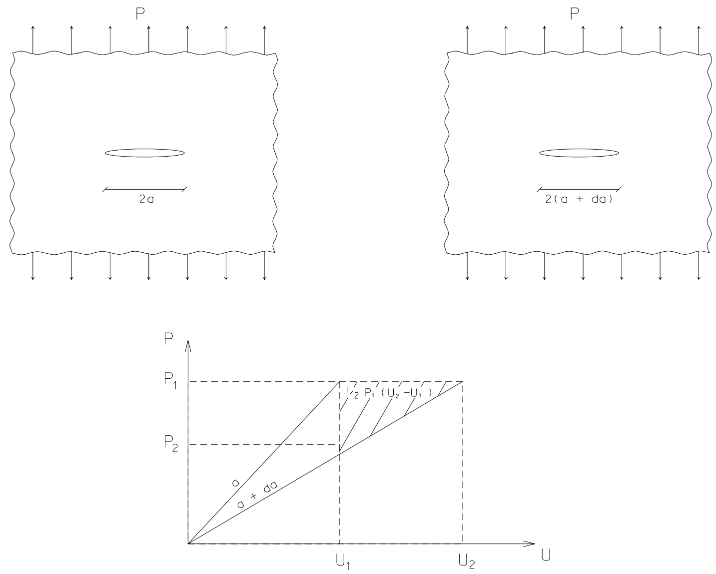


Figure 13.1: Energy Transfer in a Cracked Plate

Furthermore, under fixed grip there is no external work ($u_2 = u_1$), so the decrease in potential energy is the same as the decrease in stored internal strain energy, hence

$$\Pi_2 - \Pi_1 = \Delta W - \Delta U \tag{13.10}$$

$$= -\frac{1}{2}(P_2 - P_1)u_1 = \frac{1}{2}(P_1 - P_2)u_1 \tag{13.11}$$

Fixed Load: $P_2 = P_1$ the situation is slightly more complicated. Here there is both external work

$$\Delta W = P_1(u_2 - u_1) \tag{13.12}$$

and a release of internal strain energy. Thus the net effect is a change in potential energy given by:

$$\Pi_2 - \Pi_1 = \Delta W - \Delta U \tag{13.13}$$

$$= P_1\left(u_2 - u_1\right) - \frac{1}{2}P_1\left(u_2 - u_1\right) \tag{13.14}$$

$$= \frac{1}{2}P_1\left(u_2 - u_1\right) \tag{13.15}$$

15 Thus under fixed grip conditions there is a decrease in strain energy of magnitude $\frac{1}{2}u_1(P_1 - P_2)$ as the crack extends from a to $(a + \Delta a)$, whereas under constant load, there is a net decrease in potential energy of magnitude $\frac{1}{2}P_1(u_2 - u_1)$.

16 At the limit as $\Delta a \rightarrow da$, we define:

$$dP = P_1 - P_2 \tag{13.16}$$

$$du = u_2 - u_1 \tag{13.17}$$

then as $da \rightarrow 0$, the decrease in strain energy (and potential energy in this case) for the fixed grip would be

$$d\Pi = \frac{1}{2}udP \tag{13.18}$$

and for the constant load case

$$d\Pi = \frac{1}{2}Pdu \tag{13.19}$$

17 Furthermore, defining the compliance as

$$u = CP \tag{13.20}$$
$$du = CdP \tag{13.21}$$

18 Then the decrease in potential energy for both cases will be given by

$$\boxed{d\Pi = \frac{1}{2}CP\,dP} \tag{13.22}$$

19 In summary, as the crack extends there is a release of excess energy. Under fixed grip conditions, this energy is released from the strain energy. Under fixed load condition, external work is produced, half of it is consumed into strain energy, and the other half released. In either case, the energy released is consumed to form *surface energy*.

20 Thus a criteria for crack propagation would be

$$\boxed{d\Pi \geq 2\gamma da} \tag{13.23}$$

The difference between the two sides of the inequality will appear as kinetic energy at a real crack propagation.

$$\text{Energy Release Rate per unit crack extension} = \text{Surface energy}$$
$$\frac{d\Pi}{da} = 2\gamma \tag{13.24}$$

21 Using Inglis solution, Griffith has shown that for plane stress infinite plates with a central crack of length $2a$[1]

$$-\frac{d\Pi}{da} = \frac{\pi a \sigma_{cr}^2}{E} \tag{13.25}$$

note that the negative sign is due to the decrease in energy during crack growth. Combining with Eq. 13.24, and for incipient crack growth, this reduces to

$$\frac{\sigma_{cr}^2 \pi a da}{E'} = 2\gamma da \tag{13.26}$$

or

$$\boxed{\sigma_{cr} = \sqrt{\frac{2E'\gamma}{\pi a}}} \tag{13.27}$$

This equation derived on the basis of global fracture should be compared with Eq. 12.11 derived from local stress analysis.

13.2 Energy Release Rate Determination

13.2.1 From Load-Displacement

22 With reference to Fig. 13.2 The energy released from increment i to increment j is given by

[1]This equation will be rederived in Sect. 13.4 using Westergaard's solution.

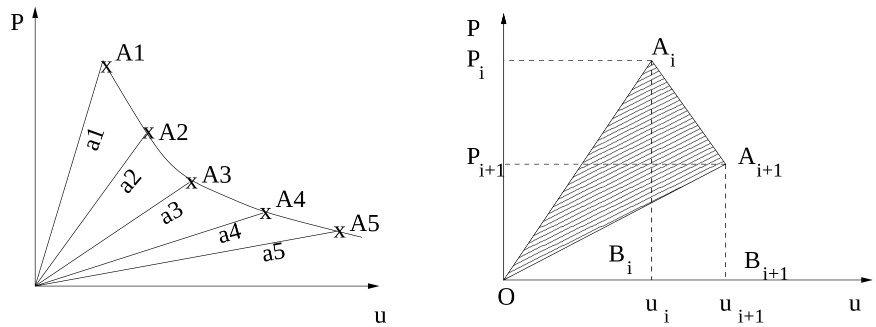


Figure 13.2: Determination of G_c From Load Displacement Curves

$$G = \sum_{i=1,n} \frac{OA_iA_{i+1}}{a_{i+1} - a_i} \tag{13.28}$$

where

$$\begin{aligned} OA_iA_{i+1} &= (OA_iB_i) + (A_iB_iB_{i+1}A_{i+1}) - (OA_{i+1}B_{i+1}) && \text{(13.29-a)} \\ &= \frac{1}{2}P_iu_i + \frac{1}{2}(P_i + P_{i+1})(u_{i+1} - u_i) - \frac{1}{2}P_{i+1}u_{i+1} && \text{(13.29-b)} \\ &= \frac{1}{2}(P_iu_{i+1} - P_{i+1}u_i) && \text{(13.29-c)} \end{aligned}$$

Thus, the critical energy release rate will be given by

$$\boxed{G = \sum_{i=1,n} \frac{1}{2B} \frac{P_iu_{i+1} - P_{i+1}u_i}{a_{i+1} - a_i}} \tag{13.30}$$

13.2.2 From Compliance

23 Under constant load we found the energy release needed to extend a crack by da was $\frac{1}{2}Pdu$. If G is the energy release rate, B is the thickness, and $u = CP$, (where u, C and P are the point load displacement, copliance and the load respectively), then

$$GBda = \frac{1}{2}Pd(CP) = \frac{1}{2}P^2dC \tag{13.31}$$

at the limit as $da \to 0$, then we would have:

$$\boxed{G = \frac{1}{2}\frac{P^2}{B}\left(\frac{dC}{da}\right)} \tag{13.32}$$

24 Thus we can use an experimental technique to obtain G and from $G = \frac{K_I^2}{E'}$ to get K_I, Fig. 13.3

25 With regard to accuracy since we are after K which does not depend on E, a low modulus plate material (i.e. high strength aluminum) can be used to increase observed displacement.

26 As an example, let us consider the double cantilever beam problem, Fig. 13.4. From strength of materials:

$$C = \underbrace{\frac{24}{EB}\int_0^a \frac{x^2}{h^3}dx}_{\text{flexural}} + \underbrace{\frac{6(1+\nu)}{EB}\int_0^a \frac{1}{h}dx}_{\text{shear}} \tag{13.33}$$

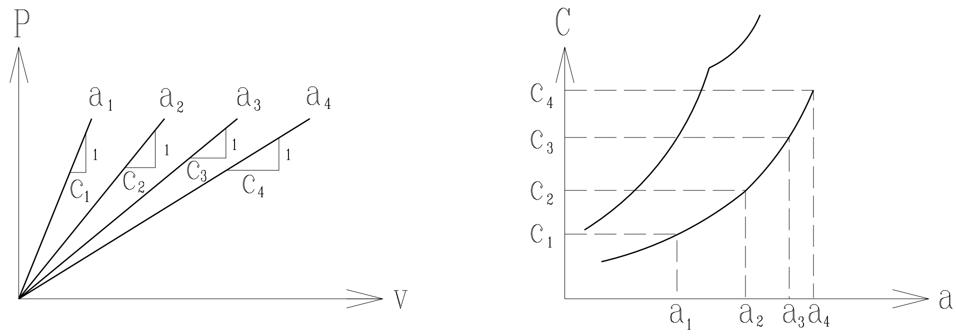


Figure 13.3: Experimental Determination of K_I from Compliance Curve

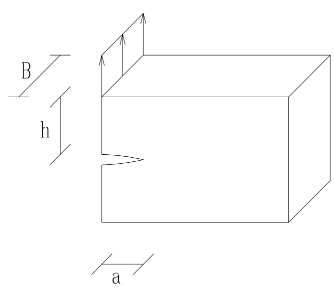


Figure 13.4: K_I for DCB using the Compliance Method

Taking $\nu = \frac{1}{3}$ we obtain

$$C = \frac{8}{EB}\int_0^a \left(\frac{3x^2}{h^3}+\frac{1}{h}\right)dx \tag{13.34}$$

$$\frac{dC}{da} = \frac{8}{EB}\left(\frac{3a^2}{h^3}+\frac{1}{h}\right) \tag{13.35}$$

Substituting in Eq. 13.32

$$G = \frac{1}{2}\frac{P^2}{B}\left(\frac{dc}{da}\right) \tag{13.36}$$

$$= \frac{1}{2}\frac{P^2 8}{EB^2}\left(\frac{3a^2}{h^3}+\frac{1}{h}\right) \tag{13.37}$$

$$= \frac{4P^2}{EB^2h^3}\left(3a^2+h^2\right) \tag{13.38}$$

Thus the stress intensity factor will be

$$K = \sqrt{GE} = \frac{2P}{B}\left(\frac{3a^2}{h^3}+\frac{1}{h}\right)^{\frac{1}{2}} \tag{13.39}$$

27 Had we kept G in terms of ν

$$G = \frac{4P^2}{EB^2h^3}\left[3a^2+\frac{3}{4}h^2(1+\nu)\right] \tag{13.40}$$

28 We observe that in this case K increases with a, hence we would have an unstable crack growth. Had we had a beam in which B increases with a, Fig. 13.5, such that

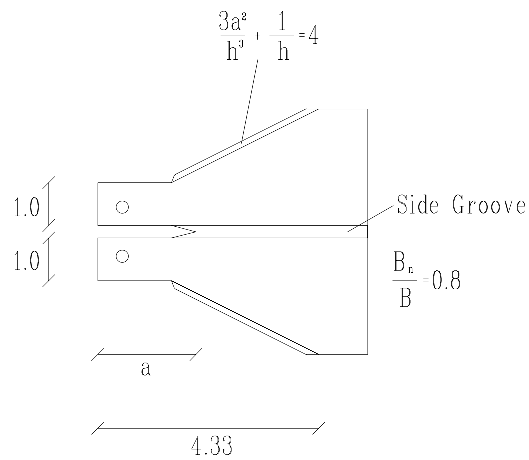


Figure 13.5: Variable Depth Double Cantilever Beam

$$\frac{3a^2}{h^3} + \frac{1}{h} = m = \text{Cst} \tag{13.41}$$

then

$$K = \frac{2P}{B} m^{\frac{1}{2}} \tag{13.42}$$

Such a specimen, in which K remains constant as a increases was proposed by Mostovoy (Mostovoy 1967) for fatigue testing.

13.3 Energy Release Rate; Equivalence with Stress Intensity Factor

29 We showed in the previous section that a transfer of energy has to occur for crack propagation. Energy is needed to create new surfaces, and this energy is provided by either release of strain energy only, or a combination of strain energy and external work. It remains to quantify energy in terms of the stress intensity factors.

30 In his original derivation, Eq. 13.25, Griffith used Inglis solution to determine the energy released. His original solution was erroneous, however he subsequently corrected it.

31 Our derivation instead will be based on Westergaard's solution. Thus, the energy released during a colinear unit crack extension can be determined by calculating the work done by the surface forces acting across the length da when the crack is closed from length $(a + da)$ to length a, Fig. 13.6.

32 This energy change is given by:

$$G = \frac{2}{\Delta a} \int_a^{a+\Delta a} \frac{1}{2} \sigma_{yy}(x) v(x - da) dx \tag{13.43}$$

33 We note that the 2 in the numerator is caused by the two crack surfaces (upper and lower), whereas the 2 in the denominator is due to the linear elastic assumption.

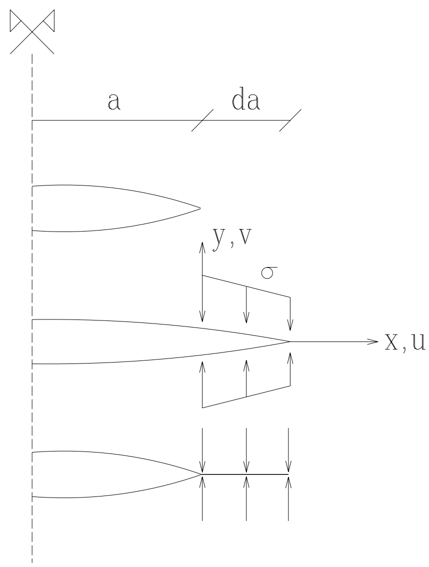


Figure 13.6: Graphical Representation of the Energy Release Rate G

34 Upon substitution for σ_{yy} and v (with $\theta = \pi$) from the Westergaard equations (Eq. 10.36-b and 10.36-f)

$$\sigma_{yy} = \frac{K_\mathrm{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \tag{13.44}$$

$$v = \frac{K_\mathrm{I}}{2\mu}\sqrt{\frac{r}{2\pi}} \sin\frac{\theta}{2}\left[\kappa + 1 - 2\cos^2\frac{\theta}{2}\right] \tag{13.45}$$

(where μ is the shear modulus); Setting $\theta = \pi$, and after and simplifying, this results in:

$$\boxed{G = \frac{K_\mathrm{I}^2}{E'}} \tag{13.46}$$

where

$$E' = E \qquad \text{plane stress} \tag{13.47}$$

and

$$E' = \frac{E}{1-\nu^2} \qquad \text{plane strain} \tag{13.48}$$

35 Substituting $K = \sigma\sqrt{\pi a}$ we obtain the energy release rate in terms of the far field stress

$$\boxed{G = \frac{\sigma^2 \pi a}{E'}} \tag{13.49}$$

we note that this is identical to Eq. 13.25 derived earlier by Griffith.

36 Finally, the total energy consumed over the crack extension will be:

$$d\Pi = \int_0^{da} G dx = \int_0^{da} \frac{\sigma^2 \pi a}{E'} dx = \frac{\sigma^2 \pi a da}{E'} \tag{13.50}$$

37 Sih, Paris and Irwin, (Sih, Paris and Irwin 1965), developed a counterpar to Equation 13.46 for anisotropic materials as

$$G = \sqrt{\left(\frac{a_{11}a_{22}}{2}\right)\left(\sqrt{\frac{a_{11}}{a_{22}}} + \frac{2a_{12}+a_{66}}{2a_{22}}\right)}K^2 \tag{13.51}$$

13.4 Crack Stability

38 Crack stability depends on both the geometry, and on the material resistance.

13.4.1 Effect of Geometry; Π Curve

39 From Eq. 13.6, crack growth is considered unstable when the energy at equilibrium is a maximum, and stable when it is a minimum. Hence, a sufficient condition for crack stability is, (Gdoutos 1993)

$$\boxed{\frac{\partial^2(\Pi+\Gamma)}{\partial A^2}\begin{cases} <0 & \text{unstable fracture} \\ >0 & \text{stable fracture} \\ =0 & \text{neutral equilibrium}\end{cases}} \tag{13.52}$$

and the potential energy is $\Pi = U - W$.

40 If we consider a line crack in an infinite plate subjected to uniform stress, Fig. 13.7, then the potential energy of the system is $\Pi = U^e$ where Eq. 13.6 yields

$$\begin{aligned} K_I &= \sigma\sqrt{\pi a} && \text{(13.53-a)} \\ G &= \frac{K_I^2}{E'} && \text{(13.53-b)} \\ &= \frac{\sigma^2\pi a}{E'} && \text{(13.53-c)} \\ U^e &= \int G da && \text{(13.53-d)} \\ &= -\frac{1}{2}\frac{\sigma^2\pi a^2}{E'} && \text{(13.53-e)} \end{aligned}$$

and $\Gamma = 4a$ (crack length is $2a$). Note that U^e is negative because there is a decrease in strain energy during crack propagation. If we plot Γ, Π and $\Gamma + \Pi$, Fig. 13.7, then we observe that the total potential energy of the system $(\Pi + \Gamma)$ is maximum at the critical crack length which corresponds to unstable equilibrium.

41 If we now consider the cleavage of mica, a wedge of thickness h is inserted under a flake of mica which is detached from a mica block along a length a. The energy of the system is determined by considering the mica flake as a cantilever beam with depth d. From beam theory

$$U^e = \frac{Ed^3h^2}{8a^3} \tag{13.54}$$

and the surface energy is $\Gamma = 2\gamma a$. From Eq. 13.52, the equilibrium crack is

$$a_c = \left(\frac{3Ed^3h^2}{16\gamma}\right)^{1/4} \tag{13.55}$$

Again, we observe from Fig. 13.7 that the total potential energy of the system at a_c is a minimum, which corresponds to stable equilibrium.

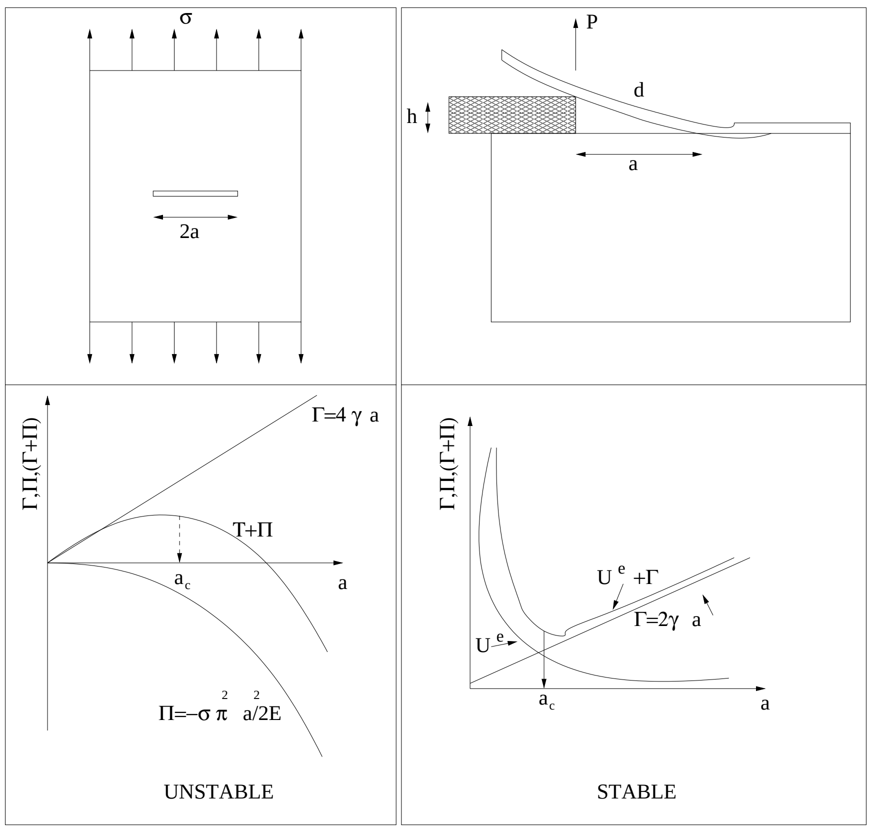


Figure 13.7: Effect of Geometry and Load on Crack Stability, (Gdoutos 1993)

13.4.2 Effect of Material; R Curve

42 As shown earlier, a crack in a linear elastic flawed structure may be characterized by its:

1. stress intensity factor determined from the near crack tip stress field
2. energy release rate determined from its global transfer of energy accompanying crack growth

43 Thus for a crack to extend, two criteria are possible:

1. Compare the stress intensity factor K with a material property called the critical stress intensity factor $K_{\rm Ic}$, or fracture toughness.
2. Compare the energy release rate G with a material property called the critical energy release rate G_{Ic}.

13.4.2.1 Theoretical Basis

44 Revisiting Eq. 13.3

$$\frac{\partial W}{\partial a} = \left(\frac{\partial U^e}{\partial a} + \frac{\partial U^p}{\partial a} \right) + \frac{\partial \Gamma}{\partial a} \tag{13.56}$$

we can rewrite it as

$$G = R \tag{13.57-a}$$

$$G = \frac{\partial W}{\partial a} - \frac{\partial U^e}{\partial a} \tag{13.57-b}$$

$$R = \frac{\partial U^p}{\partial a} + \frac{\partial \Gamma}{\partial a} \tag{13.57-c}$$

where R represents the rate of energy dissipation during stable crack growth. The first part corresponds to plastic deformation, and the second to energy consumed during crack propagation.

13.4.2.2 R vs K_{Ic}

45 Back to Eq. 13.50, crack instability will occur when for an infinitesimal crack extension da, the rate of energy released is just equal to surface energy absorbed.

$$\underbrace{\frac{\sigma_{cr}^2 \pi a da}{E'}}_{d\Pi} = 2\gamma da \tag{13.58}$$

or

$$\sigma_{cr} = \sqrt{\frac{2E'\gamma}{\pi a}} \tag{13.59}$$

Which is Eq. 13.27 as originally derived by Griffith (Griffith 1921).

46 This equation can be rewritten as

$$\underbrace{\frac{\sigma_{cr}^2 \pi a}{E'}}_{G_{cr}} \equiv \underbrace{R}_{2\gamma} \tag{13.60}$$

and as

$$\sigma_{cr}\sqrt{\pi a} = \sqrt{2E'\gamma} = K_{Ic} \tag{13.61}$$

thus

$$\boxed{G_{cr} = R = \frac{K_{Ic}^2}{E'}} \tag{13.62}$$

47 In general, the critical energy release rate is defined as R (for Resistance) and is only equal to a constant (G_{cr}) under plane strain conditions.

48 Critical energy release rate for plane stress is found not to be constant, thus K_{Ic} is not constant, and we will instead use K_{1c} and G_{1c}. Alternatively, K_{Ic}, and G_{Ic} correspond to plane strain in mode I which is constant. Hence, the shape of the R-curve depends on the plate thickness, where plane strain is approached for thick plates, and is constant; and for thin plates we do not have constant R due to plane stress conditions.

49 Using this energetic approach, we observe that contrarily to the Westergaard/Irwin criteria where we zoomed on the crack tip, a global energy change can predict a local event (crack growth).

50 The duality between energy and stress approach $G > G_{cr} = R$, or $K > K_{Ic}$, should also be noted.

51 Whereas the Westergaard/Irwin criteria can be generalized to mixed mode loading (in chapter 14), the energy release rate for mixed mode loading (where crack extension is not necessarily colinear with the crack axis) was not derived until 1974 by Hussain *et al.* (Hussain, Pu and Underwood 1974). However, should we assume a colinear crack extension under mixed mode loading, then

$$G = G_I + G_{II} + G_{III} = \frac{1-\nu^2}{E}(K_I^2 + K_{II}^2 + \frac{K_{III}^2}{1-\nu}) \tag{13.63}$$

52 From above, we have the energy release rate given by

$$G = \frac{\sigma^2 \pi a}{E'} \tag{13.64}$$

and the critical energy release rate is

$$R = G_{cr} = \frac{d\Pi}{da} = 2\gamma = \frac{K_{Ic}^2}{E'} \tag{13.65}$$

53 Criteria for crack growth can best be understood through a graphical representation of those curves under plane strain and plane stress conditions.

13.4.2.3 Plane Strain

54 For plane strain conditions, the R curve is constant and is equal to G_{Ic}. Using Fig. 13.8 From Eq. 13.64, $G = \frac{\sigma^2 \pi a}{E'}$, G is always a linear function of a, thus must be a straight line.

55 For plane strain problems, if the crack size is a_1, the energy release rate at a stress σ_2 is represented by point B. If we increase the stress from σ_2 to σ_1, we raise the G value from B to A. At A, the crack will extend. Had we had a longer crack a_2, it would have extended at σ_2.

56 Alternatively, we can plot to the right Δa, and to the left the original crack length a_i. At a stress σ_2, the G line is given by LF (really only point F). So by loading the crack from 0 to σ_2, G increases from O to F, further increase of the stress to σ_1 raises G from F to H, and then fracture occurs, and the crack goes fromH to K. On the other hand, had we had a crack of length a_2 loaded from 0 to σ_2, its G value increases from O to H (note that LF and MH are parallel). At H crack extension occurs along HN.

57 Finally, it should be noted that depending on the boundary conditions, G may increase linearly (constant load) or as a polynomila (fixed grips).

13.4.2.4 Plane Stress

58 Under plane strain R was independent of the crack length. However, under plane stress R is found to be an increasing function of a, Fig. 13.9

59 If we examine an initial crack of length a_i:

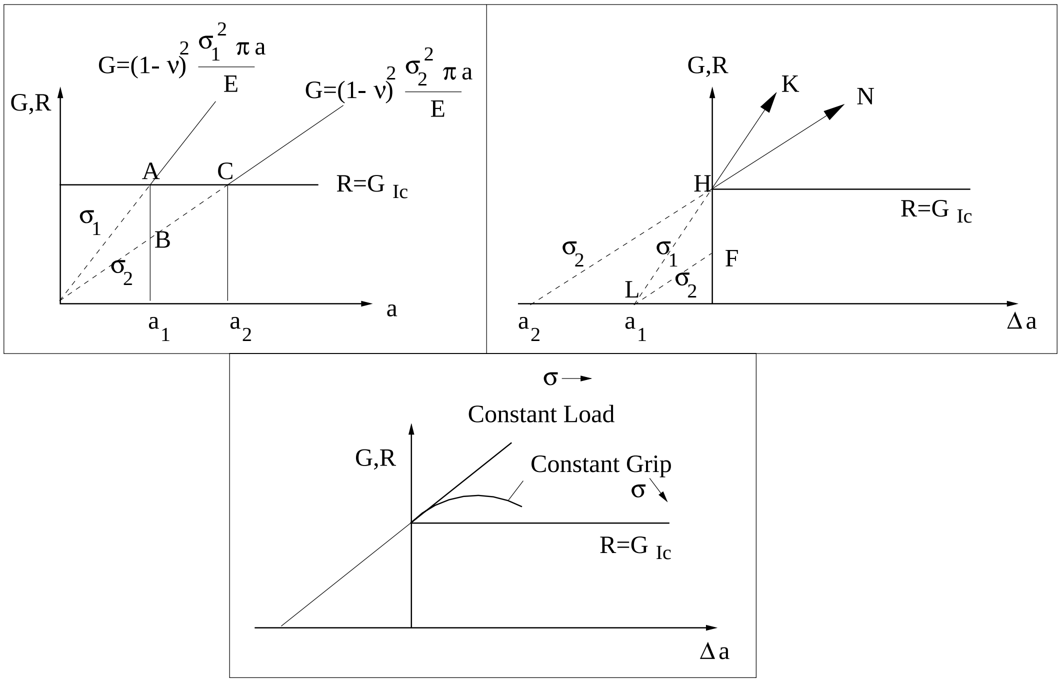


Figure 13.8: R Curve for Plane Strain

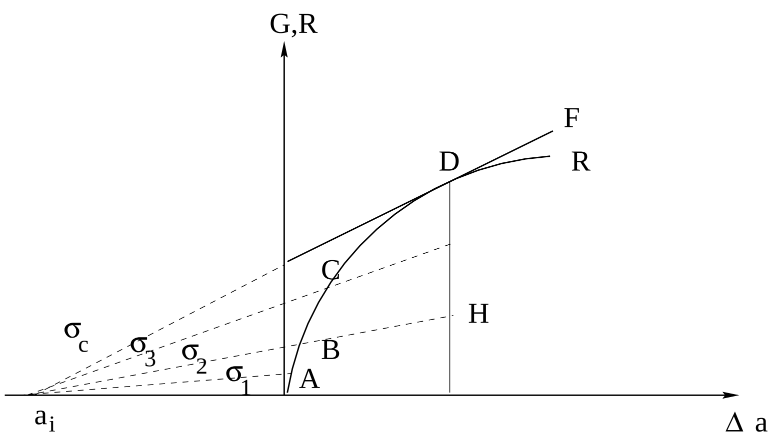


Figure 13.9: R Curve for Plane Stress

1. under σ_1 at point A, $G < R$, thus there is no crack extension.

2. If we increase σ_1 to σ_2, point B, then $G = R$ and the crack propagates by a small increment Δa and will immediately stop as G becomes smaller than R.

3. if we now increase σ_1 to σ_3, (point C) then $G > R$ and the crack extends to $a + \Delta a$. G increases to H, however, this increase is at a lower rate than the increase in R

$$\frac{dG}{da} < \frac{dR}{da} \tag{13.66}$$

thus the crack will stabilize and we would have had a stable crack growth.

4. Finally, if we increase σ_1 to σ_c, then not only is G equal to R, but it grows faster than R thus we would have an unstable crack growth.

60 From this simple illustrative example we conclude that

$$\begin{array}{|lll|}\hline \text{Stable Crack Growth:} & G > R & \frac{dG}{da} < \frac{dR}{da} \\ \text{Unstable Crack Growth:} & G > R & \frac{dG}{da} > \frac{dR}{da} \\ \hline\end{array} \tag{13.67}$$

we also observe that for unstable crack growth, excess energy is transformed into kinetic energy.

61 Finally, we note that these equations are equivalent to Eq. 13.52 where the potential energy has been expressed in terms of G, and the surface energy expressed in terms of R.

62 Some materials exhibit a flat R curve, while other have an ascending one. The shape of the R curve is a material property. For ideaally brittle material, R is flat since the surface energy γ is constant. Nonlinear material would have a small plastic zone at the tip of the crack. The driving force in this case must increase. If the plastic zone is small compared to the crack (as would be eventually the case for sufficiently long crack in a large body), then R would approach a constant value.

63 The thickness of the cracked body can also play an important role. For thin sheets, the load is predominantly plane stress, Fig. 13.10.

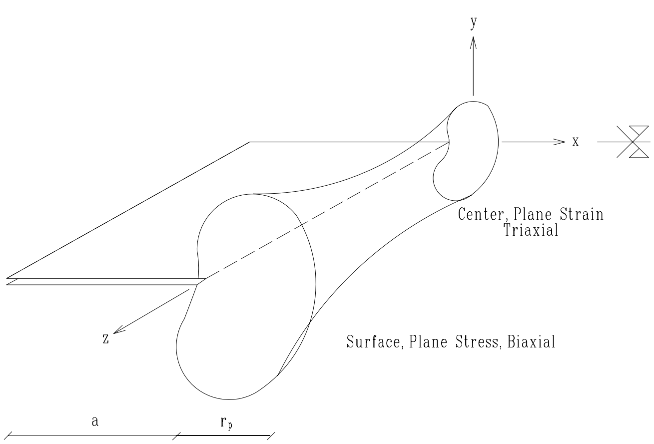


Figure 13.10: Plastic Zone Ahead of a Crack Tip Through the Thickness

64 Alternatively, for a thick plate it would be predominantly plane strain. Hence a plane stress configuration would have a steeper R curve.

Chapter 14

MIXED MODE CRACK PROPAGATION

1 Practical engineering cracked structures are subjected to mixed mode loading, thus in general K_I and K_{II} are both nonzero, yet we usually measure only mode I fracture toughness K_{Ic} (K_{IIc} concept is seldom used). Thus, so far the only fracture propagation criterion we have is for mode I only (K_I vs K_{Ic}, and G_I vs R).

2 Whereas under pure mode I in homogeneous isotropic material, crack propagation is colinear, in all other cases the propagation will be curvilinear and at an angle θ_0 with respect to the crack axis. Thus, for the general mixed mode case, we seek to formultate a criterion that will determine:

1. The angle of incipient propagation, θ_0, with respect to the crack axis.
2. If the stress intensity factors are in such a critical combination as to render the crack locally unstable and force it to propagate.

3 Once again, for pure mode I problems, fracture initiation occurs if:

$$K_{\rm I} \geq K_{\rm Ic} \tag{14.1}$$

4 The determination of a fracture initiation criterion for an existing crack in mode I and II would require a relationship between $K_{\rm I}, K_{\rm II}$, and $K_{\rm Ic}$ of the form

$$F\left(K_{\rm I}, K_{\rm II}, K_{\rm Ic}\right) = 0 \tag{14.2}$$

and would be analogous to the one between the two principal stresses and a yield stress, Fig. 14.1

$$F_{yld}(\sigma_1, \sigma_2, \sigma_y) = 0 \tag{14.3}$$

Such an equation may be the familiar Von-Mises criterion.

14.1 Maximum Circumferential Tensile Stress.

5 Erdogan and Sih (Erdogan, F. and Sih, G.C. 1963) presented the first mixed-mode fracture initiation theory, the maximum circumferential tensile stress theory. It is based on the knowledge of the stress state near the tip of a crack, written in polar coordinates.

6 The maximum circumferential stress theory states that the crack extension starts:

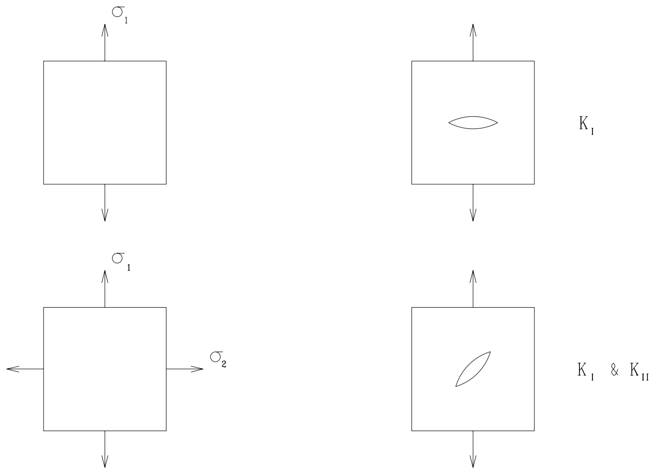


Figure 14.1: Mixed Mode Crack Propagation and Biaxial Failure Modes

1. at its tip in a radial direction
2. in the plane perpendicular to the direction of greatest tension, i.e at an angle θ_0 such that $\tau_{r\theta} = 0$
3. when $\sigma_{\theta max}$ reaches a critical material constant

7 It can be easily shown that σ_θ reaches its maximum value when $\tau_{r\theta} = 0$. Replacing $\tau_{r\theta}$ for mode I and II by their expressions given by Eq. 10.39-c and 10.40-c

$$\tau_{r\theta} = \frac{K_{\rm I}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{K_{\rm II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) \tag{14.4}$$

$$\Rightarrow \cos\frac{\theta_0}{2}\left[K_{\rm I}\sin\theta_0 + K_{\rm II}\left(3\cos\theta_0 - 1\right)\right] = 0 \tag{14.5}$$

this equation has two solutions:

$$\theta_0 = \pm\pi \qquad \text{trivial} \tag{14.6}$$

$$K_{\rm I}\sin\theta_0 + K_{\rm II}\left(3\cos\theta_0 - 1\right) = 0 \tag{14.7}$$

Solution of the second equation yields the angle of crack extension θ_0

$$\boxed{\tan\frac{\theta_0}{2} = \frac{1}{4}\frac{K_{\rm I}}{K_{\rm II}} \pm \frac{1}{4}\sqrt{\left(\frac{K_{\rm I}}{K_{\rm II}}\right)^2 + 8}} \tag{14.8}$$

8 For the crack to extend, the maximum circumferential tensile stress, σ_θ (from Eq. 10.39-b and 10.40-b)

$$\sigma_\theta = \frac{K_{\rm I}}{\sqrt{2\pi r}}\cos\frac{\theta_0}{2}\left(1 - \sin^2\frac{\theta_0}{2}\right) + \frac{K_{\rm II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta_0}{2} - \frac{3}{4}\sin\frac{3\theta_0}{2}\right) \tag{14.9}$$

must reach a critical value which is obtained by rearranging the previous equation

$$\sigma_{\theta max}\sqrt{2\pi r} = K_{\rm Ic} = \cos\frac{\theta_0}{2}\left[K_{\rm I}\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{\rm II}\sin\theta_0\right] \tag{14.10}$$

which can be normalized as

$$\boxed{\frac{K_{\rm I}}{K_{\rm Ic}}\cos^3\frac{\theta_0}{2} - \frac{3}{2}\frac{K_{\rm II}}{K_{\rm Ic}}\cos\frac{\theta_0}{2}\sin\theta_0 = 1} \tag{14.11}$$

9 This equation can be used to define an equivalent stress intensity factor K_{eq} for mixed mode problems

$$\boxed{K_{eq} = K_{\rm I} \cos^3 \frac{\theta_0}{2} - \frac{3}{2} K_{\rm II} \cos \frac{\theta_0}{2} \sin \theta_0} \tag{14.12}$$

14.1.1 Observations

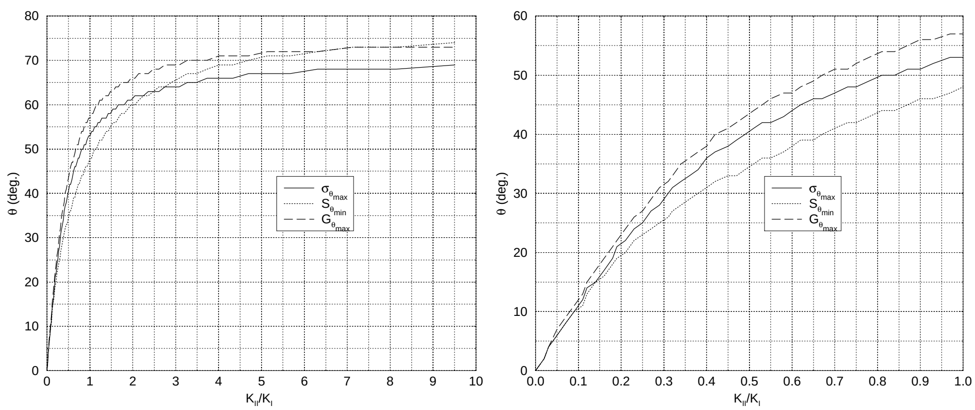


Figure 14.2: Angle of Crack Propagation Under Mixed Mode Loading

10 With reference to Fig. 14.2 and 14.3, we note the following

1. Algorithmically, the angle of crack propagation θ_0 is first obtained, and then the criteria are assessed for local fracture stability.
2. In applying $\sigma_{\theta_{max}}$, we need to define another material characteristic r_0 where to evaluate σ_θ. Whereas this may raise some fundamental questions with regard to the model, results are independent of the choice for r_0.
3. $S_{\theta_{min}}$ theory depends on ν
4. $S_{\theta_{min}}$ & $\sigma_{\theta_{max}}$ depend both on a field variable that is singular at the crack tip thus we must arbitrarily specify r_o (which cancels out).
5. It can be argued whether all materials must propagate in directions of maximum energy release rate.
6. There is a scale effect in determining the tensile strength $\Rightarrow \sigma_{\theta_{max}}$
7. Near the crack tip we have a near state of biaxial stress
8. For each model we can obtain a $K_{I\rm eq}$ in terms of K_I & K_{II} and compare it with K_{Ic}
9. All models can be represented by a normalized fracture locus.
10. For all practical purposes, all three theories give identical results for small ratios of $\frac{K_{II}}{K_I}$ and diverge slightly as this ratio increases.
11. A crack will always extend in the direction which minimizes $\frac{K_{II}}{K_I}$. That is, a crack under mixed-mode loading will tend to reorient itself so that $K_{\rm II}$ is minimized. Thus during its trajectory a crack will most often be in that portion of the normalized $\frac{K_{\rm I}}{K_{\rm Ic}} - \frac{K_{\rm II}}{K_{\rm Ic}}$ space where the three theories are in close agreement.

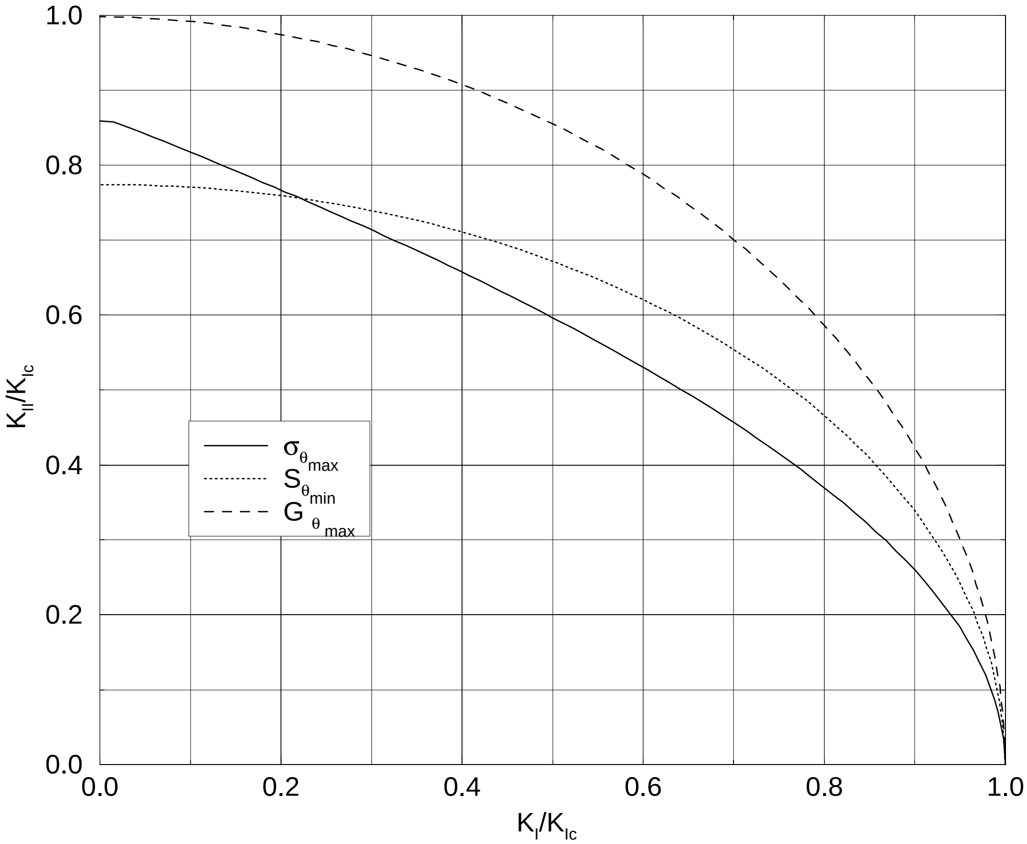


Figure 14.3: Locus of Fracture Diagram Under Mixed Mode Loading

12. If the pair of SIF is inside the fracture loci, then that crack cannot propagate without sufficient increase in stress intensity factors. If outside, then the crack is locally unstable and will continue to propagate in either of the following ways:
    (a) With an increase in the SIF (and the energy release rate G), thus resulting in a global instability, failure of the structure (crack reaching a free surface) will occur.
    (b) With a decrease in the SIF (and the energy release rate G), due to a stress redistribution, the SIF pair will return to within the locus.

Chapter 15

FATIGUE CRACK PROPAGATION

1 When a subcritical crack (a crack whose stress intensity factor is below the critical value) is subjected to either repeated or fatigue load, or is subjected to a corrosive environment, crack propagation will occur.

2 As in many structures one has to assume the presence of minute flaws (as large as the smallest one which can be detected). The application of repeated loading will cause crack growth. The loading is usually caused by vibrations.

3 Thus an important question that arises is "how long would it be before this subcritical crack grows to reach a critical size that would trigger failure?" To predict the minimum fatigue life of metallic structures, and to establish safe inspection intervals, an understanding of the rate of fatigue crack propagation is required.

Historically, fatigue life prediction was based on $S-N$ curves, Fig. 15.1 (or Goodman's Diagram)

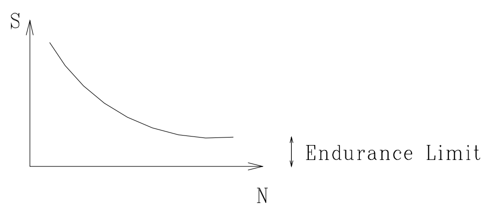


Figure 15.1: S-N Curve and Endurance Limit

using a Strength of Material Approach which did NOT assume the presence of a crack.

15.1 Experimental Observation

4 If we start with a plate that has no crack and subject it to a series of repeated loading, Fig. 15.2 between σ_{min} and σ_{max}, we would observe three distinct stages, Fig. 15.3

1. Stage 1 : Micro coalescence of voids and formation of microcracks. This stage is difficult to capture and is most appropriately investigated by metallurgists or material scientists, and compared to stage II and III it is by far the longest.

2. Stage II : Now a micro crack of finite size was formed, its SIF'well belowK_{Ic}, ($K << K_{Ic}$), and crack growth occurs after each cycle of loading.

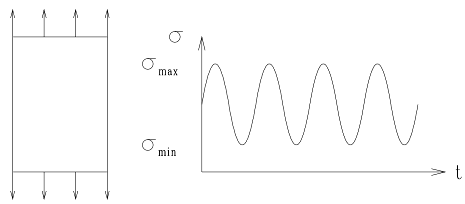


Figure 15.2: Repeated Load on a Plate

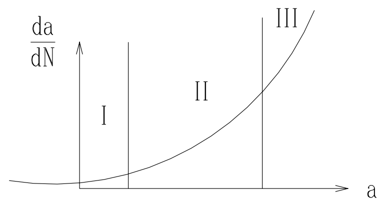


Figure 15.3: Stages of Fatigue Crack Growth

3. Stage III : Crack has reached a size a such that $a = a_c$, thus rapid unstable crack growth occurs.

5 Thus we shall primarily be concerned by stage II.

15.2 Fatigue Laws Under Constant Amplitude Loading

6 On the basis of the above it is evident that we shall be concerned with stage II only. Furthermore, fatigue crack growth can take place under:

1. Constant amplitude loading (good for testing)
2. Variable amplitude loading (in practice)

7 Empirical mathematical relationships which require the knowledge of the stress intensity factors (SIF), have been established to describe the crack growth rate. Models of increasing complexity have been proposed.

8 All of these relationships indicate that the number of cycles N required to extend a crack by a given length is proportional to the effective stress intensity factor range ΔK raised to a power n (typically varying between 2 and 9).

15.2.1 Paris Model

9 The first fracture mechanics-based model for fatigue crack growth was presented by Paris (Paris and Erdogan 1963) in the early '60s. It is important to recognize that it is an empirical law based on experimental observations. Most other empirical fatigue laws can be considered as direct extensions, or refinements of this one, given by

$$\boxed{\frac{da}{dN} = C\left(\Delta K\right)^n} \tag{15.1}$$

which is a straight line on a log-log plot of $\frac{da}{dN}$ vs ΔK, and

$$\Delta K = K_{max} - K_{min} = (\sigma_{max} - \sigma_{min}) f(g)\sqrt{\pi a} \tag{15.2}$$

a is the crack ength; N the number of load cycles; C the intercept of line along $\frac{da}{dN}$ and is of the order of 10^{-6} and has units of length/cycle; and n is the slope of the line and ranges from 2 to 10.

10 Equation 15.1 can be rewritten as :

$$\Delta N = \frac{\Delta a}{C\left[\Delta K(a)\right]^n} \tag{15.3}$$

or

$$\boxed{N = \int dN = \int_{a_i}^{a_f} \frac{da}{C\left[\Delta K(a)\right]^n}} \tag{15.4}$$

11 Thus it is apparent that a small error in the SIF calculations would be magnified greatly as n ranges from 2 to 6. Because of the sensitivity of N upon ΔK, it is essential to properly determine the numerical values of the stress intensity factors.

12 However, in most practical cases, the crack shape, boundary conditions, and load are in such a combination that an analytical solution for the SIF does not exist and large approximation errors have to be accepted. Unfortunately, analytical expressions for K are available for only few simple cases. Thus the stress analyst has to use handbook formulas for them (Tada et al. 1973). A remedy to this problem is the usage of numerical methods, of which the finite element method has achieved greatest success.

15.2.2 Foreman's Model

13 When compared with experimental data, it is evident that Paris law does not account for:

1. Increase in crack growth rate as K_{max} approaches K_{Ic}
2. Slow increase in crack growth at $K_{min} \approx K_{th}$

thus it was modified by Foreman (Foreman, Kearney and Engle 1967), Fig. 15.4

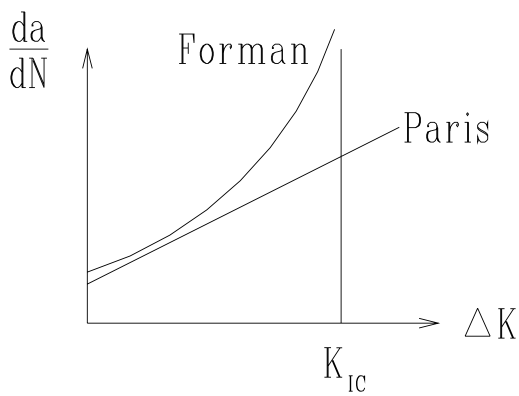


Figure 15.4: Forman's Fatigue Model

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1-R)K_c - \Delta K} \tag{15.5}$$

15.2.2.1 Modified Walker's Model

14 Walker's (Walker 1970) model is yet another variation of Paris Law which accounts for the stress ratio $R = \frac{K_{min}}{K_{max}} = \frac{\sigma_{min}}{\sigma_{max}}$

$$\frac{da}{dN} = C\left[\frac{\Delta K}{(1-R)^{(1-m)}}\right]^n \tag{15.6}$$

15.2.3 Table Look-Up

15 Whereas most methods attempt to obtain numerical coefficients for empirical models which best approximate experimental data, the table look-up method extracts directly from the experimental data base the appropriate coefficients. In a "round-robin" contest on fatigue life predictions, this model was found to be most satisfactory (Miller and Gallagher 1981).

16 This method is based on the availability of the information in the following table:

$\frac{da}{dN}$	ΔK			
	R =-1	R = .1	R = .3	R = .4

17 For a given $\frac{da}{dN}$ and R, ΔK is directly read (or rather interpolated) for available data.

15.2.4 Effective Stress Intensity Factor Range

18 All the empirical fatigue laws are written in terms of $\Delta K_{\rm I}$; however, in general a crack will be subjected to a mixed-mode loading resulting in both $\Delta K_{\rm I}$ and $\Delta K_{\rm II}$. Thus to properly use a fatigue law, an effective stress intensity factor is sought.

19 One approach, consists in determining an effective stress intensity factor ΔK_{eff} in terms of $\Delta K_{\rm I}$ and $\Delta K_{\rm II}$, and the angle of crack growth θ_0. In principle each of the above discussed mixed-mode theories could yield a separate expression for the effective stress intensity factor.

20 For the case of maximum circumferential stress theory, an effective stress intensity factor is given by (Broek 1986):

$$\Delta K_{\rm Ieff} = \Delta K_{\rm I}\cos^3\frac{\theta_0}{2} - 3\Delta K_{\rm II}\cos\frac{\theta_0}{2}\sin\theta_0 \tag{15.7}$$

15.2.5 Examples

15.2.5.1 Example 1

An aircraft flight produces 10 gusts per flight (between take-off and landing). It has two flights per day. Each gust has a $\sigma_{max} = 200$ MPa and $\sigma_{min} = 50$ MPa. The aircraft is made up of aluminum which has $R = 15\frac{kJ}{m^2}$, $E = 70GPa$, $C = 5\times 10^{-11}\frac{m}{cycle}$, and $n = 3$. The smallest detectable flaw is 4mm. How long would it be before the crack will propagate to its critical length?

Assuming $K = \sigma\sqrt{\pi a}$ and $K_c = \sqrt{ER}$, then $a_c = \frac{K_c^2}{\sigma_{max}^2\pi} = \frac{ER}{\sigma_{max}^2\pi}$ or

$$a_c = \frac{(70\times 10^9)(15\times 10^3)}{(200\times 10^6)^2\pi} = 0.0084m = 8.4mm \tag{15.8}$$

$$\begin{aligned}
\Rightarrow N &= \int_{a_i}^{a_f} \frac{da}{C[\Delta K(a)]^n} = \int_{a_i}^{a_f} \frac{da}{C\underbrace{(\sigma_{max}-\sigma_{min})^n}_{(\Delta\sigma)^n}((\pi a)^{\frac{1}{2}})^n} \\
&= \int_{4\times 10^{-3}}^{8.4\times 10^{-3}} \frac{da}{\underbrace{(5\times 10^{-11})}_{C}\underbrace{(200-50)^3}_{(\Delta\sigma)^3}\underbrace{(\pi a)^{1.5}}_{((\pi a)^{.5})^3}} = 1064\int_{.004}^{.0084} a^{-1.5}da \qquad (15.9) \\
&= -2128a^{-.5}\,|_{.004}^{.0084} = 2128[-\frac{1}{\sqrt{.0084}}+\frac{1}{\sqrt{.004}}] \\
&= 10,428 \text{ cycles}
\end{aligned}$$

thus the time t will be: t = (10,428) cycles $\times\frac{1}{10}\ \frac{\text{flight}}{\text{cycle}} \times\frac{1}{2}\ \frac{\text{day}}{\text{flight}} \times\frac{1}{30}\ \frac{\text{month}}{\text{day}} \approx$ 17.38 month ≈ 1.5 years.

If a longer lifetime is desired, then we can:

1. Employ a different material with higher K_{Ic}, so as to increase the critical crack length a_c at instability.
2. Reduce the maximum value of the stress σ_{max}.
3. Reduce the stress range $\Delta\sigma$.
4. Improve the inspection so as to reduce tha ssumed initial crack length a_{min}.

15.2.5.2 Example 2

21 Repeat the previous problem except that more sophisticated (and expensive) NDT equipment is available with a resolution of .1 mm thus $a_i = .1$mm

$t = 2128[-\frac{1}{\sqrt{.0084}}+\frac{1}{\sqrt{.0001}}] = 184,583$cycles

$t = \frac{1738}{10,428}(189,583) = 316$ months ≈ 26 years!

15.2.5.3 Example 3

Rolfe and Barsoum p.261-263.

15.3 Variable Amplitude Loading

15.3.1 No Load Interaction

22 Most Engineering structures are subjected to variable amplitude repeated loading, however, most experimental data is based on constant amplitude load test. Thus, the following questions arise:

1. How do we put the two together?
2. Is there an interaction between high and low amplitude loading?

1. Root Mean Square Model (Barsoum)

$$\frac{da}{dN} = C(\Delta K_{rms})^n \qquad (15.10)$$

$$\Delta K_{rms} = \sqrt{\frac{\sum_{i=1}^{k}\Delta K_i^2}{n}} \qquad (15.11)$$

where ΔK_{rms} is the square root of the mean of the squares of the individual stress intensity factors cycles in a spectrum.

2. Accurate "block by block" numerical integration of the fatigue law

$$\Delta a = C(\Delta K)^n \Delta N \qquad (15.12)$$

solve for a instead of N.

15.3.2 Load Interaction

15.3.2.1 Observation

23 Under aircraft flight simulation involving random load spectrum:

- High wind related gust load, N_H
- Without high wind related gust load, N_L

$N_H > N_L$, thus "Aircraft that logged some bad weather flight time could be expected to possess a longer service life than a plane having a better flight weather history."

24 Is this correct? Why? Under which condition overload is damaging!

15.3.2.2 Retardation Models

25 Baseline fatigue data are derived under constant amplitude loading conditions, but many structural components are subjected to variable amplitude loading. If interaction effects of high and low loads did not exist in the sequence, it would be relatively easy to establish a crack growth curve by means of a cycle-by-cycle integration. However crack growth under variable amplitude cycling is largely complicated by interaction effects of high and low loads.

26 A high level load occurring in a sequence of low amplitude cycles significantly reduces the rate of crack growth during the cycles applied subsequent to the overload. This phenomena is called Retardation, Fig. 15.5.

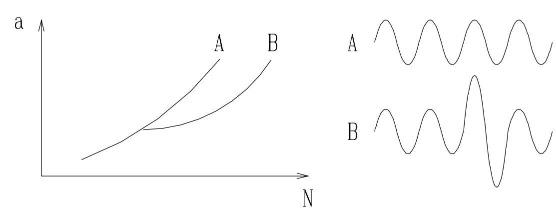


Figure 15.5: Retardation Effects on Fatigue Life

27 During loading, the material at the crack tip is plastically deformed and a tensile plastic zone is formed. Upon load release, the surrounding material is elastically unloaded and a part of the plastic zone experiences compressive stresses.

28 The larger the load, the larger the zone of compressive stresses. If the load is repeated in a constant amplitude sense, there is no observable direct effect of the residual stresses on the crack growth behavior; in essence, the process of growth is steady state.

29 Measurements have indicated, however, that the plastic deformations occurring at the crack tip remain as the crack propagates so that the crack surfaces open and close at non-zero (positive) levels.

30 When the load history contains a mix of constant amplitude loads and discretely applied higher level loads, the patterns of residual stress and plastic deformation are perturbed. As the crack propagates through this perturbed zone under the constant amplitude loading cycles, it grows slower (the crack is retarded) than it would have if the perturbation had not occurred. After the crack has propagated through the perturbed zone, the crack growth rate returns to its typical steady-state level, Fig. 15.6.

15.3.2.2.1 Wheeler's Model

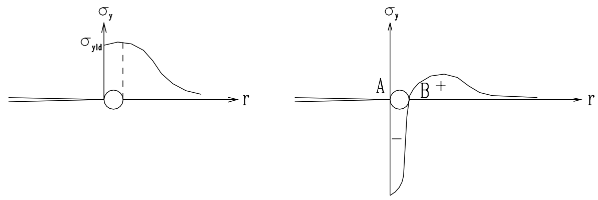


Figure 15.6: Cause of Retardation in Fatigue Crack Grwoth

31 Wheeler (Wheeler 1972) defined a crack-growth retardation factor C_p:

$$\frac{da}{dN}_{\text{retarded}} = C_p \left(\frac{da}{dN}\right)_{\text{linear}} \tag{15.13}$$

$$C_p = \left(\frac{r_{pi}}{a_{oL} + r_{poL} - a_i}\right)^m \tag{15.14}$$

in which r_{pi} is the current plastic zone size in the i^{th} cycle under consideration, a_i is the current crack size, r_{poL} is the plastic size generated by a previous higher load excursion, a_{oL} is the crack size at which the higher load excursion occurred, and m is an empirical constant, Fig. 15.7.

32 Thus there is retardation as long as the current plastic zone is contained within the previously generated one.

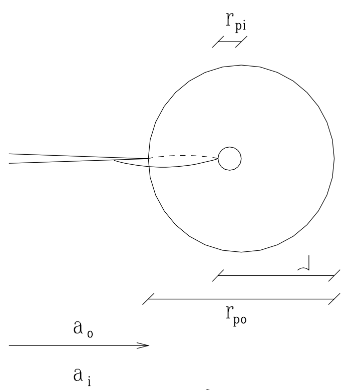


Figure 15.7: Yield Zone Due to Overload

15.3.2.2.2 Generalized Willenborg's Model

33 In the generalized Willenborg model (Willenborg, Engle and Wood 1971), the stress intensity factor $K_{\rm I}$ is replaced by an effective one:

$$K_{\rm I}^{\rm eff} = K_{\rm I} - K_{\rm R} \tag{15.15}$$

in which $K_{\rm R}$ is equal to:

$$K_R = \phi K_{\rm R}^{\rm w} \tag{15.16}$$

$$\phi = \frac{1 - \frac{K_{\rm max,th}}{K_{\rm max,i}}}{s^{oL} - 1} \tag{15.17}$$

$$K_{\mathrm{R}} = K_{\mathrm{R}}^{\mathrm{w}} = K_{\max}^{oL}\sqrt{1 - \frac{a_i - a_{oL}}{r_{poL}}} - K_{\mathrm{max,i}} \tag{15.18}$$

and a_i is the current crack size, a_{oL} is the crack size at the occurrence of the overload, r_{poL} is the yield zone produced by the overload, K_{max}^{oL} is the maximum stress intensity of the overload, and $K_{max,i}$ is the maximum stress intensity for the current cycle.

34 This equation shows that retardation will occur until the crack has generated a plastic zone size that reaches the boundary of the overload yield zone. At that time, $a_i - a_{oL} = r_{poL}$ and the reduction becomes zero.

35 Equation 15.15 indicates that the complete stress-intensity factor cycle, and therefore its maximum and minimum levels ($K_{max,i}$ and $K_{min,i}$), are reduced by the same amount (K_R). Thus, the retardation effect is sensed by the change in the effective stress ratio calculated from:

$$R_{\mathrm{eff}} = \frac{K_{\mathrm{min,i}}^{\mathrm{eff}}}{K_{\mathrm{max,i}}^{\mathrm{eff}}} = \frac{K_{\mathrm{min,i}} - K_{\mathrm{R}}}{K_{\mathrm{max,i}} - K_{\mathrm{R}}} \tag{15.19}$$

because the range in stress intensity factor is unchanged by the uniform reduction.

36 Thus, for the i^{th} load cycle, the crack growth increment Δa_i is:

$$\Delta a_i = \frac{da}{dN} = f(\Delta K, R_{\mathrm{eff}}) \tag{15.20}$$

37 In this model there are two empirical constants: $K_{max,th}$, which is the threshold stress intensity factor level associated with zero fatigue crack growth rate, and S^{oL}, which is the overload (shut-off) ratio required to cause crack arrest for the given material.

Part IV

PLASTICITY

Chapter 16

PLASTICITY; Introduction

16.1 Laboratory Observations

1 The typical stress-strain behavior of most metals in simple tension is shown in Fig. 16.1

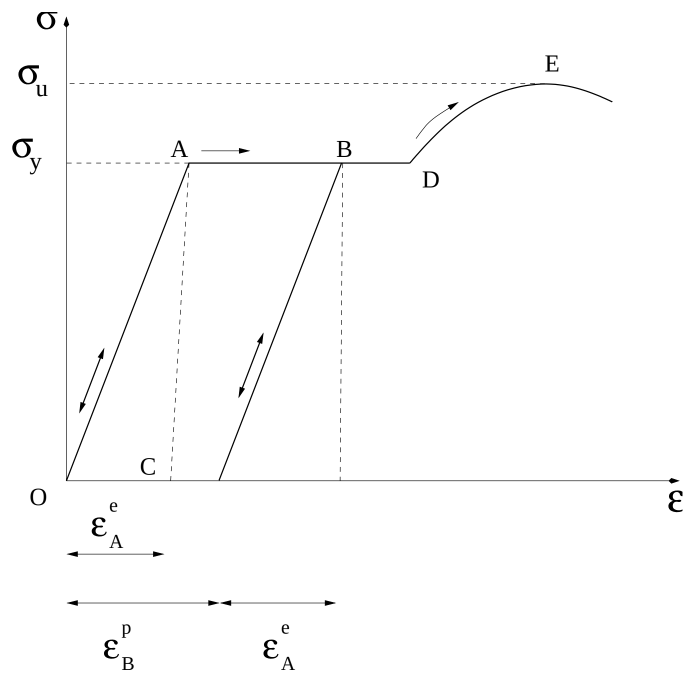


Figure 16.1: Typical Stress-Strain Curve of an Elastoplastic Bar

2 Up to A, the response is **linearly elastic**, and unloading follows the initial loading path. $O-A$ represents the **elastic range** where the behavior is load path independent.

3 At point A, the material has reached its **elastic limit**, from A onward the material becomes plastic and behaves **irreversibly**. First the material is yielding $(A-D)$ and then it hardens.

4 Unloading from any point after A results in a proportionally decreasing stress and strain parallel to the initial elastic loading $O-A$. A complete unloading would leave a **permanent strain** or a **plastic strain** ε_1^p. Thus only part of the total strain ε_B at B is recovered upon unloading, that is the *elastic strain* ε_1^e.

5 In case of reversed loading, Fig. 16.2, when the material is loaded in compression after it has been loaded in tension, the stress-strain curve will be different from the one obtained from pure tension or compression. The new yield point in compression at B corresponds to stress σ_B which is smaller than

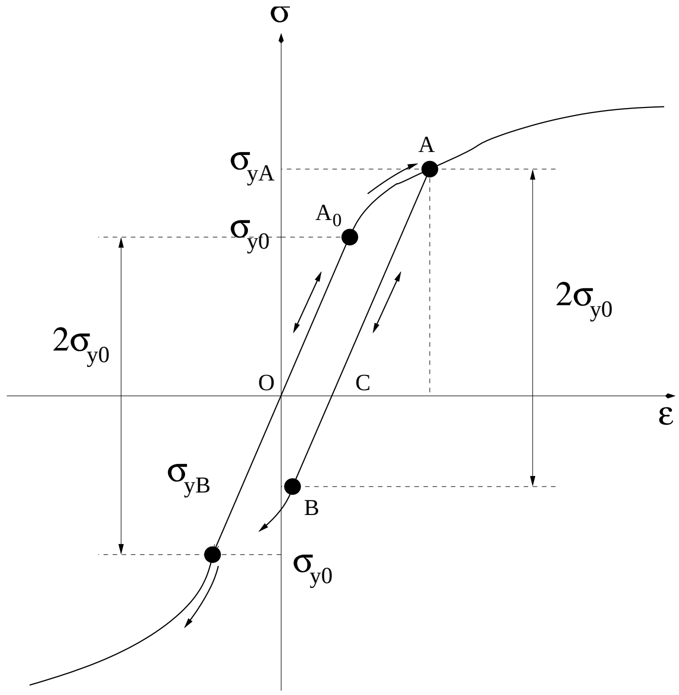


Figure 16.2: Bauschinger Effect on Reversed Loading

σ_{y0} and is much smaller than the previous yield stress at A. This phenomena is called **Bauschinger effect**.

6 It is thus apparent that the stress-strain behavior in the plastic range is **path dependent**. In general the strain will not depend on the the current stress state, but also on the entire loading history, i.e. **stress history** and **deformation history**.

7 Plasticity will then play a major role in the required level of analysis sophistication, Fig. 16.3.

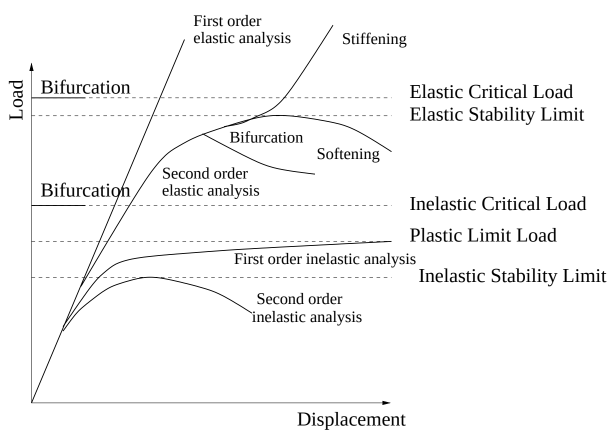


Figure 16.3: Levels of Analysis

8 In this chapter, and the subsequent ones, we seek to examine the reasons for the material nonlinearity.

16.2 Physical Plasticity

16.2.1 Chemical Bonds

9 To properly understand plasticity, it is important to revisit some basic notions of chemistry, more specifically chemical bonds. There are three type of primary bonds, Fig. 16.4:

Ionic Bond: electrons are transferred from one atom to a neighbouring one. The atom giving up the electron, becomes positively charged and the atom receiving it becomes negatively charged. The opposite ionic charges thus created bind the atoms together (NaCl, ceramics).

Covalent Bond: electrons are shared more or less equally between neighbouring atoms. Although the electrostatic force of attraction between adjacent atoms is less than it is in ionic bonds, covalent bonds tend to be highly directional, meaning that they resist motion of atoms past one another. Diamond has covalent bonds.

Metalic Bond: electrons are distributed equally through a metallic crystal, bond is not localized between two atoms. Best described as positive ions in a sea of electrons.

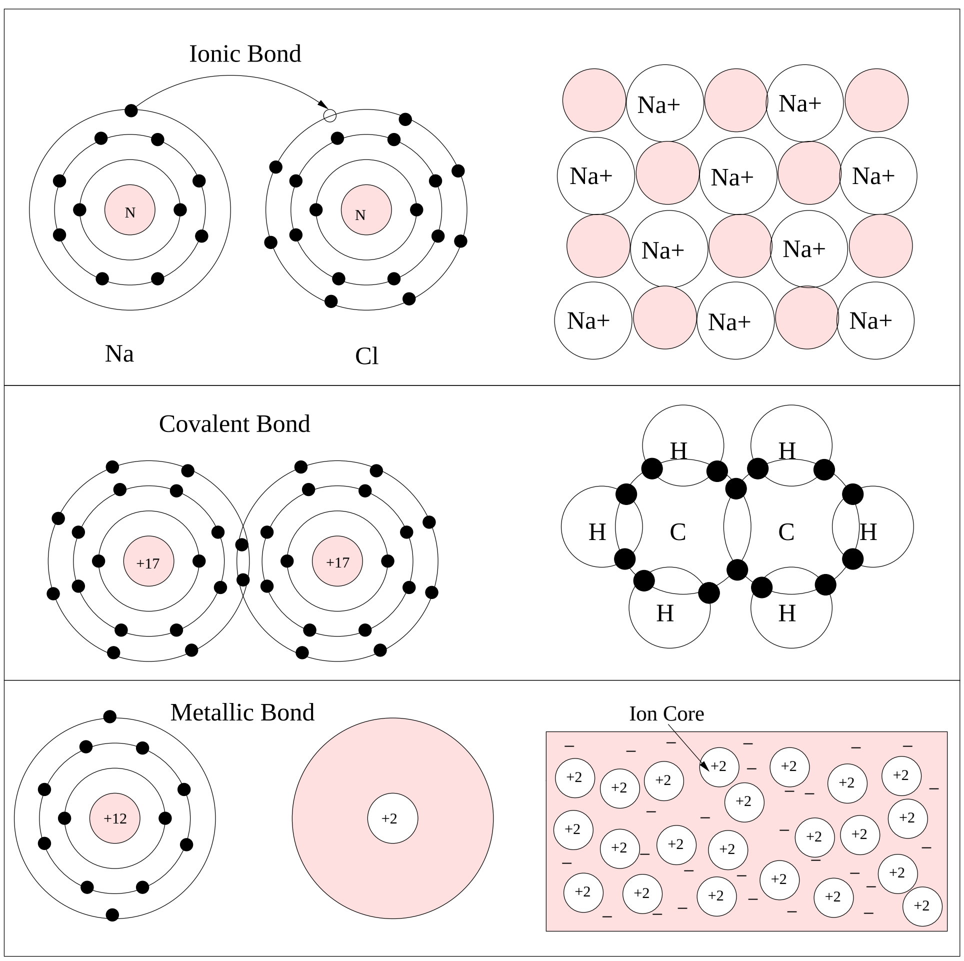


Figure 16.4: Metallic, Covalent, and Ionic Bonds

10 Hence, when a shear stress is applied on a metal bond, the atoms can slip and slide past one another without regard to electrical charge constraint, and thus it gives rise to a **ductile** response. On the other hand in a ioninc solid, each ion is surrounded by oppositely charged ions, thus the ionic slip may lead to like charges moving into adjacent positions causing coulombic repulsion. This makes slipping much more difficult to achieve, and the material respond by breaking in a **brittle** behavior, Fig. 16.5.

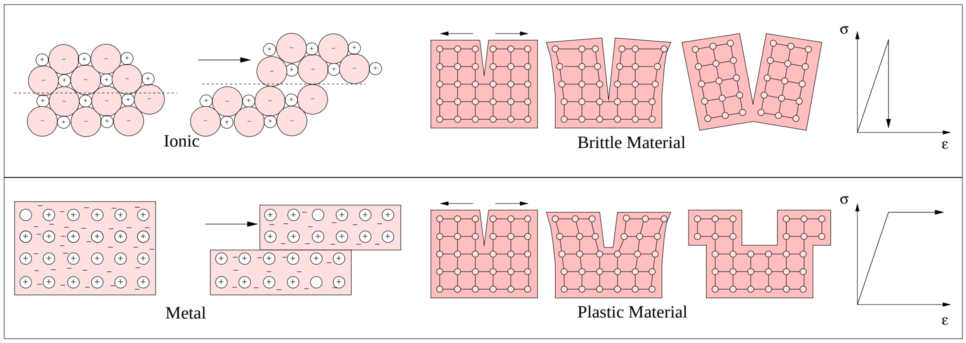


Figure 16.5: Brittle and Ductile Response as a Function of Chemical Bond

11 In the preceding chapters, we have examined the response of brittle material through fracture mechanics, we shall next examine the response of ductile ones through plasticity.

16.2.2 Causes of Plasticity

12 The permanent displacement of atoms within a crystal resulting from an applied load is known as **plastic deformation**. It occurs when a force of sufficient magnitude displaces atoms from one equilibrium position to another, Fig. 16.6. The plane on which deformation occurs is the **slip plane**.

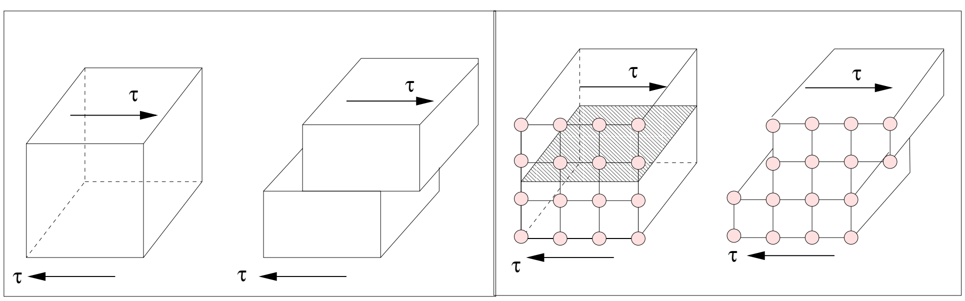


Figure 16.6: Slip Plane in a Perfect Crystal

13 Following a similar derivation as the one for the theoretical (normal) strength (Eq. 12.19), it can be shown that the theoretical shear strength to break all the atomic bonds across a slip plane is on the order of $E/10$. However, in practice we never reach this value.

14 This can be explained through the a defect arising from the insertion of part of an atomic plane as shown in Fig. 16.7. This defect is called an **edge dislocation**. Dislocations are introduced into a crystal in several ways, including: 1) "accidents" in the growth process during solidification of the crystal; 2) internal stresses associated with other defects in the crystal; and 3) interaction between existing dislocations that occur during plastic deformation.

15 With respect to Fig. 16.7, if a shear stress τ is applied to the crystal, there is a driving force for

breaking the bonds between the atoms marked A and C and the formation of bonds between the atoms in rows A and B. The process of breaking and reestablishing one row of atomic bonds may continue until the dislocation passes entirely out of the crystal. This is called a **dislocation glide**. When the dislocation leaves the crystal, the top half of the crystal is permanently offset by one atomic unit relative to the bottom half.

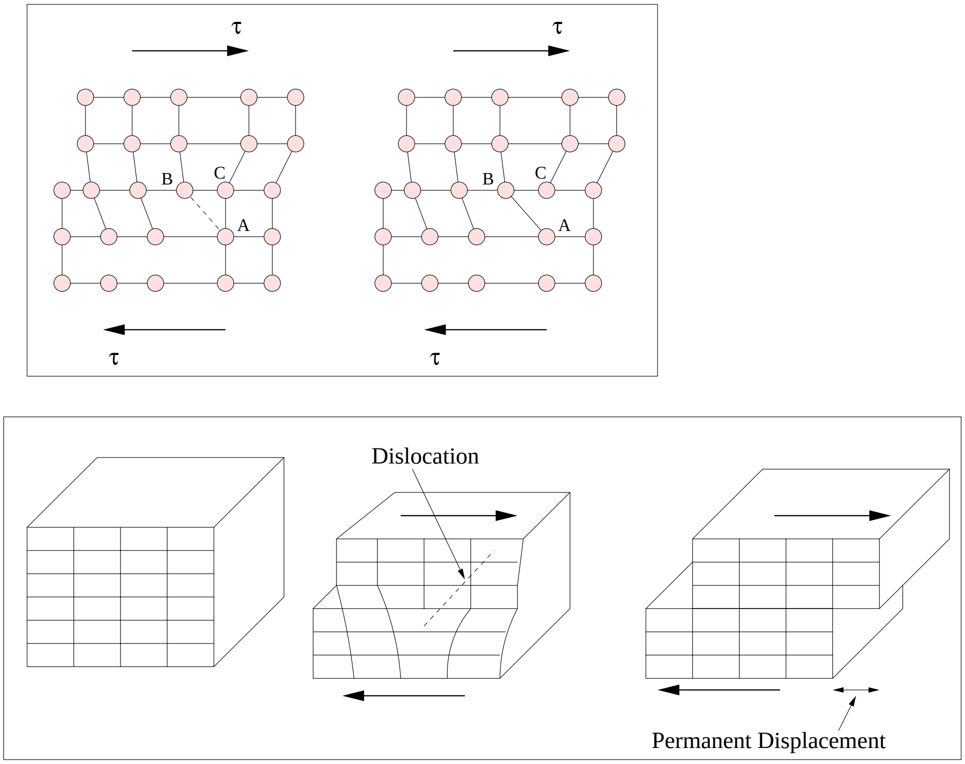


Figure 16.7: Dislocation Through a Crystal

16 Since the permanent deformation (i.e. irreversible motion of atoms from one equilibrium position to another) via dislocation glide was produced by breaking only one row of atomic bonds at any one time, the corresponding theoretical shear strength should be much lower than when all the bonds are broken simultaneously.

17 Other types of dislocations include **screw dislocation**which can be envisioned as forming a helical ramp that runs through the crystal.

18 In light of the above, we redefine

Yield stress: is essentially the applied **shear** stress necessary to provide the dislocations with enough energy to overcome the short range forces exerted by the obstacles.

Work-Hardening: As plastic deformation proceeds, dislocations multiply and eventually get stuck. The stress field of these dislocations acts as a back stress on moving dislocations, whose movement accordingly becomes progressively more difficult, and thus even greater stresses are needed to overcome their resistance.

Bauschinger Effect: The dislocations in a pile-up are in equilibrium under the applied stress σ, the internal stress σ_i due to various obstacles, and the back stress σ_b. σ_i may be associated with the elastic limit, when the applied stress is reduced, the dislocations back-off a little, with very little plastic deformation in order to reduce the internal stress acting on them. They can do so, until they are in positions in which the internal stress on them is $-\sigma_i$. When this occurs, they can move freely backward, resulting in reverse plastic flow when the applied stress has been reduced to $2\sigma_i$, Fig. ??.

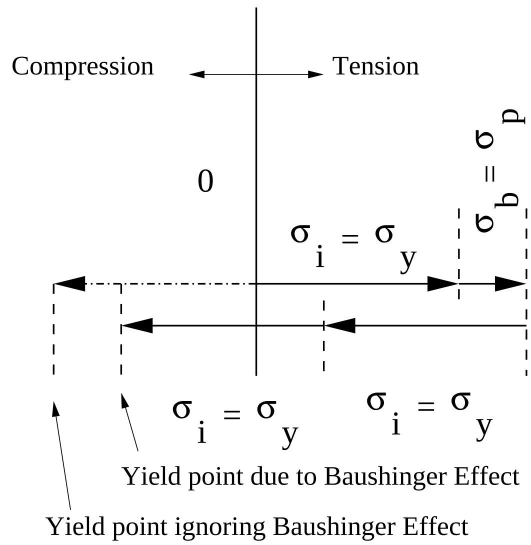


Figure 16.8: Baushinger Effect

16.3 Rheological Models

16.3.1 Elementary Models

19 Rheological models are used to describe the response of different materials. These, in turn, are easily visualized through **analogical** models, which are assemblies of simple mechanical elements with responses similar to those expected in the real material. They are used to provide a simple and concrete illustration of the constitutive equation.

20 The simplest model, Fig. 16.9, is for a linear spring and nonlinear spring, respectively characterized by:

$$\sigma = E\varepsilon \quad (16.1\text{-a})$$
$$d\sigma = E d\varepsilon \quad (16.1\text{-b})$$

21 We may also have a strain or a stress **threshold**, Fig. 16.10, given by

$$-\varepsilon_s \leq \varepsilon \leq \varepsilon_s \quad (16.2\text{-a})$$
$$-\sigma_s \leq \sigma \leq \sigma_s \quad (16.2\text{-b})$$

22 Finally, material response may be a function of the displacement velocity, **Newtonian dashpot**, Fig. 16.11, where

$$d\sigma = \eta d\dot{\varepsilon} \quad (16.3\text{-a})$$
or
$$\sigma = \lambda\dot{\varepsilon}^{1/N} \quad (16.3\text{-b})$$

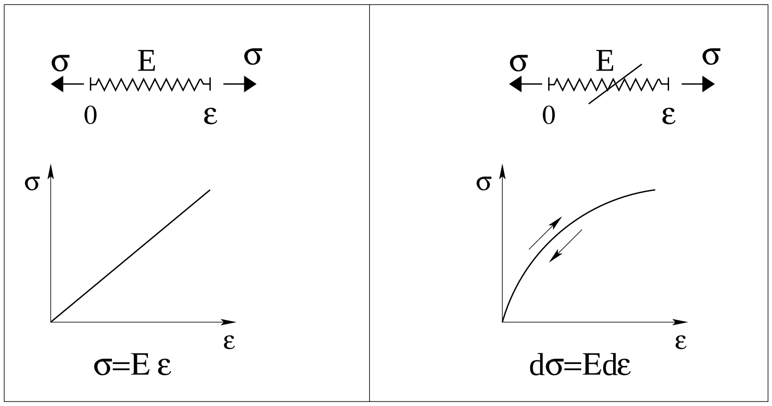


Figure 16.9: Linear (Hooke) and Nonlinear (Hencky) Springs

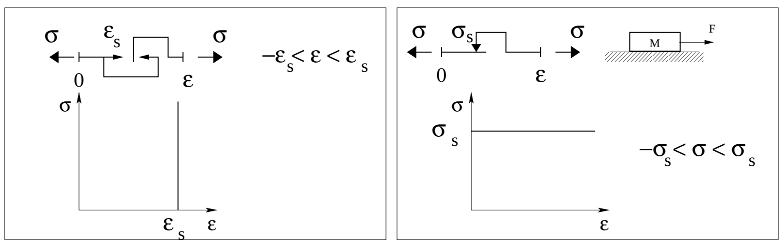


Figure 16.10: Strain Threshold

and η is the viscosity (Pa.sec).

16.3.2 One Dimensional Idealized Material Behavior

23 All the preceding elementary models can be further assembled either in

$$\text{Series} \quad \varepsilon = \sum_i \varepsilon_i \quad \sigma = \sigma_i \tag{16.4-a}$$

$$\text{Parallel} \quad \sigma = \sum_i \sigma_i \quad \varepsilon = \varepsilon_i \tag{16.4-b}$$

as in actuality, real materials exhibit a response which seldom can be reprepresented by a single elementary model, but rather by an assemblage of them.

Plasticity models are illustrated in Fig. 16.12. More (much more) about plasticity in subsequent chapters.

Visco-Elasticity In visco-elasticity, we may have different assemblages too, Fig. 16.13.

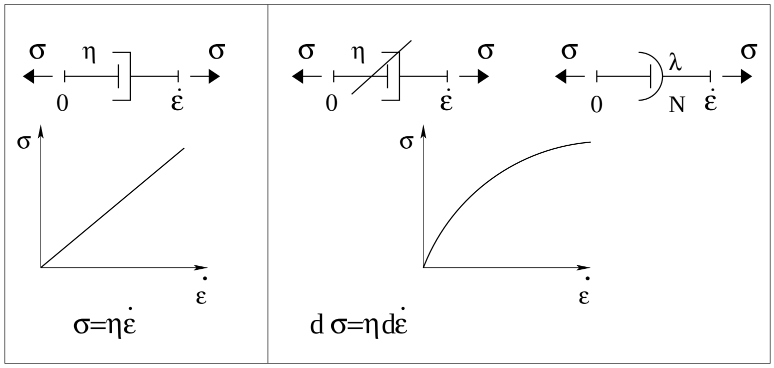


Figure 16.11: Ideal Viscous (Newtonian), and Quasi-Viscous (Stokes) Models

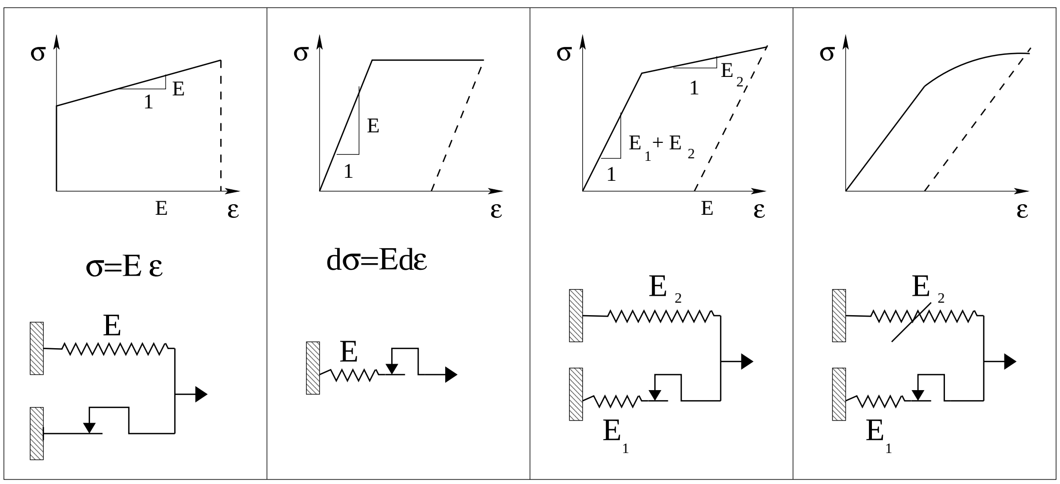
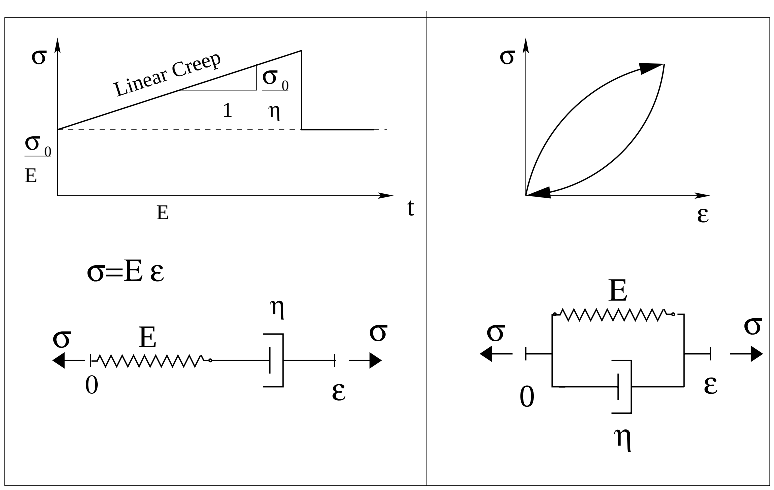


Figure 16.12: a) Rigid Plastic with Linear Strain Hardening; b) Linear Elastic, Perfectly Plastic; c) Linear Elastic, Plastic with Strain Hardening; d) Linear Elastic, Plastic with Nonlinear Strain Hardening



Maxwell Kelvin (Voigt)

Figure 16.13: Linear Kelvin and Maxwell Models

Chapter 17

LIMIT ANALYSIS

1 The design of structures based on plastic approach is called **limit design** and is at the core of most modern design codes (ACI, AISC).

17.1 Review

2 The stress distribution on a typical wide-flange shape subjected to increasing bending moment is shown in Fig.17.1. In the service range (that is before we multiplied the load by the appropriate factors in the LRFD method) the section is elastic. This elastic condition prevails as long as the stress at the extreme fiber has not reached the yield stress F_y. Once the strain ε reaches its yield value ε_y, increasing strain induces no increase in stress beyond F_y.

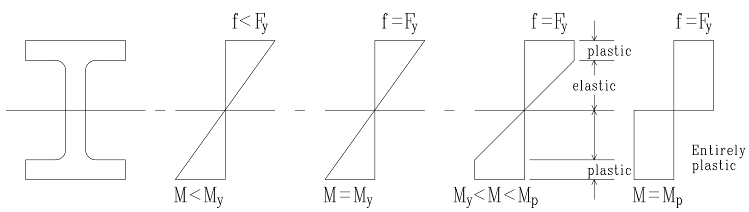


Figure 17.1: Stress distribution at different stages of loading

3 When the yield stress is reached at the extreme fiber, the nominal moment strength M_n, is referred to as the **yield moment** M_y and is computed as

$$\boxed{M_n = M_y = S_x F_y} \tag{17.1}$$

(assuming that bending is occurring with respect to the $x-x$ axis).

4 When across the entire section, the strain is equal or larger than the yield strain ($\varepsilon \geq \varepsilon_y = F_y/E_s$) then the section is fully plastified, and the nominal moment strength M_n is therefore referred to as the **plastic moment** M_p and is determined from

$$\boxed{M_p = F_y \int_A y dA = F_y Z} \tag{17.2}$$

$$\boxed{Z = \int y dA} \tag{17.3}$$

is the **Plastic Section Modulus**.

5 If all the forces acting on a structure vary proportionally to a certain load parameter μ, then we have **proportional loading**.

17.2 Limit Theorems

6 Beams and frames typically fail after a sufficient number of plastic hinges form, and the structures turns into a **mechanism**, and thus collapse (partially or totally).

7 There are two basic theorems.

17.2.1 Upper Bound Theorem; Kinematics Approach

8 A load computed on the basis of an assumed mechanism will always be greater than, or at best equal to, the true ultimate load.

9 Any set of loads in equilibrium with an assumed kinematically admissible field is larger than or at least equal to the set of loads that produces collapse of the strucutre.

10 The safety factor is the smallest kinematically admissible multiplier.

11 Note similartiy with principle of Virtual Work (or displacement) A deformable system is in equilibrium if the sum of the external virtual work and the internal virtual work is zero for virtual displacements $\delta\mathbf{u}$ which are kinematically admissible.

12 A kinematically admissible field is one where the external work W_e done by the forces $\mathbf{F}$ on the deformation $\boldsymbol{\Delta}_F$ and the internal work W_i done by the moments $\mathbf{M}_p$ on the rotations $\boldsymbol{\theta}$ are positives.

13 The collapse of a structure can be determined by equating the external and internal work during a virtual movement of the collapsed mechanism. If we consider a possible mechanism, i, equilibrium requires that

$$U_i = \lambda_i W_i \tag{17.4}$$

where W_i is the external work of the applied service loada, λ_i is a kinematic multipplier, U_i is the total internal energy dissipated by plastic hinges

$$U_i = \sum_{j=1}^{n} M_{p_j} \theta_{ij} \tag{17.5}$$

where M_{p_j} is the plastic moment, θ_{ij} the hinge rotation, and n the number of potential plastic hinges or critical sections.

14 According to the kinematic theorem of plastic analysis (Hodge 1959) the load factor λ and the associated collapse mode of the structure satisfy the following condition

$$\lambda = \min_i(\lambda_i) = \min_i \left(\frac{U_i}{W_i}\right) \min_i \left(\sum_{j=1}^{n} M_{p_j} \frac{\theta_{ij}}{W - W_i}\right) \qquad i = 1, \cdots, p \tag{17.6}$$

where p is the total number of possible mechanisms.

15 It can be shown that all possible mechanisms can be generated by linear combination of m independent mechanisms, where

$$m = n - NR \tag{17.7}$$

where NR is the degree of static indeterminancy.

16 The analysis procedure described in this chapter is only approximate because of the following assumptions

1. Response of a member is elastic perfectly plastic.
2. Plasticity is localized at specific points.
3. Only the plastic moment capacity M_p of a cross section is governing.

17.2.1.1 Example; Frame Upper Bound

17 Considering the portal frame shown in Fig. 17.2, there are five critical sections and the number of independent mechanisms is $m = 5 - 3 = 2$. The total number of possible mechanisms is three. From Fig. 17.2 we note that only mechanisms 1 and 2 are independent, whereas mechanisms 3 is a combined one.

18 Writing the expression for the virtual work equation for each mechanism we obtain

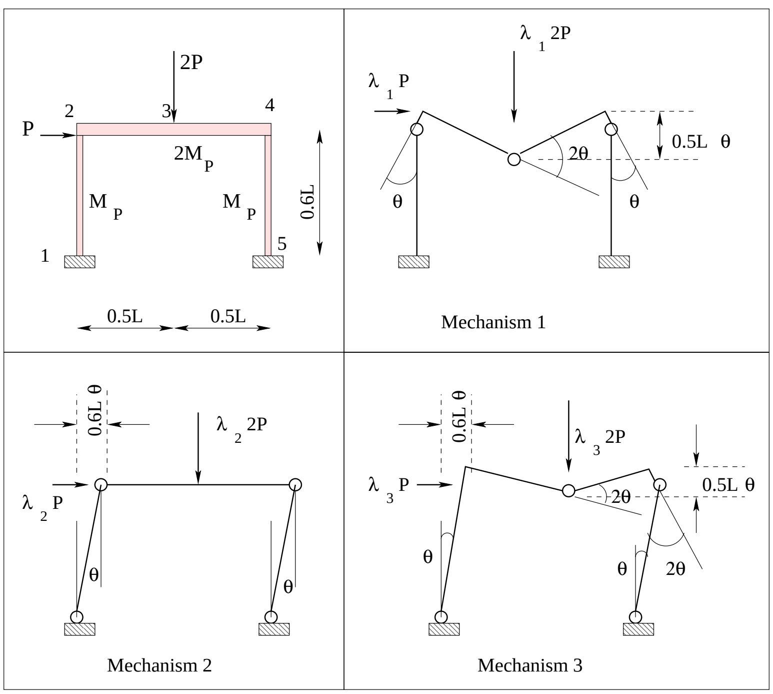


Figure 17.2: Possible Collapse Mechanisms of a Frame

Mechanism 1 $$M_p(\theta+\theta)+2M_p(2\theta)=\lambda_1(2P)(0.5L\theta) \quad \Rightarrow \lambda_1 = 6\frac{M_p}{PL} \tag{17.8-a}$$

Mechanism 2 $$M_p(\theta+\theta+\theta+\theta)=\lambda_2(P)(0.6L\theta) \quad \Rightarrow \lambda_2 = 6.67\frac{M_p}{PL} \tag{17.8-b}$$

Mechanism 3 $$M_p(\theta+\theta+2\theta)+2M_p(2\theta)=\lambda_3\left(P(0.6L\theta)+2P(0.5L\theta)\right) \quad \Rightarrow \lambda_3 = 5\frac{M_p}{PL} \tag{17.8-c}$$

(17.8-d)

Thus we select the smallest λ as

$$\lambda = \min_i(\lambda_i) = \boxed{5\frac{M_p}{PL}} \tag{17.9}$$

and the failure of the frame will occur through mechanism 3. To verify if this indeed the lower bound on λ, we may draw the corresponding moment diagram, and verify that at no section is the moment greater than M_p.

17.2.1.2 Example; Beam Upper Bound

19 Considering the beam shown in Fig. 17.3, the only possible mechanism is given by Fig. 17.4.

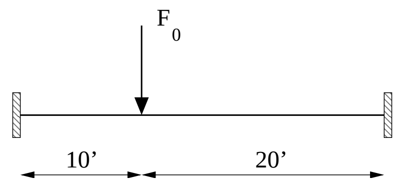


Figure 17.3: Limit Load for a Rigidly Connected Beam

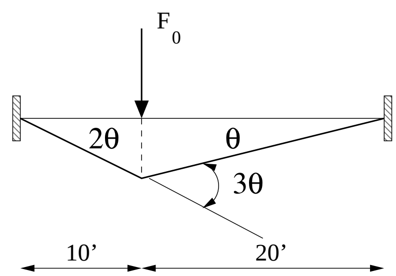


Figure 17.4: Failure Mechanism for Connected Beam

$$W_{int} = W_{ext} \tag{17.10-a}$$
$$M_p(\theta + 2\theta + 3\theta) = F_0\Delta \tag{17.10-b}$$
$$M_p = \frac{F_0\Delta}{6\theta} \tag{17.10-c}$$
$$= \frac{20\theta}{6\theta}F_0 \tag{17.10-d}$$
$$= 3.33F_0 \tag{17.10-e}$$
$$F_0 = 0.30M_p \tag{17.10-f}$$

17.2.2 Lower Bound Theorem; Statics Approach

20 A simple (engineering) statement of the lower bound theorem is

> A load computed on the basis of an assumed moment distribution, which is in equilibrium with the applied loading, and where no moment exceeds M_p is less than, or at best equal to the true ultimate load.

21 Note similartiy with principle of complementary virtual work: A deformable system satisfies all kinematical requirements if the sum of the external complementary virtual work and the internal complementary virtual work is zero for all statically admissible virtual stresses $\delta\sigma_{ij}$.

22 If the loads computed by the two methods coincide, the true and unique ultimate load has been found.

23 At ultimate load, the following conditions must be met:

1. The applied loads must be in equilibrium with the internaql forces.
2. There must be a sufficient number of plastic hinges for the formation of a mechanism.

17.2.2.1 Example; Beam lower Bound

We seek to determine the failure load of the rigidly connected beam shown in Fig. 17.5.

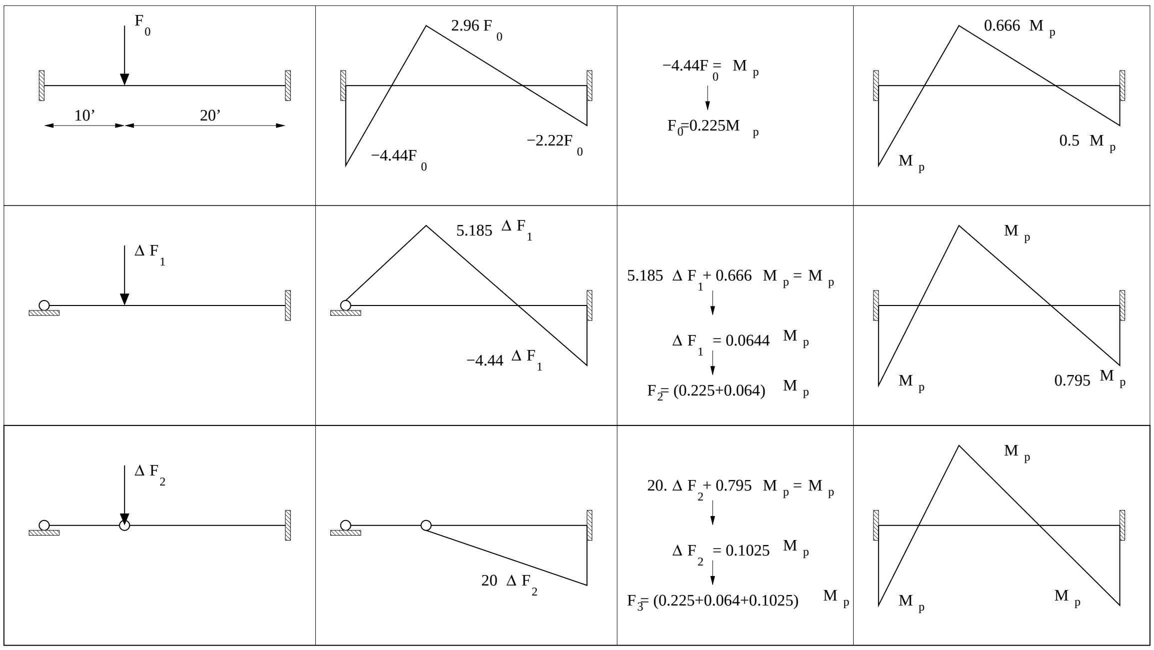


Figure 17.5: Limit Load for a Rigidly Connected Beam

1. First we consider the original structure
   (a) Apply a load F_0., and determine the corresponding moment diagram.
   (b) We identify the largest moment (-4.44F_0) and set it equal to M_P. This is the first point where a plastic hinge will form.
   (c) We redraw the moment diagram in terms of M_P.
2. Next we consider the structure with a plastic hinge on the left support.
   (a) We apply an **incremental** load ΔF_1.
   (b) Draw the corresponding moment diagram in terms of ΔF_1.
   (c) Identify the point of maximum **total** moment as the point under the load $5.185\Delta F_1 + 0.666 M_P$ and set it equal to M_P.
   (d) Solve for ΔF_1, and determine the total externally applied load.
   (e) Draw the updated total moment diagram. We now have two plastic hinges, we still need a third one to have a mechanism leading to collapse.
3. Finally, we analyse the revised structure with the two plastic hinges.

(a) Apply an incremental load ΔF_2.

(b) Draw the corresponding moment diagram in terms of ΔF_2.

(c) Set the total moment node on the right equal to M_P.

(d) Solve for ΔF_2, and determine the total external load. This load will correspond to the failure load of the structure.

17.2.2.2 Example; Frame Lower Bound

24 We now seek to determine the lower bound limit load of the frame previously analysed, Fig 17.6.

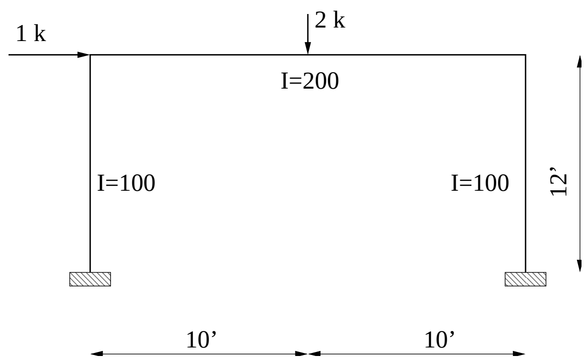


Figure 17.6: Limit Analysis of Frame

25 Fig . 17.7 summarizes the various analyses

1. First plastic hinge is under the 2k load, and $6.885F_0 = M_p \Rightarrow F_0 = 0.145M_p$.
2. Next hinge occurs on the right connection between horizontal and vertical member with $M_{max} = M_p - 0.842M_p = 0.158M_p$, and $\Delta F_1 = \frac{0.158}{12.633}M_p = 0.013M_p$
3. As before $M_{max} = M_p - 0.818M_p = 0.182M_p$ and $\Delta F_2 = \frac{0.182}{20.295}M_p = 0.009M_p$
4. Again $M_{max} = M_p - 0.344M_p = 0.656M_p$ and $\Delta F_3 = \frac{0.656}{32}M_p = 0.021M_p$
5. Hence the final collapse load is $F_0+\Delta F_1+\Delta F_2+\Delta F_3 = (0.145+0.013+0.009+0.021)M_p = 0.188M_p$ or $F_{max} = 3.76\frac{M_p}{L}$

17.3 Shakedown

26 A structure subjected to a general variable load can collapse even if the loads remain inside the elastoplastic domain of the load space. THus the elastoplastic domain represents a safe domain only for monotonic loads.

27 Under general, non-monotonic loading, a structure can nevertheless fail by incremental collapse or plastic fatigue.

28 The behavior of a structure is termed **shakedown**

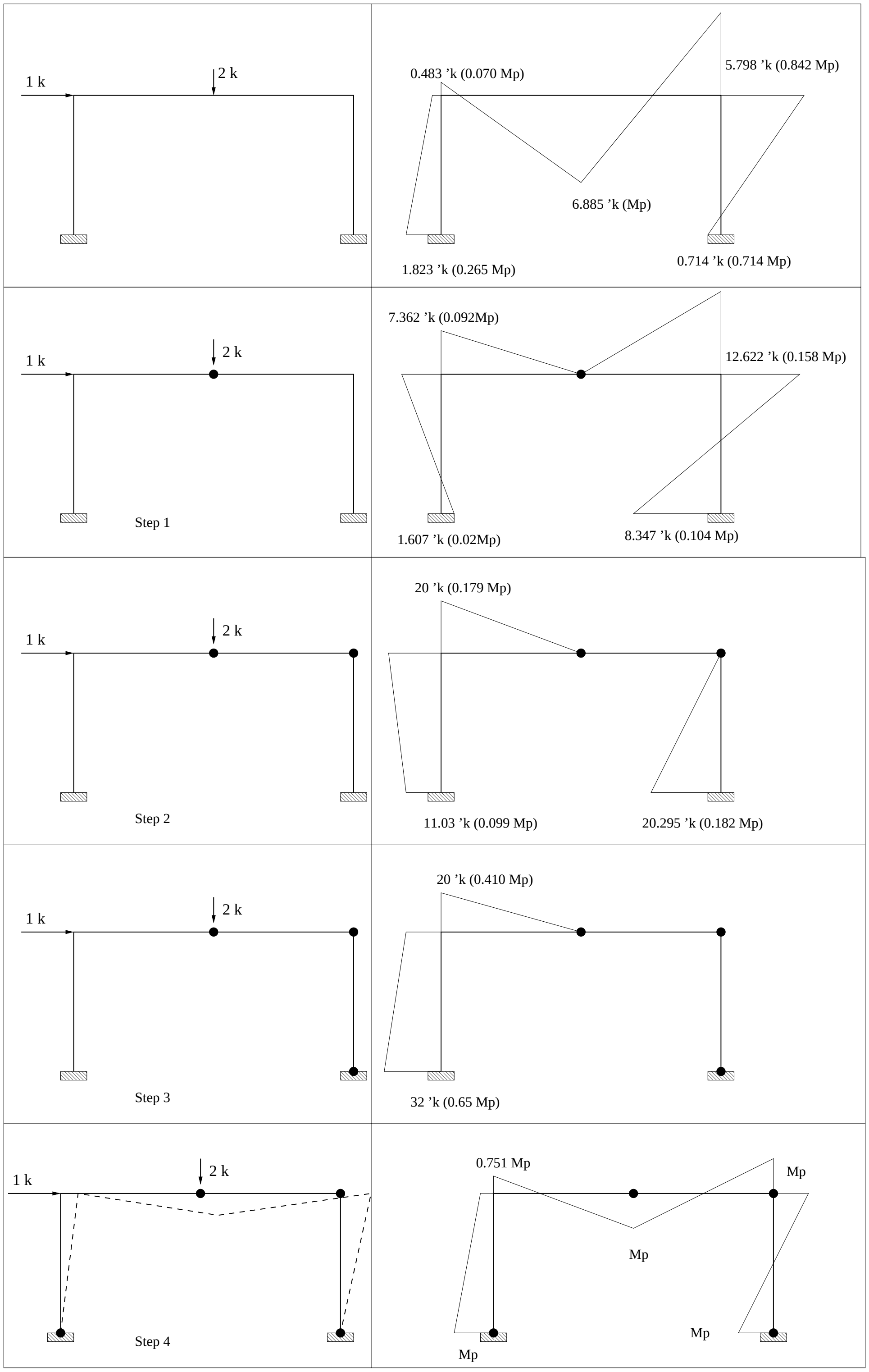


Figure 17.7: Limit Analysis of Frame; Moment Diagrams

Chapter 18

CONSTITUTIVE EQUATIONS; Part II A Thermodynamic Approach

1 When thermal effects are disregarded, the use of thermodynamics in order to introduce constitutive equations is not necessary. Those can be developed through a purely phenomenological (based on laboratory experiments and physical deduction) manner, as was done in a preceding chapter.

2 On the other hand, contrarily to the phenomenological approach, a thermodynamics one will provide a rigorous framework to formulate constitutive equations, identify variables that can be linked.

3 From the preceding chapter, the first law of thermodynamic expresses the conservation of energy, irrespective of its form. It is the second law , though expressed as an inequality, which addresses the "type" of energy; its transformatbility into efficient mechanical work (as opposed to lost heat) can only diminish. Hence, the entropy of a system, a measure of the deterioration, can only increase.

4 A **constitutive law** seeks to express $(\mathbf{X}, t)$ in terms of $\boldsymbol{\sigma}, \mathbf{q}, u, s$ in terms of (or rather up to) time t; In other words we have a deterministic system (the past determines the present) and thus the solid has a "memory".

18.1 State Variables

5 The method of local state postulates that the thermodynamic **state of a continuum** at a given point and instant is completely defined by several **state variables** (also known as **thermodynamic or independent variables**). A change in time of those state variables constitutes a **thermodynamic process**. Usually state variables are not all independent, and functional relationships exist among them through **equations of state**. Any state variable which may be expressed as a single valued function of a set of other state variables is known as a **state function**.

6 We differentiate between observable (i.e. which can be measured in an experiment), internal variables (or hidden variables), and associated variables, Table 18.1.

7 The time derivatives of these variables are not involved in the definition of the state, this postulate implies that any evolution can be considered as a succession of equilibrium states (therefore ultra rapid phenomena are excluded).

8 The **thermodynamic state** is specified by $n + 1$ variables $\nu_1, \nu_2, \cdots, \nu_n$ and s where ν_i are the **thermodynamic substate variables** and s the specific entropy. The former have mechanical (or electromagnetic) dimensions, but are otherwise left arbitrary in the general formulation. In ideal elasticity we have nine substate variables the components of the strain or deformation tensors.

9 The **basic assumption of thermodynamics** is that in addition to the n substate variables, just

State Variables		
Observable	Internal	Associated Variable
$\boldsymbol{\varepsilon}$		$\boldsymbol{\sigma}$
θ		s
	$\boldsymbol{\varepsilon}^e$	$\boldsymbol{\sigma}$
	$\boldsymbol{\varepsilon}^p$	$-\boldsymbol{\sigma}$
	$\boldsymbol{\nu}_k$	A_k

Table 18.1: State Variables

one additional dimensionally independent scalar paramerter suffices to determine the specific internal energy u. This assumes that there exists a **caloric equation of state**

$$\boxed{u = u(s, \boldsymbol{\nu}, \mathbf{X})} \tag{18.1}$$

10 In general the internal energy u can not be experimentally measured but rather its derivative.

11 For instance we can define the **thermodynamic temperature** θ and the **thermodynamic "tension"** τ_j through the following state functions

$$\theta \equiv \left(\frac{\partial u}{\partial s}\right)_{\nu} \tag{18.2}$$

$$\tau_j \equiv \left(\frac{\partial u}{\partial \nu_j}\right)_{s,\nu_{i(i\neq j)}} \quad j = 1, 2, \cdots, n \tag{18.3}$$

where the subscript outside the parenthesis indicates that the variables are held constant, and (by extension)

$$A_i = -\rho\tau_i \tag{18.4}$$

would be the **thermodynamic "force"** and its dimension depends on the one of ν_i.

12 Thus, in any real or hypothetical change in the thermodynamic state of a given particle $\mathbf{X}$

$$\boxed{\mathrm{d}u = \theta \mathrm{d}s + \tau_p \mathrm{d}h\nu_p} \tag{18.5}$$

this is **Gibbs equation**. It is the maximum amount of work a system can do at a constant pressure and and temperature.

18.2 Clausius-Duhem Inequality

13 According to the Second Law, the time rate of change of total entropy $\mathcal{S}$ in a continuum occupying a volume V is never less than the rate of heat supply divided by the absolute temperature (i.e. sum of the entropy influx through the continuum surface plus the entropy produced internally by body sources).

14 We restate the definition of entropy as heat divided by temperature, and write the second principle

$$\overbrace{\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho s \mathrm{d}V}_{\text{Rate of Entropy Increase}}}^{\text{Internal}} \geq \overbrace{\underbrace{\int_V \rho\frac{r}{\theta}\mathrm{d}V}_{\text{Sources}} - \underbrace{\int_S \frac{\mathbf{q}}{\theta}\cdot\mathbf{n}\mathrm{d}S}_{\text{Exchange}}}^{\text{External}} \tag{18.6}$$

The second term on the right hand side corresponds to the heat provided by conduction through the surface S, the first term on the right corresponds to the volume entropy rate associated with external heat. Both terms correspond to the rate of externally supplied entropy. Hence, the difference between the left and right hand side terms corresponds to the rate of internal production of entropy relative to the matter contained in V. The second law thus stipulates that intenal entropy rate, which corresponds to an uncontrolled spontaneous production, can not be negative. For an "entropically" isolated system ($\mathbf{q} = 0$ and $r = 0$), the entropy can not decrease.

15 The dimension of $\mathcal{S} = \int_v \rho s \mathrm{d}V$ is one of energy divided by temperature or $L^2MT^{-2}\theta^{-1}$, and the SI unit for entropy is Joule/Kelvin.

16 Applying Gauss theorem to the last term

$$\int_S \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\mathrm{d}S = \int_V \boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{\theta}\right)\mathrm{d}V = \int_V \left(\frac{\boldsymbol{\nabla}\cdot\mathbf{q}}{\theta} - \frac{\mathbf{q}\cdot\boldsymbol{\nabla}\theta}{\theta^2}\right)\mathrm{d}V \tag{18.7}$$

17 The previous inequality holds for any arbitrary volume, thus after transformation of the surface integral into a volume integral, we obtain the following local version of the Clausius-Duhem inequality which must holds at every point

$$\boxed{\underbrace{\rho\frac{\mathrm{d}s}{\mathrm{d}t}}_{\text{Rate of Entropy Increase}} \geq \underbrace{\frac{\rho r}{\theta}}_{\text{Sources}} \underbrace{-\boldsymbol{\nabla}\cdot\frac{\mathbf{q}}{\theta} + \frac{\mathbf{q}\cdot\boldsymbol{\nabla}\theta}{\theta^2}}_{\text{Exchange}}} \tag{18.8}$$

The left hand side is the rate of entropy, the right hand side is called the rate of *external entropy supply* and the difference between the two is called the **rate of internal entropy production**. Again, the entropy is a measure of complexity and disorder of the internal state.

18 Since θ is always positive, we rewrite the previous equation as

$$\rho\theta\frac{\mathrm{d}s}{\mathrm{d}t} \geq -\boldsymbol{\nabla}\cdot\mathbf{q} + \rho r + \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \tag{18.9}$$

where $-\boldsymbol{\nabla}\cdot\mathbf{q} + \rho r$ is the heat input into V and appeared in the first principle Eq. 6.32

$$\rho\frac{\mathrm{d}u}{\mathrm{d}t} = \mathbf{T}{:}\mathbf{D} + \rho r - \boldsymbol{\nabla}\cdot\mathbf{q} \tag{18.10}$$

hence, substituting, we obtain the **Clausius-Duhem inequality**

$$\boxed{\mathbf{T}{:}\mathbf{D} - \rho\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \theta\frac{\mathrm{d}s}{\mathrm{d}t}\right) - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0} \tag{18.11}$$

18.3 Thermal Equation of State

19 From the caloric equation of state, Eq. 18.1, and the the definitions of Eq. 18.2 and 18.3 it follows that the temperature and the thermodynamic tensions are functions of the thermodynamic state:

$$\theta = \theta(s, \boldsymbol{\nu}); \qquad \tau_j = \tau_j(s, \boldsymbol{\nu}) \tag{18.12}$$

we assume the first one to be invertible

$$s = s(\theta, \boldsymbol{\nu}) \tag{18.13}$$

and substitute this into Eq. 18.1 to obtain an alternative form of the caloric equation of state with corresponding **thermal equations of state**. Repeating this operation, we obtain

$$u = u(\theta, \boldsymbol{\nu}, \mathbf{X}) \quad \leftarrow \quad (18.14)$$
$$\tau_i = \tau_i(\theta, \boldsymbol{\nu}, \mathbf{X}) \quad (18.15)$$
$$\nu_i = \nu_i(\theta, \boldsymbol{\theta}, \mathbf{X}) \quad (18.16)$$

20 The thermal equations of state resemble stress-strain relations, but some caution is necessary in interpreting the tensions as stresses and the ν_j as strains.

18.4 Thermodynamic Potentials

21 We can postulate the existence of a thermodynamic potential from which the state laws can be derived. The specification of a function with a scalar value, concave with respect to θ, and convex with respect to other variables allow us to satisfy a priori the conditions of thermodynamic stability imposed by the inequalities that can be derived from the second principle (**?**).

22 Based on the assumed existence of a caloric equation of state, four thermodynamic potentials are introduced, Table 18.2. Those potentials are derived through the **Legendre-Fenchel transformation**

Potential		**Relation to u**	**Independent Variables**
Internal energy	u	u	s, ν_j
Helmholtz free energy	Ψ	$\Psi = u - s\theta$	$\theta, \nu_j \quad \leftarrow$
Enthalpy	h	$h = u - \tau_j \nu_j$	s, τ_j
Gibbs free energy	g	$g = u - s\theta - \tau_j \nu_j$	θ, τ_j

or

$\rightarrow -s\theta$

$\downarrow$	u	Ψ
$-\tau\nu$	h	g

Table 18.2: Thermodynamic Potentials

on the basis of selected state variables best suited for a given problem.

23 The thermodynamic potentials allow us to write relations between observable state variables and associated variables. However, for internal variables it allows only the definition of their associated variables.

24 By means of the preceding equations, any one of the potentials can be expressed in terms of any of the four choices of state variables listed in Table 18.2.

25 In any actual or hypothetical change obeying the equations of state, we have

$$\mathrm{d}u = \theta \mathrm{d}s + \tau_j \mathrm{d}\nu_j \quad (18.17\text{-a})$$
$$\mathrm{d}\Psi = -s\mathrm{d}\theta + \tau_j \mathrm{d}\nu_j \quad \leftarrow \quad (18.17\text{-b})$$
$$\mathrm{d}h = \theta \mathrm{d}s - \nu_j \mathrm{d}\tau_j \quad (18.17\text{-c})$$
$$\mathrm{d}g = -s\mathrm{d}\theta - \nu_j \mathrm{d}\tau_j \quad (18.17\text{-d})$$

We note that the second equation is an alternate form of Gibbs equation.

26 The **complementary laws** provide the **thermodynamic forces** through the **normality properties**:

$$\theta = \left(\frac{\partial u}{\partial s}\right)_{\boldsymbol{\nu}}; \qquad \tau_j = \left(\frac{\partial u}{\partial \nu_j}\right)_{s, \nu_{i(i \neq j)}} \quad (18.18\text{-a})$$
$$s = -\left(\frac{\partial \Psi}{\partial \theta}\right)_{\boldsymbol{\nu}}; \qquad \tau_j = \left(\frac{\partial \Psi}{\partial \nu_j}\right)_{\theta} \quad \leftarrow \quad (18.18\text{-b})$$

$$\theta = \left(\frac{\partial h}{\partial s}\right)_{\boldsymbol{\tau}}; \qquad \nu_j = -\left(\frac{\partial h}{\partial \tau_j}\right)_{s,\nu_{i(i\neq j)}} \tag{18.18-c}$$

$$= -\left(\frac{\partial g}{\partial \theta}\right)_{\boldsymbol{\tau}}; \qquad \nu_j = -\left(\frac{\partial g}{\partial \tau_j}\right)_{\theta} \tag{18.18-d}$$

27 We focus our attention on the **free energy** Ψ which is the portion of the internal energy available for doing work at constant temperature (and thus eventually recoverable). Helmholtz free energy describes the capacity of a system to do work.

28 † The **enthalpy** h (as defined here) is the portion of the internal energy that can be released as heat when the thermodynamic tensions are held constant.

29 It can be shown that the work of the thermodynamic tensions is recoverable and that the external stress power equals the rate of work of the thermodynamic tensions, i.e.

$$T_{ij}D_{ij} = \rho \sum_{j=1}^{n} \tau_j \dot{\nu}_j \tag{18.19}$$

for both the adiabatic and isentropic (s = constant) deformation or isothermal deformation with reversible heat transfer.

18.5 Linear Thermo-Elasticity

30 In order to maintain a linear theory, it is enough to select as a convex thermodynamic potential a positive definite quadratic function in the components of the strain tensor

$$\Psi = \frac{1}{2\rho}\mathbf{a}{:}\boldsymbol{\varepsilon}{:}\boldsymbol{\varepsilon} \tag{18.20}$$

and by definition

$$\boldsymbol{\sigma} = \rho\frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \mathbf{a}{:}\boldsymbol{\varepsilon} \tag{18.21}$$

which is Hooke's law.

31 We can define the dual potential

$$\Psi^* = \frac{1}{2\rho}\mathbf{A}{:}\boldsymbol{\sigma}{:}\boldsymbol{\sigma} \tag{18.22}$$

and

$$\boldsymbol{\varepsilon} = \rho\frac{\partial \Psi}{\partial \boldsymbol{\sigma}} = \mathbf{A}{:}\boldsymbol{\sigma} \tag{18.23}$$

32 Isotropy and linearity require that the potential Ψ be a quadratic invariant of the strain tensor, i.e. a linear combination of the square of the first invariant $\varepsilon_I^2 = [\text{tr }(\boldsymbol{\varepsilon})]^2$, and the second invariant $\varepsilon_{II}^2 = \frac{1}{2}\text{tr }(\boldsymbol{\varepsilon}^2)$

$$\Psi = \frac{1}{\rho}\left[\frac{1}{2}\left(\lambda\varepsilon_I^2 + 4\mu\varepsilon_{II}\right) - (3\lambda+2\mu)\alpha\theta\varepsilon_I\right] - \frac{C_\varepsilon}{2T_0}\theta^2 \tag{18.24}$$

33 Differentiating accroding to Eq. 18.18-b we obtain the stress

$$\boldsymbol{\sigma} = \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \lambda\text{tr }(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon} - (3\lambda+2\mu)\alpha\theta\mathbf{I} \tag{18.25}$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} - (3\lambda+2\mu)\alpha\theta\delta_{ij} \tag{18.26}$$

which are identical to the expressions previously derived.

18.5.1 †Elastic Potential or Strain Energy Function

34 Green defined an elastic material as one for which a strain-energy function exists. Such a material is called **Green-elastic** or **hyperelastic** if there exists an **elastic potential function** W or **strain energy function**, a scalar function of one of the strain or deformation tensors, whose derivative with respect to a strain component determines the corresponding stress component.

35 For the fully recoverable case of isothermal deformation with reversible heat conduction we have

$$\tilde{T}_{IJ} = \rho_0 \left(\frac{\partial \Psi}{\partial E_{IJ}} \right)_\theta \tag{18.27}$$

hence $W = \rho_0 \Psi$ is an elastic potential function for this case, while $W = \rho_0 u$ is the potential for adiabatic isentropic case (s = constant).

36 Hyperelasticity ignores thermal effects and assumes that the elastic potential function always exists, it is a function of the strains alone and is purely mechanical

$$\tilde{T}_{IJ} = \frac{\partial W(\mathbf{E})}{\partial E_{IJ}} \tag{18.28}$$

and $W(\mathbf{E})$ is the **strain energy per unit undeformed volume**. If the displacement gradients are small compared to unity, then we obtain

$$\boxed{T_{ij} = \frac{\partial W}{\partial E_{ij}}} \tag{18.29}$$

which is written in terms of Cauchy stress T_{ij} and small strain E_{ij}.

37 We assume that the elastic potential is represented by a power series expansion in the small-strain components.

$$W = c_0 + c_{ij}E_{ij} + \frac{1}{2}c_{ijkm}E_{ij}E_{km} + \frac{1}{3}c_{ijkmnp}E_{ij}E_{km}E_{np} + \cdots \tag{18.30}$$

where c_0 is a constant and $c_{ij}, c_{ijkm}, c_{ijkmnp}$ denote tensorial properties required to maintain the invariant property of W. Physically, the second term represents the energy due to residual stresses, the third one refers to the **strain energy** which corresponds to linear elastic deformation, and the fourth one indicates nonlinear behavior.

38 Neglecting terms higher than the second degree in the series expansion, then W is quadratic in terms of the strains

$$\begin{aligned} W = {} & c_0 + c_1E_{11} + c_2E_{22} + c_3E_{33} + 2c_4E_{23} + 2c_5E_{31} + 2c_6E_{12} \\ & + \tfrac{1}{2}c_{1111}E_{11}^2 + c_{1122}E_{11}E_{22} + c_{1133}E_{11}E_{33} + 2c_{1123}E_{11}E_{23} + 2c_{1131}E_{11}E_{31} + 2c_{1112}E_{11}E_{12} \\ & + \tfrac{1}{2}c_{2222}E_{22}^2 + c_{2233}E_{22}E_{33} + 2c_{2223}E_{22}E_{23} + 2c_{2231}E_{22}E_{31} + 2c_{2212}E_{22}E_{12} \\ & + \tfrac{1}{2}c_{3333}E_{33}^2 + 2c_{3323}E_{33}E_{23} + 2c_{3331}E_{33}E_{31} + 2c_{3312}E_{33}E_{12} \\ & + 2c_{2323}E_{23}^2 + 4c_{2331}E_{23}E_{31} + 4c_{2312}E_{23}E_{12} \\ & + 2c_{3131}E_{31}^2 + 4c_{3112}E_{31}E_{12} \\ & + 2c_{1212}E_{12}^2 \end{aligned} \tag{18.31}$$

we require that W vanish in the unstrained state, thus $c_0 = 0$.

39 We next apply Eq. 18.29 to the quadratic expression of W and obtain for instance

$$T_{12} = \frac{\partial W}{\partial E_{12}} = 2c_6 + c_{1112}E_{11} + c_{2212}E_{22} + c_{3312}E_{33} + c_{1212}E_{12} + c_{1223}E_{23} + c_{1231}E_{31} \tag{18.32}$$

if the stress must also be zero in the unstrained state, then $c_6 = 0$, and similarly all the coefficients in the first row of the quadratic expansion of W. Thus the elastic potential function is a **homogeneous quadratic** function of the strains and we obtain **Hooke's law**

18.6 Dissipation

40 We next introduce the density of energy dissipation rate, henceforth called **dissipation**, Φ defined as the rate of internal entopy production per unit volume multiplied by the absolute temperature. Since the absolute temperature is always positive, the Second Law is equivalent to the condition of non-negative dissipation.

41 Rewriting Eq. 18.5 as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{\left(\frac{\partial u}{\partial s}\right)_{\nu}}_{\theta} \frac{\mathrm{d}s}{\mathrm{d}t} + \underbrace{\left(\frac{\partial u}{\partial \nu_j}\right)_{s,\nu_{i(i\neq j)}}}_{\tau_p} \frac{\mathrm{d}\nu_p}{\mathrm{d}t} \tag{18.33}$$

and substituting Eq. 18.33 into the Clasius-Duhem inequality of Eq. 18.11, and finally recalling that $A_i = -\rho\tau_i$, we obtain

$$\boxed{\mathbf{T}{:}\mathbf{D} + A_p \frac{\mathrm{d}\nu_p}{\mathrm{d}t} - \frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0} \tag{18.34}$$

42 Based on the above, we define two volumetric dissipations

$$\text{Intrinsic: } \Phi_1 = \mathbf{T}{:}\mathbf{D} + A_p \frac{\mathrm{d}\nu_p}{\mathrm{d}t} \geq 0 \tag{18.35}$$

$$\text{Thermal: } \Phi_2 = -\frac{1}{\theta}\mathbf{q}\cdot\boldsymbol{\nabla}\theta \geq 0 \tag{18.36}$$

the intrinsic dissipation (or mechanical dissipation) consists of plastic dissipation plus the dissipation associated with the evolution of the other internal variables; it is generally dissipated by the volume element in the form of heat. The second term is the thermal dissipation due to heat conduction.

43 $\Phi_1 = 0$ for reversible material, and $\Phi_2 = 0$ if the thermal conductivity is zero.

18.6.1 Dissipation Potentials

44 We next postulate the existence of a **dissipation potential** expressed as a continuous and convex scalar valued function of the flux variables

$$\phi(\dot{\boldsymbol{\varepsilon}}, \dot{\nu}_k, \mathbf{q}/\theta) \tag{18.37}$$

where through the normality property we have the complementary laws

$$\boldsymbol{\sigma} = \frac{\partial \phi}{\partial \dot{\boldsymbol{\varepsilon}}^p} \tag{18.38}$$

$$A_k = -\frac{\partial \phi}{\partial \dot{\nu}_k} \tag{18.39}$$

$$\boldsymbol{\nabla}\theta \tag{18.40}$$

in other words the thermodynamic forces are the components of the vector $\boldsymbol{\nabla}\phi$ normal to $\phi =$ constant in the space of the flux variables.

45 Dissipation variables are shown in Table 18.3.

46 Alternatively, the Legendre-Fenchel transformation enables us to define the corresponding potential

$$\phi^*(\dot{\boldsymbol{\sigma}}, A_k, \mathbf{g}) \tag{18.41}$$

and the compelmentary laws of evolution can be written as

Variables	
Flux	Dual
$\dot{\boldsymbol{\varepsilon}}^p$	$\boldsymbol{\sigma}$
$\dot{\nu}_k$	A_k
$-\mathbf{q}/\theta$	$\mathbf{g} = \boldsymbol{\nabla}\theta$

Table 18.3: Dissipation Variables

$$\dot{\boldsymbol{\varepsilon}}^p = \frac{\partial \phi^*}{\partial \boldsymbol{\sigma}} \quad (18.42)$$

$$-\dot{\nu}_k = -\frac{\partial \phi^*}{\partial A_k} \quad (18.43)$$

$$-\frac{\mathbf{q}}{\theta} = -\frac{\partial \phi^*}{\partial \mathbf{g}} \quad (18.44)$$

The first relation yields the plasticity and viscoplasticity laws (later used in Eq. 19.78), the second equation expresses the evolution laws of the internal variables, and the third one leads to the Fourrier law of thermostatics.

47 The whole problem of modelling a phenomenon lies in the determination of the thermodynamic potential Ψ and for the dissipation potential ϕ or its dual ϕ^*, and their identification in characteristic experiments.

48 In practice ϕ and ϕ^* are almost impossible to measure as they represent energy dissipated as heat.

Chapter 19

3D PLASTICITY

19.1 Introduction

1 There are two major theories for elastoplasticity, Fig. 19.1

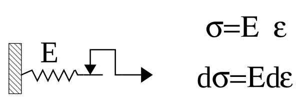


Figure 19.1: Rheological Model for Plasticity

Deformation Theory (or Total) of Hencky and Nadai, where the total strain ε_{ij} is a function of the current stress.

$$\epsilon = \epsilon_e + \epsilon_p \tag{19.1}$$

It leads to a secant-type formulation of plasticity that is based on the additive decomposition of total strain into elastic and plastic components (Hencky). This theory results in discontinuities in the transition region between elasticity and plasticity under unloading (or repeated loading).

Rate Theory (or incremental) of Prandtl-Reuss, defined by

$$\boxed{\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p} \tag{19.2}$$

if $\sigma \leq \sigma_y$(elasticity), then

$$\dot{\epsilon} = \dot{\epsilon}_e = \frac{\dot{\sigma}}{E} \tag{19.3}$$

if $\sigma > \sigma_y$(plasticity), then

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_p \tag{19.4}$$

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\dot{\sigma}}{E_p} = \frac{\dot{\sigma}}{E_T} \tag{19.5}$$

where

$$E_T = \frac{EE_p}{E + E_p} \tag{19.6}$$

We note that,

$$E_T = \begin{cases} > 0, & \text{Hardening} \\ = 0, & \text{Perfectly Plastic} \\ < 0, & \text{Softening} \end{cases}$$

, Fig. 19.2.

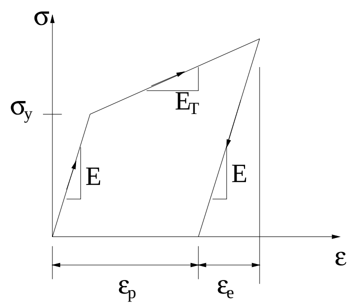


Figure 19.2: Stress-Strain diagram for Elastoplasticity

19.2 Elastic Behavior

2 Revisiting Eq. 18.20 and 18.20, we can rewrite the Helmhotz free energy Ψ as $\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_p, \kappa$ where κ accounts for possible hardening, Sect. 19.7.2

$$\boldsymbol{\sigma} = \rho \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}_e} \tag{19.7}$$

19.3 Idealized Uniaxial Stress-Strain Relationships

3 There are many stress-strain models for the elastic-plastic behavior under monotonic loading:

Elastic-Perfectly Plastic where hardening is neglected, and plastic flows begins when the yield stress is reached

$$\varepsilon = \frac{\sigma}{E} \text{ for } \sigma < \sigma_y \tag{19.8-a}$$
$$\varepsilon = \frac{\sigma}{E} + \lambda \text{ for } \sigma = \sigma_y \tag{19.8-b}$$

Elastic-Linearly Hardening model, where the tangential modulus is assumed to be constant

$$\varepsilon = \frac{\sigma}{E} \text{ for } \sigma < \sigma_y \tag{19.9-a}$$
$$\varepsilon = \frac{\sigma}{E} + \frac{1}{E_t}(\sigma - \sigma_y) \text{ for } \sigma = \sigma_{Y0} \tag{19.9-b}$$

Elastic-Exponential Hardening where a power law is assumed for the plastic region

$$\varepsilon = \frac{\sigma}{E} \text{ for } \sigma < \sigma_y \tag{19.10-a}$$
$$\varepsilon = k\varepsilon^n \text{ for } \sigma = \sigma_y \tag{19.10-b}$$

Ramberg-Osgood which is a nonlinear smooth single expression

$$\varepsilon = \frac{\sigma}{E} + a\left(\frac{\sigma}{b}\right)^n \tag{19.11}$$

19.4 Plastic Yield Conditions (Classical Models)

19.4.1 Introduction

4 Yielding in a uniaxially loaded structural element can be easily determined from $|\frac{\sigma}{\sigma_{yld}}| \geq 1$. But what about a general three dimensional stress state?

5 We introduce a **yield function** as a function of all six stress components of the stress tensor

$$F = F(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}) \begin{cases} < 0 & \text{Elastic} \\ = 0 & \text{Plastic} \\ > 0 & \text{Impossible} \end{cases} \quad \left| \begin{matrix} \frac{d\varepsilon^P}{dt} \\ \frac{d\varepsilon^P}{dt} \end{matrix} \right| \begin{matrix} = 0 \\ \geq 0 \end{matrix} \tag{19.12}$$

note, that f can not be greater than zero, for the same reason that a uniaxial stress can not exceed the yield stress, Fig. 19.3 in $\sigma_1 - \sigma_2$ (principal) stress space.

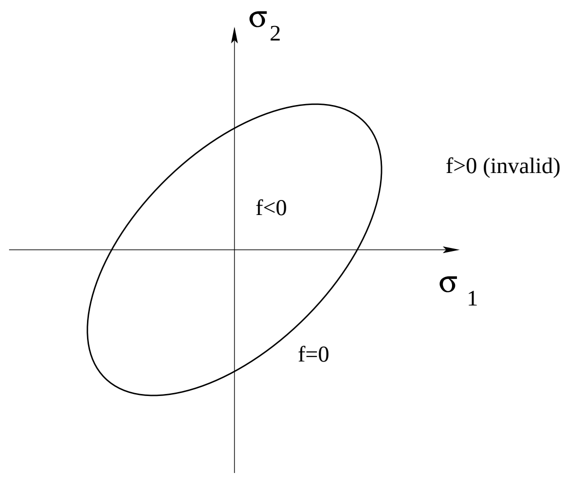


Figure 19.3: Yield Criteria

6 In uniaxial stress states, the elastic limit is obtained by a well-defined yield stress point σ_0. In biaxial or triaxial state of stresses, the elastic limit is defined mathematically by a certain **yield criterion** which is a function of the stress state σ_{ij} expressed as

$$F(\sigma_{ij}) = 0 \tag{19.13}$$

For isotropic materials, the stress state can be uniquely defined by either one of the following set of variables

$$F(\sigma_1, \sigma_2, \sigma_3) = 0 \tag{19.14-a}$$
$$F(I_1, J_2, J_3) = 0 \tag{19.14-b}$$
$$F(\xi, \rho, \theta) = 0 \tag{19.14-c}$$

those equations represent a surface in the principal stress space, this surface is called the **yield surface**. Within it, the material behaves elastically, on it it begins to yield. The elastic-plastic behavior of most metals is essentially hydrostatic pressure insensitive, thus the yield criteria will not depend on I_1, and the yield surface can generally be expressed by any one of the following equations.

$$F(J_2, J_3) = 0 \tag{19.15-a}$$
$$F(\rho, \theta) = 0 \tag{19.15-b}$$

19.4.1.1 Deviatoric Stress Invariants

7 If we let σ denote the mean normal stress p

$$\sigma = -p = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\text{tr}\,\boldsymbol{\sigma} \tag{19.16}$$

then the stress tensor can be written as the sum of two tensors:

Hydrostatic stress in which each normal stress is equal to $-p$ and the shear stresses are zero. The hydrostatic stress produces volume change without change in shape in an isotropic medium.

$$\sigma_{hyd} = -p\mathbf{I} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \tag{19.17}$$

Deviatoric Stress: which causes the change in shape.

$$\mathbf{s} = \begin{bmatrix} s_{11}-\sigma & s_{12} & s_{13} \\ s_{21} & s_{22}-\sigma & s_{23} \\ s_{31} & s_{32} & s_{33}-\sigma \end{bmatrix} \tag{19.18}$$

8 The principal stresses are physical quantities, whose values do not depend on the coordinate system in which the components of the stress were initially given. They are therefore **invariants** of the stress state.

9 If we examine the stress invariants,

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0 \tag{19.19-a}$$

$$|\sigma_{rs} - \lambda\delta_{rs}| = 0 \tag{19.19-b}$$

$$|\boldsymbol{\sigma} - \lambda\mathbf{I}| = 0 \tag{19.19-c}$$

When the determinant in the characteristic Eq. 19.21-c is expanded, the cubic equation takes the form

$$\boxed{\lambda^3 - I_1\lambda^2 - I_2\lambda - I_3 = 0} \tag{19.20}$$

10 Similarly, we can determine the invariants of the deviatoric stresses from

$$\begin{vmatrix} s_{11}-\lambda & s_{12} & s_{13} \\ s_{21} & s_{22}-\lambda & s_{23} \\ s_{31} & s_{32} & s_{33}-\lambda \end{vmatrix} = 0 \tag{19.21-a}$$

$$|s_{rs} - \lambda\delta_{rs}| = 0 \tag{19.21-b}$$

$$|\boldsymbol{\sigma} - \lambda\mathbf{I}| = 0 \tag{19.21-c}$$

or

$$\boxed{\lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0} \tag{19.22}$$

11 The invariants are defined by

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} = \text{tr}\,\boldsymbol{\sigma} \tag{19.23-a}$$

$$I_2 = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 \tag{19.23-b}$$

$$= \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) = \frac{1}{2}\sigma_{ij}\sigma_{ij} - \frac{1}{2}I_\sigma^2 \tag{19.23-c}$$

$$= \frac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_\sigma^2) \tag{19.23-d}$$

$$I_3 = \det\boldsymbol{\sigma} = \frac{1}{6}e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} \tag{19.23-e}$$

12 In terms of the principal stresses, those invariants can be simplified into

$$\boxed{\begin{array}{lcl} I_1 &=& \sigma_1+\sigma_2+\sigma_3 \quad (19.24)\\ I_2 &=& -(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \quad (19.25)\\ I_3 &=& \sigma_1\sigma_2\sigma_3 \quad (19.26)\end{array}}$$

13 Similarly,

$$\boxed{\begin{array}{lcl} J_1 &=& s_1+s_2+s_3 \quad (19.27)\\ J_2 &=& -(s_1s_2+s_2s_3+s_3s_1) \quad (19.28)\\ J_3 &=& s_1s_2s_3 \quad (19.29)\end{array}}$$

19.4.1.2 Physical Interpretations of Stress Invariants

14 If we consider a plane which makes equal angles with respect to each of the principal-stress directions, π plane, or **octahedral plane**, the normal to this plane is given by

$$\mathbf{n} = \frac{1}{\sqrt{3}}\left\{\begin{array}{c}1\\1\\1\end{array}\right\} \tag{19.30}$$

The vector of traction on this plane is

$$\mathbf{t}_{oct} = \frac{1}{\sqrt{3}}\left\{\begin{array}{c}\sigma_1\\\sigma_2\\\sigma_3\end{array}\right\} \tag{19.31}$$

and the normal component of the stress on the octahedral plane is given by

$$\sigma_{oct} = \mathbf{t}_{oct}\cdot\mathbf{n} = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{1}{3}\mathbf{I}_1 = \sigma_{hyd} \tag{19.32}$$

or

$$\boxed{\sigma_{oct} = \frac{1}{3}\mathbf{I}_1} \tag{19.33}$$

15 Finally, the octahedral shear stress is obtained from

$$\tau_{oct}^2 = |\mathbf{t}_{oct}|^2 - \sigma_{oct}^2 = \frac{\sigma_1^2}{3}+\frac{\sigma_2^2}{3}+\frac{\sigma_3^2}{3}-\frac{(\sigma_1+\sigma_2+\sigma_3)^2}{9} \tag{19.34}$$

Upon algebraic manipulation, it can be shown that

$$9\tau_{oct}^2 = (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2 = 6J_2 \tag{19.35}$$

or

$$\boxed{\tau_{oct} = \sqrt{\frac{2}{3}J_2}} \tag{19.36}$$

and finally, the direction of the octahedral shear stress is given by

$$\cos 3\theta = \sqrt{2}\frac{J_3}{\tau_{oct}^3} \tag{19.37}$$

16 The **elastic strain energy** (total) per unit volume can be decomposed into two parts

$$U = U_1 + U_2 \tag{19.38}$$

where

$$U_1 = \frac{1-2\nu}{E} I_1^2 \quad \text{Dilational energy} \tag{19.39-a}$$

$$U_2 = \frac{1+\nu}{E} J_2 \quad \text{Distortional energy} \tag{19.39-b}$$

19.4.1.3 Geometric Representation of Stress States

`Adapted from (Chen and Zhang 1990)`

17 Using the three principal stresses σ_1, σ_2, and σ_3, as the coordinates, a three-dimensional stress space can be constructed. This stress representation is known as the **Haigh-Westergaard stress space**, Fig. 19.4.

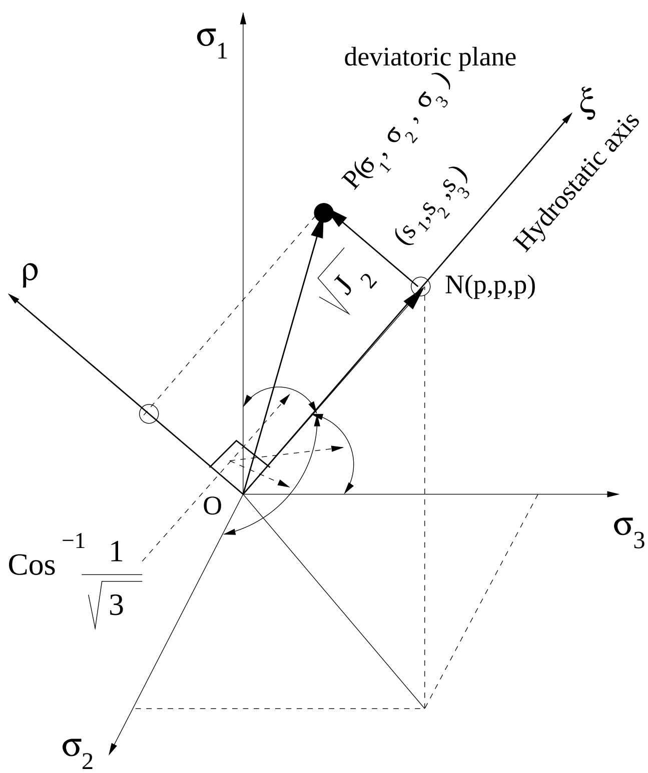


Figure 19.4: Haigh-Westergaard Stress Space

18 The decomposition of a stress state into a hydrostatic, $p\delta_{ij}$ and deviatoric s_{ij} stress components can be geometrically represented in this space. Considering an arbitrary stress state **OP** starting from $O(0,0,0)$ and ending at $P(\sigma_1, \sigma_2, \sigma_3)$, the vector **OP** can be decomposed into two components **ON** and **NP**. The former is along the direction of the unit vector $(1\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, and **NP**$\perp$**ON**.

19 Vector **ON** represents the hydrostatic component of the stress state, and axis $O\xi$ is called the **hydrostatic axis** ξ, and every point on this axis has $\sigma_1 = \sigma_2 = \sigma_3 = p$, or

$$\xi = \sqrt{3}p \tag{19.40}$$

20 Vector **NP** represents the deviatoric component of the stress state (s_1, s_2, s_3) and is perpendicular to the ξ axis. Any plane perpendicular to the hydrostatic axis is called the **deviatoric plane** and is expressed as

$$\frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) = \xi \tag{19.41}$$

and the particular plane which passes through the origin is called the π **plane** and is represented by $\xi = 0$. Any plane containing the hydrostatic axis is called a **meridian plane**. The vector **NP** lies in a meridian plane and has

$$\rho = \sqrt{s_1^2 + s_2^2 + s_3^2} = \sqrt{2J_2} \tag{19.42}$$

21 The projection of **NP** and the coordinate axes σ_i on a deviatoric plane is shown in Fig. 19.5. The

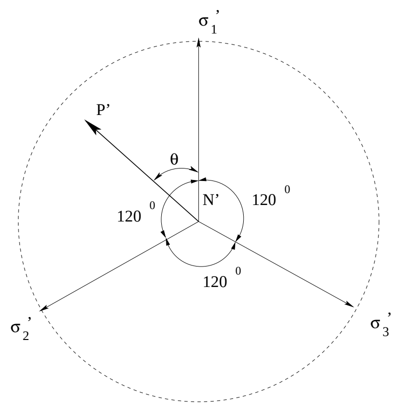


Figure 19.5: Stress on a Deviatoric Plane

projection of $\mathbf{N'P'}$ of **NP** on this plane makes an angle θ with the axis σ'_1.

$$\cos 3\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}} \tag{19.43}$$

22 The three new variables ξ, ρ and θ can all be expressed in terms of the principal stresses through their invariants. Hence, the general state of stress can be expressed either in terms of $(\sigma_1, \sigma_2, \sigma_3)$, or (ξ, ρ, θ). For $0 \leq \theta \leq \pi/3$, and $\sigma_1 \geq \sigma_2 \geq \sigma_3$, we have

$$\left\{\begin{array}{c}\sigma_1\\ \sigma_2\\ \sigma_3\end{array}\right\} = \left\{\begin{array}{c}p\\ p\\ p\end{array}\right\} + \frac{2}{\sqrt{3}}\sqrt{J_2}\left\{\begin{array}{c}\cos\theta\\ \cos(\theta - 2\pi/3)\\ \cos(\theta + 2\pi/3)\end{array}\right\} \tag{19.44-a}$$

$$= \left\{\begin{array}{c}\xi\\ \xi\\ \xi\end{array}\right\} + \sqrt{\frac{2}{3}}\rho\left\{\begin{array}{c}\cos\theta\\ \cos(\theta - 2\pi/3)\\ \cos(\theta + 2\pi/3)\end{array}\right\} \tag{19.44-b}$$

(19.44-c)

19.4.2 Hydrostatic Pressure Independent Models

`Adapted from (Chen and Zhang 1990)`

23 For hydrostatic pressure independent yield surfaces (such as for steel), their meridians are straigth lines parallel to the hydrostatic axis. Hence, shearing stress must be the major cause of yielding for

this type of materials. Since it is the magnitude of the shear stress that is important, and not its direction, it follows that the elastic-plastic behavior in tension and in compression should be equivalent for hydorstatic-pressure independent materials (such as steel). Hence, the cross-sectional shapes for this kind of materials will have six-fold symmetry, and $\rho_t = \rho_c$.

19.4.2.1 Tresca

24 Tresca criterion postulates that yielding occurs when the maximum shear stress reaches a limiting value k.

$$\boxed{\max\left(\frac{1}{2}|\sigma_1-\sigma_2|, \frac{1}{2}|\sigma_2-\sigma_3|, \frac{1}{2}|\sigma_3-\sigma_1|\right) = k} \tag{19.45}$$

from uniaxial tension test, we determine that $k = \sigma_y/2$ and from pure shear test $k = \tau_y$. Hence, in Tresca, tensile strength and shear strength are related by

$$\sigma_y = 2\tau_y \tag{19.46}$$

25 Tresca's criterion can also be represented as

$$2\sqrt{J_2}\sin\left(\theta+\frac{\pi}{3}\right) - \sigma_y = 0 \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{3} \tag{19.47}$$

26 Tresca is on, Fig. 19.6:

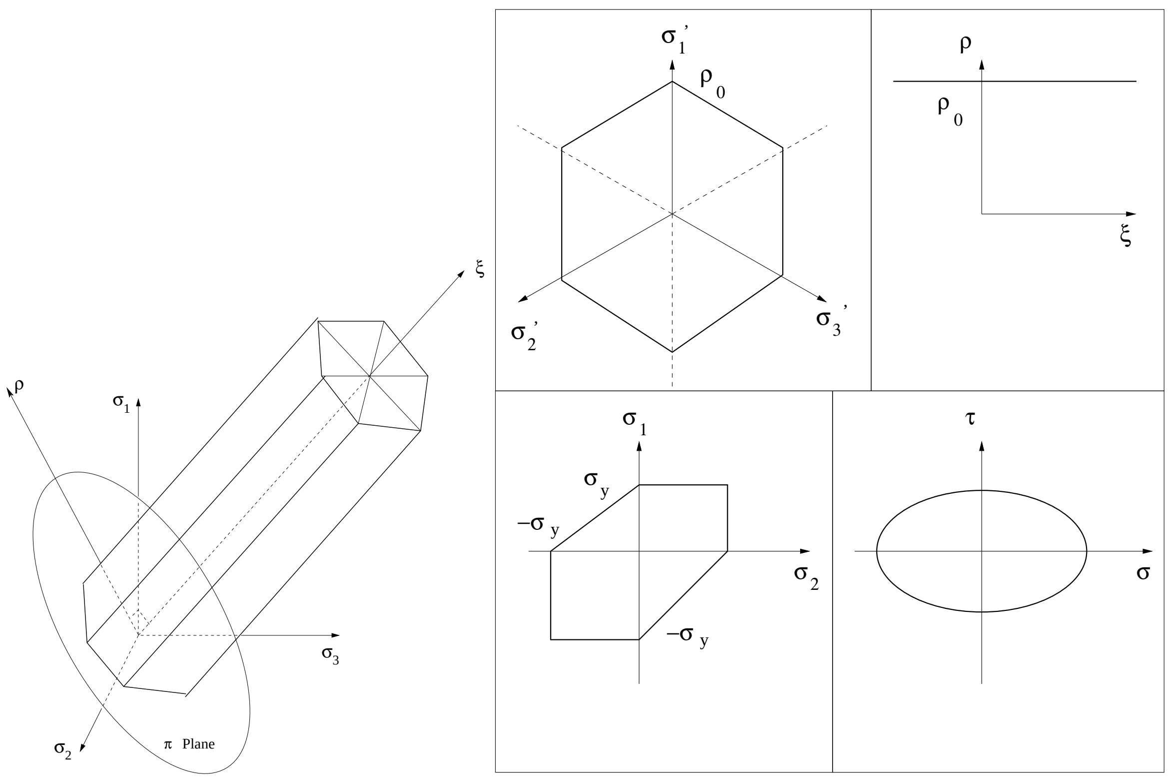


Figure 19.6: Tresca Criterion

- $\sigma_1\sigma_2\sigma_3$ **space** represented by an infinitly long regular hexagonal cylinder.

- π **(Deviatoric) Plane**, the yield criterion is

$$\rho = 2\sqrt{J_2} = \frac{\sigma_y}{\sqrt{2}\sin\left(\theta + \frac{\pi}{3}\right)} \quad \text{for} \quad 0 \leq \theta \leq \frac{\pi}{3} \tag{19.48}$$

  a regular hexagon with six singular corners.

- **Meridian plane:** a straight line parallel to the ξ axis.

- $\sigma_1\sigma_2$ **sub-space** (with $\sigma_3 = 0$) an irregular hexagon. Note that in the $\sigma_1 \geq 0, \sigma_2 \leq 0$ the yield criterion is

$$\sigma_1 - \sigma_2 = \sigma_y \tag{19.49}$$

- $\sigma\tau$ **sub-space** (with $\sigma_3 = 0$) is an ellipse

$$\left(\frac{\sigma}{\sigma_y}\right)^2 + \left(\frac{\tau}{\tau_y}\right)^2 = 1 \tag{19.50}$$

27 The Tresca criterion is the first one proposed, used mostly for elastic-plastic problems. However, because of the singular corners, it causes numerous problems in numerical analysis.

19.4.2.2 von Mises

28 There are two different physical interpretation for the von Mises criteria postulate:

1. Material will yield when the distorsional (shear) energy reaches the same critical value as for yield as in uniaxial tension.

$$F(J_2) = J_2 - k^2 = 0 \tag{19.51}$$

$$= \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}} - \sigma_y = 0 \tag{19.52}$$

2. ρ, or the octahedral shear stress (**CHECK**) τ_{oct}, the distance of the corresponding stress point from the hydrostatic axis, ξ is constant and equal to:

$$\rho_0 = \tau_y\sqrt{2} \tag{19.53}$$

29 Using Eq. 19.52, and from the uniaxial test, k is equal to $k = \sigma_y/\sqrt{3}$, and from pure shear test $k = \tau_y$. Hence, in von Mises, tensile strength and shear strength are related by

$$\sigma_y = \sqrt{3}\tau_y \tag{19.54}$$

Hence, we can rewrite Eq. 19.51 as

$$f(J_2) = J_2 - \frac{\sigma_y^2}{3} = 0 \tag{19.55}$$

30 von Mises is on, Fig. **??**:

- $\sigma_1\sigma_2\sigma_3$ **space** represented by an infinitly long regular circular cylinder.

- π **(Deviatoric) Plane**, the yield criterion is

$$\rho = \sqrt{\frac{2}{3}}\sigma_y \tag{19.56}$$

  a circle.

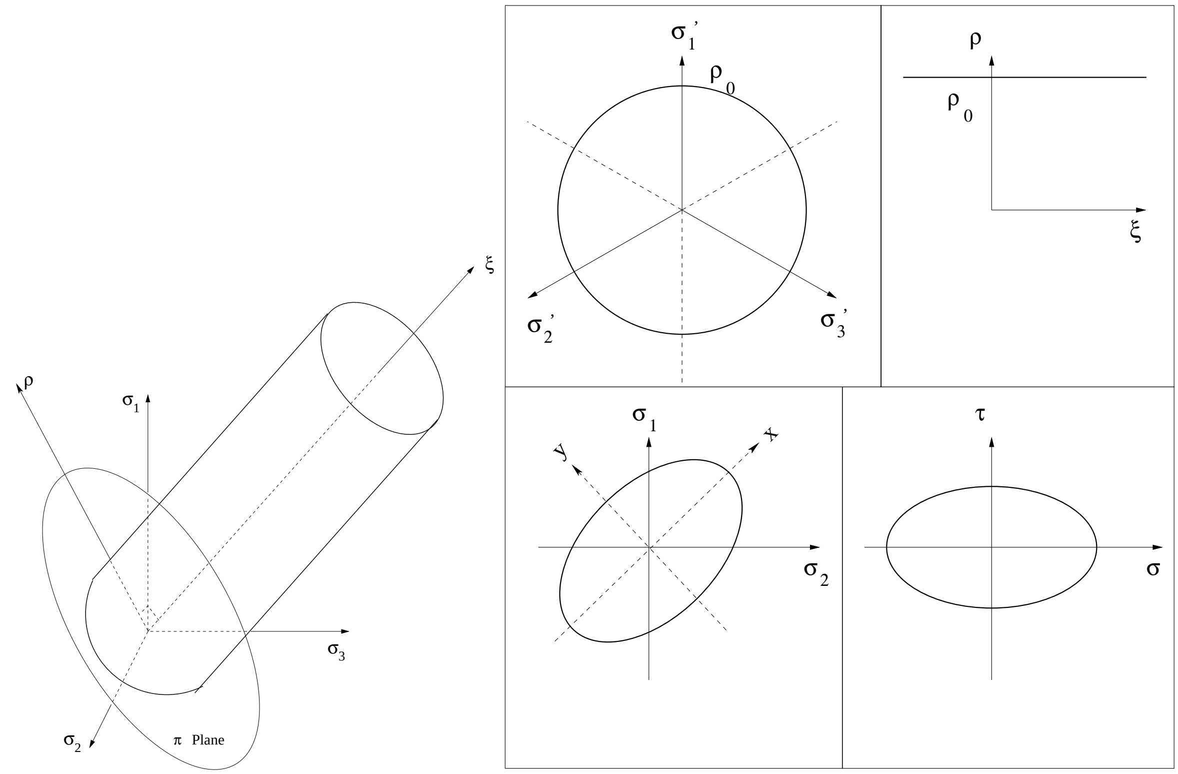


Figure 19.7: von Mises Criterion

- **Meridian plane:** a straight line parallel to the ξ axis.
- $\sigma_1\sigma_2$ **sub-space** (with $\sigma_3 = 0$) an ellipse

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2 \tag{19.57}$$

- $\sigma\tau$ **sub-space** (with $\sigma_3 = 0$) is an ellipse

$$\left(\frac{\sigma}{\sigma_y}\right)^2 + \left(\frac{\tau}{\tau_y}\right)^2 = 1 \tag{19.58-a}$$

Note that whereas this equation is similar to the corresponding one for Tresca, Eq. 19.50, the difference is in the relationships between σ_y and τ_y.

19.4.3 Hydrostatic Pressure Dependent Models

31 Pressure sensitive frictional materials (such as soil, rock, concrete) need to consider the effects of both the first and second stress invariants. frictional materials such as concrete.

32 The cross-sections of a yield surface are the intersection curves between the yield surface and the deviatoric plane (ρ, θ) which is perpendicular to the hydrostatic axis ξ and with ξ = constant. The cross-sectional shapes of this yield surface will have threefold symmetry, Fig. 19.8.

33 The meridians of a yield surface are the intersection curves between the surface and a meridian plane (ξ, ρ) which contains the hydrostatic axis. The meridian plane with $\theta = 0$ is the **tensile meridian**, and passes through the uniaxial tensile yield point. The meridian plane with $\theta = \pi/3$ is the **compressive meridian** and passes through the uniaxial compression yield point.

34 The radius of a yield surface on the tensile meridian is ρ_t, and on the compressive meridian is ρ_c.

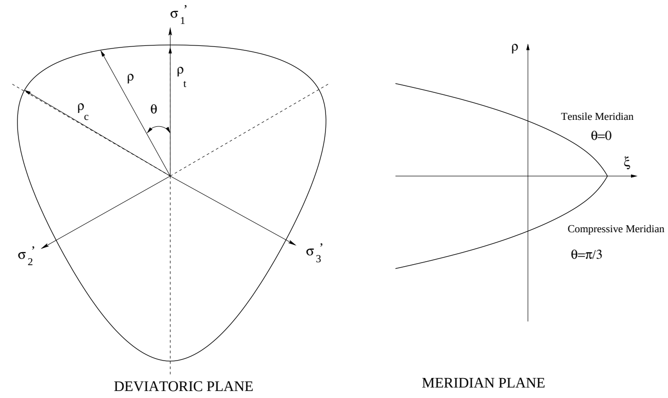


Figure 19.8: Pressure Dependent Yield Surfaces

19.4.3.1 Rankine

35 The Rankine criterion postulates that yielding occur when the maximum principal stress reaches the tensile strength.

$$\boxed{\sigma_1 = \sigma_y; \qquad \sigma_2 = \sigma_y; \qquad \sigma_3 = \sigma_y;} \tag{19.59}$$

36 Rankine is on, Fig. 19.9:

- $\sigma_1\sigma_2\sigma_3$ **space** represented by XXX
- π **(Deviatoric) Plane**, the yield criterion is

$$\begin{cases} \rho_t &= \frac{1}{\sqrt{2}}(\sqrt{3}\sigma_Y - \xi) \\ \rho_c &= \sqrt{2}(\sqrt{3}\sigma_y - \xi) \end{cases} \tag{19.60}$$

  a regular triangle.

- **Meridian plane:** Two straight lines which intersect the ξ axis $\xi_y = \sqrt{3}\sigma_y$
- $\sigma_1\sigma_2$ **sub-space** (with $\sigma_3 = 0$) two straight lines.

$$\begin{cases} \sigma + 1 &= \sigma_y \\ \sigma_2 &= \sigma_y \end{cases} \tag{19.61}$$

- $\sigma\tau$ **sub-space** (with $\sigma_3 = 0$) is a parabola

$$\frac{\sigma}{\sigma_y} + \left(\frac{\tau}{\sigma_y}\right)^2 = 1 \tag{19.62-a}$$

19.4.3.2 Mohr-Coulomb

37 The Mohr-Coulomb criteria can be considered as an extension of the Tresca criterion. The maximum shear stress is a constant plus a function of the normal stress acting on the same plane.

$$\boxed{|\tau| = c - \sigma \tan\phi} \tag{19.63}$$

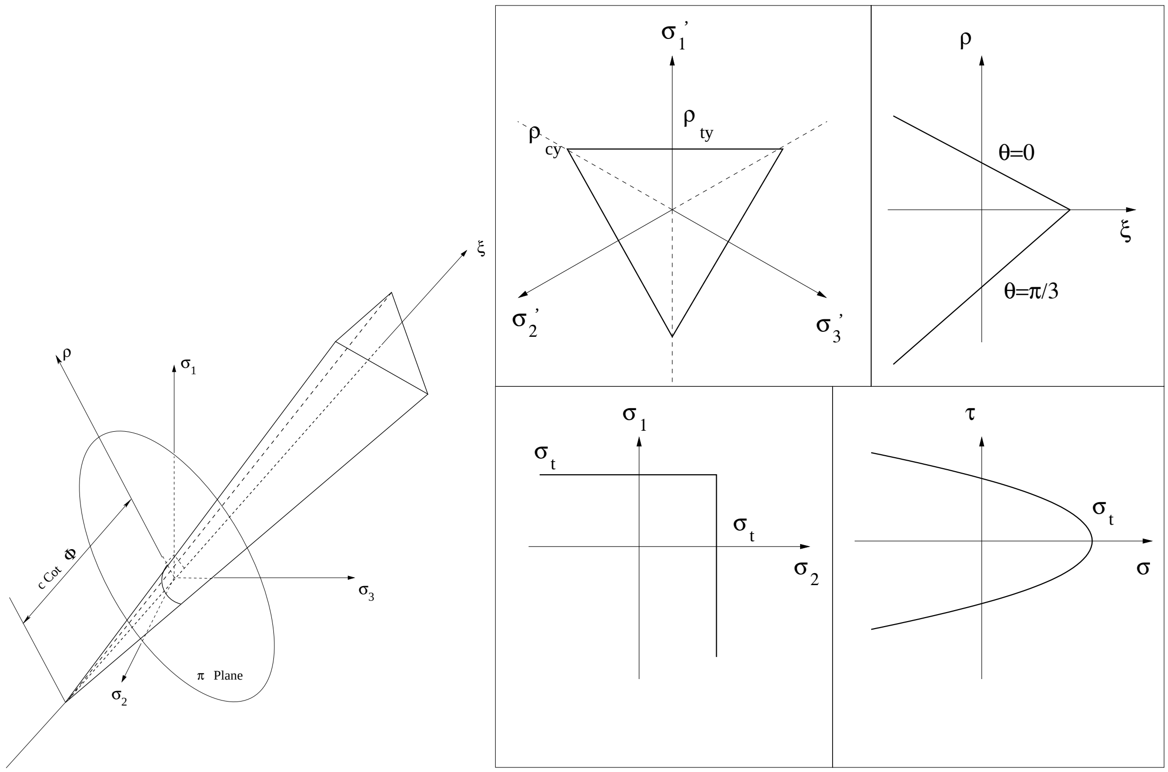


Figure 19.9: Rankine Criterion

where c is the cohesion, and ϕ the angle of internal friction.

38 Both c and ϕ are material properties which can be calibrated from uniaxial tensile and uniaxail compressive tests.

$$\left\{ \begin{array}{lcl} \sigma_t & = & \frac{2c\cos\phi}{1+\sin\phi} \\ \sigma_c & = & \frac{2c\cos\phi}{1-\sin\phi} \end{array} \right. \tag{19.64}$$

39 In terms of invariants, the Mohr-Coulomb criteria can be expressed as:

$$\frac{1}{3}I_1\sin\phi + \sqrt{J_2}\sin\left(\theta+\frac{\pi}{3}\right) + \sqrt{\frac{J_2}{3}}\cos\left(\theta+\frac{\pi}{3}\right)\sin\phi - c\cos\phi = 0 \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{3} \tag{19.65}$$

40 Mohr-Coulomb is on, Fig. 19.10:

- $\sigma_1\sigma_2\sigma_3$ **space** represented by a conical yield surface whose normal section at any point is an irregular hexagon.
- π **(Deviatoric) Plane**, the cross-section of the surface is an irregular hexagon.
- **Meridian plane:** Two straight lines which intersect the ξ axis $\xi_y = 2\sqrt{3}c/\tan\phi$, and the two characteristic lengths of the surface on the deviatoric and meridian planes are

$$\left\{ \begin{array}{lcl} \rho_t & = & \frac{2\sqrt{6}c\cos\phi - 2\sqrt{2}\xi\sin\phi}{3+\sin\phi} \\ \rho_c & = & \frac{2\sqrt{6}c\cos\phi - 2\sqrt{2}\xi\sin\phi}{3-\sin\phi} \end{array} \right. \tag{19.66}$$

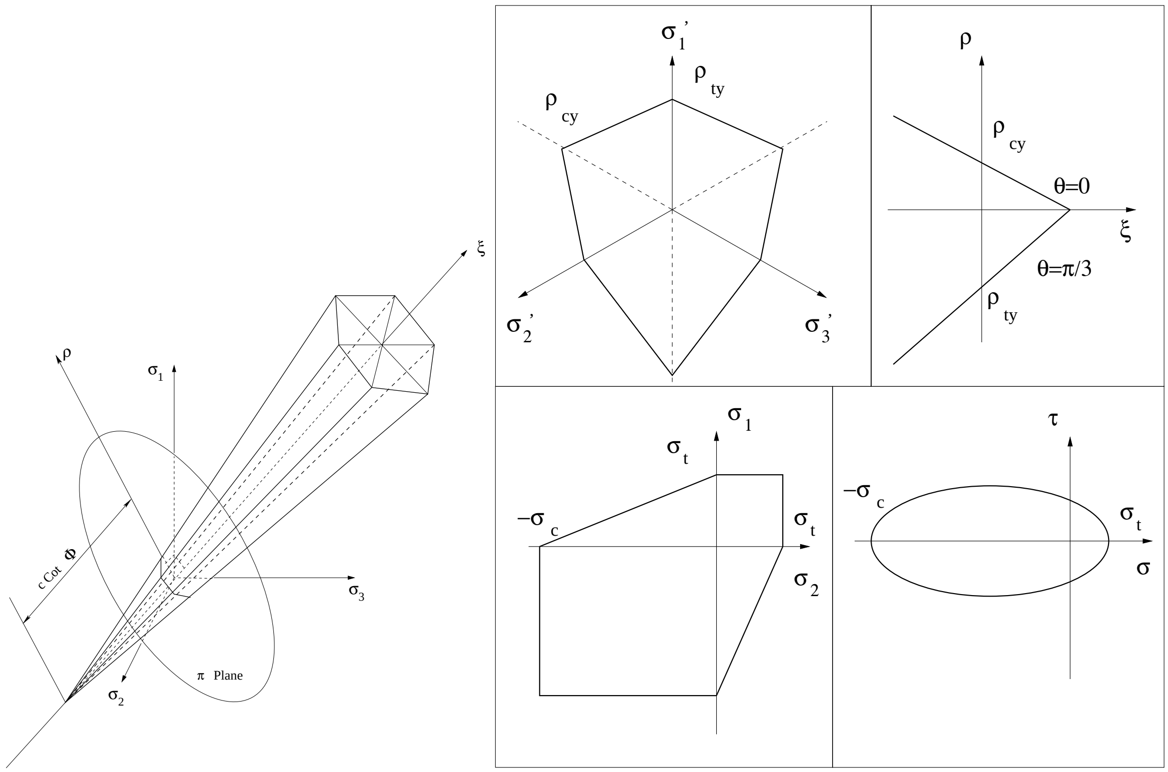


Figure 19.10: Mohr-Coulomb Criterion

- $\sigma_1\sigma_2$ **sub-space** (with $\sigma_3 = 0$) the surface is an irregular hexagon. In the quarter $\sigma_1 \geq 0, \sigma_2 \leq 0$, of the plane, the criterios is

$$m\sigma_1 - \sigma_2 = \sigma_c \tag{19.67}$$

where

$$m = \frac{\sigma_c}{\sigma_t} = \frac{1+\sin\phi}{1-\sin\phi} \tag{19.68}$$

- $\sigma\tau$ **sub-space** (with $\sigma_3 = 0$) is an ellipse

$$\left(\frac{\sigma + \frac{m-1}{2m}\sigma_c}{\frac{m+1}{2m}\sigma_c}\right)^2 + \left(\frac{\tau}{\frac{\sqrt{m}}{2m}\sigma_c}\right)^2 \tag{19.69-a}$$

19.4.3.3 Drucker-Prager

41 The Drucker-Prager postulates is a simple extension of the von Mises criterion to include the effect of hydrostatic pressure on the yielding of the materials through I_1

$$\boxed{F(I_1, J_2) = \alpha I_1 + J_2 - k} \tag{19.70}$$

The strength parameters α and k can be determined from the uni axial tension and compression tests

$$\left\{ \begin{array}{lcl} \sigma_t & = & \frac{\sqrt{3}k}{1+\sqrt{3}\alpha} \\ \sigma_c & = & \frac{\sqrt{3}k}{1-\sqrt{3}\alpha} \end{array} \right. \tag{19.71}$$

or

$$\begin{cases} \alpha = \frac{m-1}{\sqrt{3}(m+1)} \\ k = \frac{2\sigma_c}{\sqrt{3}(m+1)} \end{cases} \tag{19.72}$$

42 Drucker-Prager is on, Fig. 19.11:

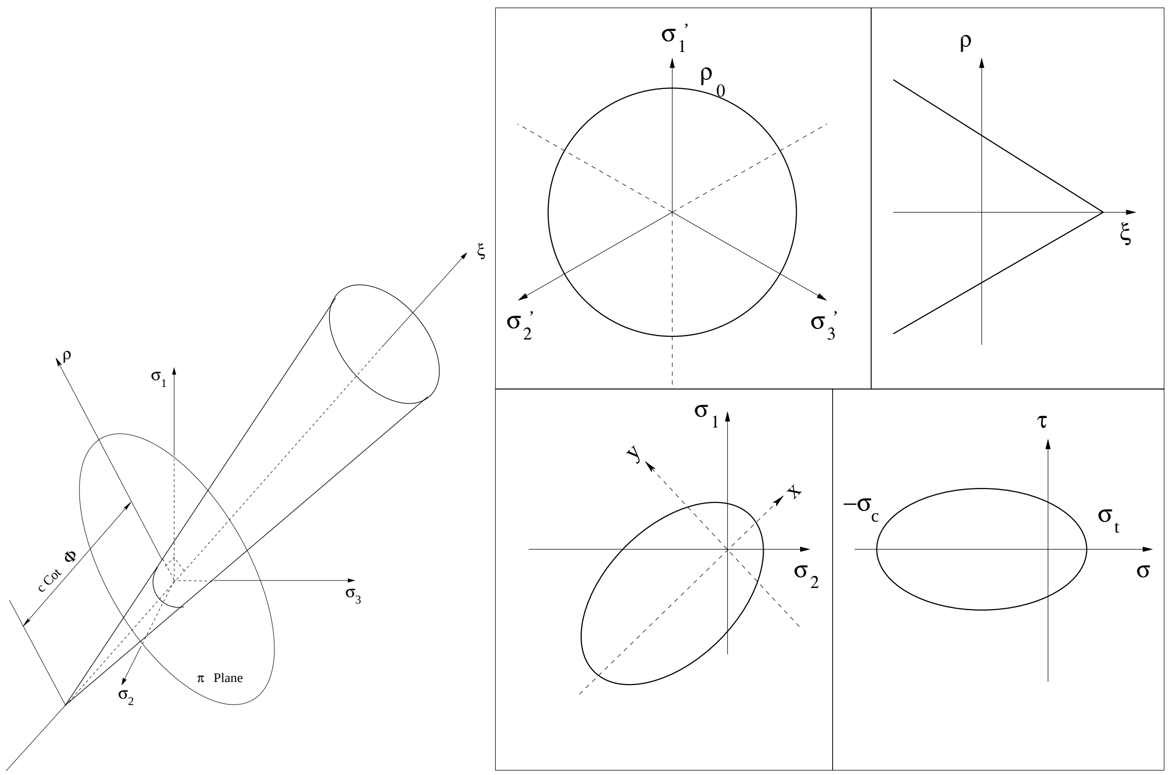


Figure 19.11: Drucker-Prager Criterion

- $\sigma_1\sigma_2\sigma_3$ **space** represented by a circular cone.
- π **(Deviatoric) Plane**, the cross-section of the surface is a circle of radius ρ.

$$\rho = \sqrt{2}k - \sqrt{6}\alpha\xi \tag{19.73}$$

- **Meridian plane:** The meridians of the surface are straight lines which intersect with the ξ axis at $\xi_y = k/\sqrt{3}\alpha$.
- $\sigma_1\sigma_2$ **sub-space** (with $\sigma_3 = 0$) the surface is an ellipse

$$\left(\frac{x + \frac{6\sqrt{2}k\alpha}{1-12\alpha^2}}{\frac{\sqrt{6}k}{1-12\alpha^2}}\right)^2 + \left(\frac{y}{\frac{\sqrt{2}k}{\sqrt{1-12\alpha^2}}}\right)^2 \tag{19.74}$$

where

$$x = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2) \tag{19.75-a}$$

$$y = \frac{1}{\sqrt{2}}(\sigma_2 - \sigma_1) \tag{19.75-b}$$

- $\sigma\tau$ **sub-space** (with $\sigma_3 = 0$) is also an ellipse

$$\left(\frac{\sigma + \frac{3k\alpha}{1-3\alpha^2}}{\frac{\sqrt{3}k}{1-3\alpha^2}}\right)^2 + \left(\frac{\tau}{\frac{k}{\sqrt{1-3\alpha^2}}}\right)^2 = 1 \tag{19.76-a}$$

43 In order to make the Drucker-Prager circle coincide with the Mohr-Coulomb hexagon at any section

$$\text{Outer} \quad \left\{ \begin{array}{rcl} \alpha &=& \frac{2\sin\phi}{\sqrt{3}(3-\sin\phi)} \\ k &=& \frac{6c\cos\phi}{\sqrt{3}(3-\sin\phi)} \end{array} \right. \tag{19.77-a}$$

$$\text{Inner} \quad \left\{ \begin{array}{rcl} \alpha &=& \frac{2\sin\phi}{\sqrt{3}(3+\sin\phi)} \\ k &=& \frac{6c\cos\phi}{\sqrt{3}(3+\sin\phi)} \end{array} \right. \tag{19.77-b}$$

19.5 Plastic Potential

44 Once a material yields, it exhibits permanent deformations through the generation of plastic strains. According to the flow theory of plasticity, the rate of generation of these plastic strains is governed by the flow rule. In order to define the direction of the plastic flow (which in turn determines the magnitudes of the plastic strain components), it must be assumed that a scalar plastic potential function Q exists such that $Q = Q(\boldsymbol{\sigma}_p, \varepsilon_p)$.

45 In the case of an associated flow rule, $Q = F$, and in the case of a non-associated flow rule, $Q \neq F$. Since the plastic potential function helps define the plastic strain rate, it is often advantageous to define a non-associated flow rule in order to control the amount of plastic strain generated by the plasticity formulation. For example, when modeling plain concrete, excess plastic strains may lead to excess dilatancy (too much volume expansion) which is undesirable. The plastic potential function can be formulated to decrease the plastic strain rate, producing better results.

19.6 Plastic Flow Rule

46 We have established a yield criterion. When the stress is inside the yield surface, it is elastic, Hooke's law is applicable, strains are recoverable, and there is no dissipation of energy. However, when the load on the structure pushes the stress tensor to be beyond the yield surface, the stress tensor locks up on the yield surface, and the structure deforms plastically (if the material exhibits hardening as opposed to elastic-perfectly plastic response, then the yield surface expands or moves with the stress point still on the yield surface). At this point, the crucial question is what will be direction of the plastic flow (that is the relative magnitude of the components of $\boldsymbol{\varepsilon}^P$. This question is addressed by the **flow rule**, or **normality rule**.

47 We will assume that the direction of the plastic flow is given by a unit vector $\mathbf{m}$, thus the incremental plastic strain is written as

$$\boxed{\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}_p \underbrace{\frac{\partial Q}{\partial \boldsymbol{\sigma}}}_{\mathbf{m}_p}} \tag{19.78}$$

where $\dot{\lambda}_p$ is the plastic multiplier which scales the unit vector $\mathbf{m}_p$, in the direction of the plastic flow evolution, to give the actual plastic strain in the material. Note analogy with Eq. 18.42.

48 We now must determine **m**, it is clearly a function of the stress state, and for convenience we represent this vector as the gradient of a scalar potential Q which itself is a function of of the stresses

$$\mathbf{m} = \frac{\partial Q}{\partial \boldsymbol{\sigma}} = \lfloor \begin{array}{cccccc} \frac{\partial Q}{\partial \sigma_{11}} & \frac{\partial Q}{\partial \sigma_{22}} & \frac{\partial Q}{\partial \sigma_{33}} & \frac{\partial Q}{\partial \sigma_{12}} & \frac{\partial Q}{\partial \sigma_{23}} & \frac{\partial Q}{\partial \sigma_{31}} \end{array} \rfloor^T \tag{19.79}$$

where Q is called the **plastic flow potential** and is yet to be defined. Hence, the direction of plastic flow is always perpendicular to the plastic potential function.

49 We have two cases

Non-Associated Flow when $F_p \neq Q_p$ which corresponds to the general case.

Associated Flow when when $F_p = Q_p$ which is a special case. This gives rise to the **Associated flow rule**

$$\boxed{\dot{\boldsymbol{\epsilon}}_p = \dot{\lambda}_p \frac{\partial F_p}{\partial \boldsymbol{\sigma}}} \tag{19.80}$$

In this context, the difficulty in determining the normal of a yield surface with a sharp corner should be noted.

50 The incremental plastic work (irrecoverable) is given by

$$\mathrm{d}W^P = \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}}^P \tag{19.81}$$

which can be rewritten as

$$\mathrm{d}W^P = \boldsymbol{\sigma}\cdot\lambda\mathbf{m} \tag{19.82}$$

51 Hence, the evolution of the stress state $\boldsymbol{\sigma}$ is given by the stress rate relation,

$$\boxed{\dot{\boldsymbol{\sigma}} = \mathbf{E}_o : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p)} \tag{19.83}$$

19.7 Post-Yielding

19.7.1 Kuhn-Tucker Conditions

52 In the elastic regime, the yield function F must remain negative, and the rate of plastic multiplier $\dot{\lambda}_p$ is zero. On the other hand, during plastic flow the yield function must be equal to zero while the rate of plastic multiplier is positive.

$$\left\{ \begin{array}{lcl} \text{plastic loading} & : & \dot{F}_p = 0; \quad \dot{\lambda}_p > 0 \\ \text{elastic (un)loading} & : & \dot{F}_p < 0; \quad \dot{\lambda}_p = 0 \end{array} \right. \tag{19.84}$$

53 Hence, both cases can be simultaneously covered by the loading-unloading conditions called **Kuhn-Tucker conditions**

$$\boxed{\dot{F}_p \leq 0, \quad \dot{\lambda}_p \geq 0 \quad \text{and} \quad \dot{F}_p \dot{\lambda}_p = 0} \tag{19.85}$$

19.7.2 Hardening Rules

54 A hardening rule describes a specific relationship between the subsequent yield stress σ_y of a material and the plastic deformation accumulated during prior loadings.

55 We define a **hardening parameter** or **plastic internal variable**, which is often denoted by κ.

$$\kappa = \varepsilon_p = \int \sqrt{d\varepsilon^p d\varepsilon^p} \quad \text{Equivalent Plastic Strain} \tag{19.86-a}$$
$$\kappa = W_p = \int \sigma d\varepsilon^p \quad \text{Plastic Work} \tag{19.86-b}$$
$$\kappa = \varepsilon^p = \int d\varepsilon^p \quad \text{Plastic Strain} \tag{19.86-c}$$

56 A **hardening rule** expresses the relationship of the subsequent yield stress σ_y, tangent modulus E_t and plastic modulus E_p with the hardening parameter κ.

19.7.2.1 Isotropic Hardening

57 In isotropic hardening, the yield surface simply increases in size but maintains its original shape. Hence, the the progressively increasing yield stresses under both tension and compression loadings are always the same.

$$|\sigma| = |\sigma(\kappa)| \tag{19.87}$$

19.7.2.2 Kinematic Hardening

58 In kinematic hardening, the initial yield surface is translated to a new location in stress space without change in size or shape. Hence, the difference between the yield stresses under tension loading and under compression loading remains constant. If we denote by σ_y^t and σ_y^c the yield stress under tension and compression respectively, then

$$\sigma_y^t(\kappa) - \sigma_y^c(\kappa) = 2\sigma_{y0} \tag{19.88}$$

or alternatively

$$|\sigma - c(\kappa)| = \sigma_{y0} \tag{19.89}$$

where $c(\kappa)$ represents the track of the elastic center and satisfies $c(0) = 0$.

59 Kinematic hardening accounts for the Baushinger effect, Fig. 16.2.

19.7.3 Consistency Condition

60 Fig. 19.12 illustrates the elastic $d\epsilon^e$ and plastic $d\epsilon^p$ strain increments for a given stress increment $d\sigma$. Unloading always follows the initial elastic stiffness E^o. The material experiences plastic loading once the stress state exceeds the yield surface. Further loading results in the development of plastic strains and stresses. However, the total stress state cannot exceed the yield surface. Thus, during plastic flow the stress must remain on the yield surface, and hence the time derivative of the yield function must vanish whenever and this limit is enforced through the consistency condition,

$$\frac{\mathrm{d}}{\mathrm{d}t} F_p(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}^p, \kappa) = \frac{\mathrm{d}}{\mathrm{d}t} F_p(\boldsymbol{\sigma}, \boldsymbol{\lambda}, \kappa) = 0 \tag{19.90}$$

or

$$\underbrace{\frac{\partial F_p}{\partial \boldsymbol{\sigma}}}_{\mathbf{n}_p} : \dot{\boldsymbol{\sigma}} + \underbrace{\frac{\partial F_p}{\partial \lambda_p}}_{-H_p} : \dot{\lambda}_p + \frac{\partial F_p}{\partial \kappa}\frac{\partial \kappa}{\partial \lambda} : \dot{\lambda}_p = 0 \tag{19.91}$$

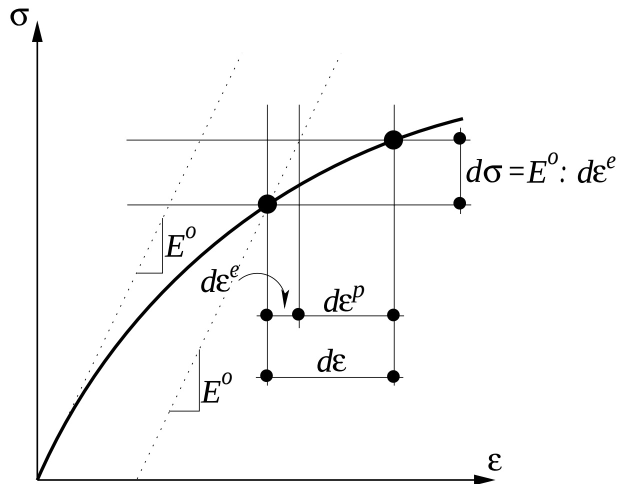


Figure 19.12: Elastic and plastic strain increments

Ignoring the last term, this relation simplifies if the normal to the yield surface $\mathbf{n}_p$ and the hardening parameter H_p are substituted, such that

$$\boxed{\dot{F}_p = \mathbf{n}_p : \dot{\boldsymbol{\sigma}} - H_p\dot{\lambda}_p = 0} \tag{19.92}$$

61 The consistency condition states that during persistent plastic flow the stresses remain on the yield surface (since the rate of change of the yield function must be equal to zero).

62 An expression for the plastic multiplier $\dot{\lambda}_p$ can be attained by the combination of the simplified consistency condition Eq. 19.92 with the stress rate Eq. 19.83 and the flow rule Eq. 19.78:

$$\dot{F}_p = \mathbf{n} : \underbrace{\mathbf{E}_o : (\dot{\boldsymbol{\epsilon}} - \overbrace{\dot{\lambda}_p\mathbf{m}_p}^{\dot{\boldsymbol{\epsilon}}_p})}_{\dot{\boldsymbol{\sigma}}} - H_p\dot{\lambda}_p = 0$$

$$\dot{\lambda}_p = \frac{\mathbf{n}_p : \mathbf{E}_o : \dot{\boldsymbol{\epsilon}}}{H_p + \mathbf{n}_p : \mathbf{E}_o : \mathbf{m}_p} \geq 0 \tag{19.93}$$

63 The hardening parameter H_p, defined above as $H_p = -\partial F_p/\partial\lambda_p$, provides insight into the state of the material in the plastic regime depending upon its sign:

$$\begin{cases} \text{hardening} & : \quad H_p > 0 \\ \text{perfect plasticity} & : \quad H_p = 0 \\ \text{softening} & : \quad H_p < 0 \end{cases} \tag{19.94}$$

19.8 Elasto-Plastic Stiffness Relation

64 Substituting the plastic multiplier expression of Eq. 19.93 into the stress rate expression Eq. 19.83 results in the following expression relating the stress and strain rate:

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_o : \left(\dot{\boldsymbol{\epsilon}} - \mathbf{m}_p : \frac{\mathbf{n}_p : \mathbf{E}_o : \mathbf{m}_p}{H_p + \mathbf{n}_p : \mathbf{E}_o : \mathbf{m}_p} : \dot{\boldsymbol{\epsilon}}\right) \tag{19.95-a}$$

$$\dot{\boldsymbol{\sigma}} = \left(\mathbf{E}_o - \frac{\mathbf{E}_o : \mathbf{m}_p \otimes \mathbf{n}_p : \mathbf{E}_o}{H_p + \mathbf{n}_p : \mathbf{E}_o : \mathbf{m}_p}\right) : \dot{\boldsymbol{\epsilon}} \tag{19.95-b}$$

from which the **elastoplastic tangent operator** may be identified as

$$\boxed{\mathbf{E}_p = \mathbf{E}_o - \frac{\mathbf{E}_o : \mathbf{m}_p \otimes \mathbf{n}_p : \mathbf{E}_o}{H_p + \mathbf{n}_p : \mathbf{E}_o : \mathbf{m}_p}} \tag{19.96}$$

19.9 †Case Study: J_2 Plasticity/von Mises Plasticity

65 For J_2 plasticity or von Mises plasticity, our stress function is perfectly plastic. Recall perfectly plastic materials have a total modulus of elasticity (E_T) which is equivalent to zero. We will deal now with deviatoric stress and strain for the J_2 plasticity stress function.

1. Yield function:
$$F(\boldsymbol{s}) = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} - \frac{1}{3}\sigma_y^2 = 0 \tag{19.97}$$

2. Flow rate (associated):
$$\dot{\boldsymbol{e}}_p = \dot{\lambda}\frac{\partial Q_p}{\partial \boldsymbol{s}} = \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{s}} = \dot{\lambda}\boldsymbol{s} \tag{19.98}$$

3. Consistency condition ($\dot{F} = 0$):
$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{s}} : \dot{\boldsymbol{s}} + \frac{\partial F}{\partial \boldsymbol{q}} : \dot{\boldsymbol{q}} = 0 \tag{19.99}$$
since $\dot{\boldsymbol{q}} = 0$ in perfect plasticity, the second term drops out and $\dot{F}$ becomes
$$\dot{F} = \boldsymbol{s} : \dot{\boldsymbol{s}} = 0 \tag{19.100}$$
Recall that
$$\dot{\boldsymbol{s}} = 2G : \dot{\boldsymbol{e}}_e = 2G : [\dot{\boldsymbol{e}} - \dot{\boldsymbol{e}}_p] \tag{19.101}$$
finally substituting $\dot{\boldsymbol{e}}_p$ in
$$\dot{\boldsymbol{s}} = 2G : [\dot{\boldsymbol{e}} - \dot{\lambda}\boldsymbol{s}] \tag{19.102}$$
substituting $\dot{\boldsymbol{s}}$ back into (19.100)
$$\dot{F} = 2G\boldsymbol{s} : [\dot{\boldsymbol{e}} - \dot{\lambda}\boldsymbol{m}] = 0 \tag{19.103}$$
and solving for $\dot{\lambda}$
$$\dot{\lambda} = \frac{\boldsymbol{s} : \dot{\boldsymbol{\epsilon}}}{\boldsymbol{s} : \boldsymbol{s}} \tag{19.104}$$

4. Tangential stress-strain relation(deviatoric):
$$\dot{\boldsymbol{s}} = 2G : [\dot{\boldsymbol{e}} - \frac{\boldsymbol{s} : \dot{\boldsymbol{e}}}{\boldsymbol{s} : \boldsymbol{s}}\boldsymbol{s}] \tag{19.105}$$
then by factoring $\dot{\boldsymbol{e}}$ out
$$\dot{\boldsymbol{s}} = 2G : [\boldsymbol{I}_4 - \frac{\boldsymbol{s} \otimes \boldsymbol{s}}{\boldsymbol{s} : \boldsymbol{s}}] : \dot{\boldsymbol{e}} \tag{19.106}$$
Now we have the simplified expression
$$\dot{\boldsymbol{s}} = \boldsymbol{G}_{ep} : \dot{\boldsymbol{e}} \tag{19.107}$$
where
$$\boldsymbol{G}_{ep} = 2G : [\boldsymbol{I}_4 - \frac{\boldsymbol{s} \otimes \boldsymbol{s}}{\boldsymbol{s} : \boldsymbol{s}}] \tag{19.108}$$
is the 4th order elastoplastic shear modulus tensor which relates deviatoric stress rate to deviatoric strain rate.

5. Solving for $\boldsymbol{E}_{ep}$ in order to relate regular stress and strain rates:
   Volumetric response in purely elastic

$$\mathrm{tr}\,(\dot{\boldsymbol{\sigma}}) = 3K\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) \tag{19.109}$$

   altogether

$$\dot{\boldsymbol{\sigma}} = \frac{1}{3}\mathrm{tr}\,(\dot{\boldsymbol{\sigma}}) : \boldsymbol{I}_2 + \dot{\boldsymbol{s}} \tag{19.110}$$

$$\dot{\boldsymbol{\sigma}} = K\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) : \boldsymbol{I}_2 + \boldsymbol{G}_{ep} : \dot{\boldsymbol{e}} \tag{19.111}$$

$$\dot{\boldsymbol{\sigma}} = K\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) : \boldsymbol{I}_2 + \boldsymbol{G}_{ep} : [\dot{\boldsymbol{\epsilon}} - \frac{1}{3}\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) : \boldsymbol{I}_2] \tag{19.112}$$

$$\dot{\boldsymbol{\sigma}} = K\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) : \boldsymbol{I}_2 + \boldsymbol{G}_{ep} : \dot{\boldsymbol{\epsilon}} - \frac{1}{3}\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}})\boldsymbol{G}_{ep} : \boldsymbol{I}_2 \tag{19.113}$$

$$\dot{\boldsymbol{\sigma}} = K\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}}) : \boldsymbol{I}_2 - \frac{2}{3}G\mathrm{tr}\,(\dot{\boldsymbol{\epsilon}})\boldsymbol{I}_2 + \boldsymbol{G}_{ep} : \dot{\boldsymbol{\epsilon}} \tag{19.114}$$

$$\dot{\boldsymbol{\sigma}} = K\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 : \dot{\boldsymbol{\epsilon}} - \frac{2}{3}G\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 : \dot{\boldsymbol{\epsilon}} + \boldsymbol{G}_{ep} : \dot{\boldsymbol{\epsilon}} \tag{19.115}$$

   and finally we have recovered the stress-strain relationship

$$\dot{\boldsymbol{\sigma}} = [[K - \frac{2}{3}G]\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 + \boldsymbol{G}_{ep}] : \dot{\boldsymbol{\epsilon}} \tag{19.116}$$

   where the elastoplastic material tensor is

$$\boldsymbol{E_{ep}} = [[K - \frac{2}{3}G]\boldsymbol{I}_2 \otimes \boldsymbol{I}_2 + \boldsymbol{G}_{ep}] \tag{19.117}$$

19.9.1 Isotropic Hardening/Softening(J_2- plasticity)

66 In isotropic hardening/softening the yield surface may shrink (softening) or expand (hardening) uniformly (see figure 19.13).

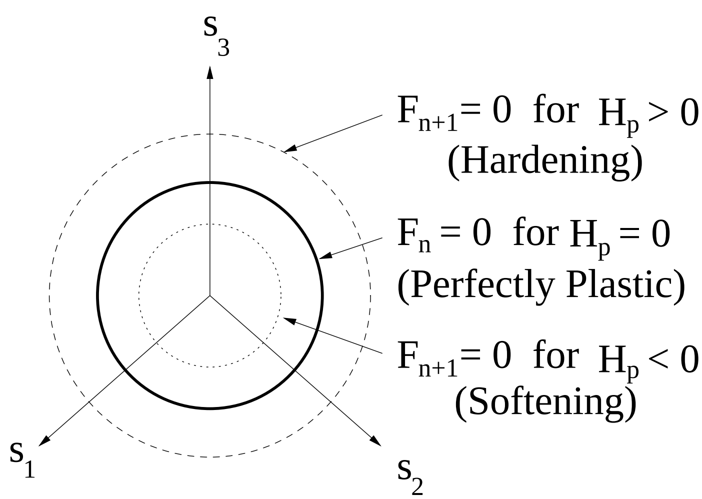


Figure 19.13: Isotropic Hardening/Softening

1. Yield function for linear strain hardening/softening:

$$F(\boldsymbol{s}, \epsilon^p_{eff}) = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} - \frac{1}{3}(\sigma^o_y + E_p\epsilon^p_{eff})^2 = 0 \tag{19.118}$$

2. Consistency condition:

$$\dot{F} = \frac{\partial F}{\partial \boldsymbol{s}} : \dot{\boldsymbol{s}} + \frac{\partial F}{\partial \boldsymbol{q}} : \frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\lambda}} \dot{\lambda} = 0 \tag{19.119}$$

from which we solve the plastic multiplier

$$\dot{\lambda} = \frac{2G\boldsymbol{s} : \dot{\boldsymbol{e}}}{2G\boldsymbol{s} : \boldsymbol{s} + \frac{2E_p}{3}(\sigma_y^o + E_p \epsilon_{eff}^p)\sqrt{\frac{2}{3}\boldsymbol{s} : \boldsymbol{s}}} \tag{19.120}$$

3. Tangential stress-strain relation(deviatoric):

$$\dot{\boldsymbol{s}} = \boldsymbol{G}_{ep} : \dot{\boldsymbol{e}} \tag{19.121}$$

where

$$\boldsymbol{G}_{ep} = 2G[\boldsymbol{I}_4 - \frac{2G\boldsymbol{s} \otimes \boldsymbol{s}}{G\boldsymbol{s} : \boldsymbol{s} + \frac{2E_p}{3}(\sigma_y^o + E_p \epsilon_{eff}^p)\sqrt{\frac{2}{3}\boldsymbol{s} : \boldsymbol{s}}}] \tag{19.122}$$

67 Note that isotropic hardening/softening is a poor representation of plastic behavior under cyclic loading because of the Bauschinger effect.

19.9.2 Kinematic Hardening/Softening(J_2- plasticity)

68 Kinematic hardening/softening, developed by Prager [1956], involves a shift of the origin of the yield surface (see figure 19.14). Here, kinematic hardening/softening captures the Bauschinger effect in a more realistic manner than the isotropic hardening/ softening.

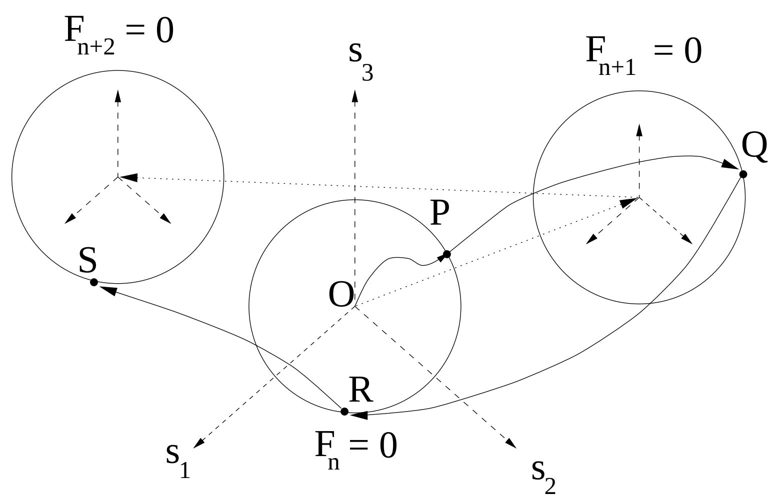


Figure 19.14: Kinematic Hardening/Softening

1. Yield function:

$$F(\boldsymbol{s}, \boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{s} - \boldsymbol{\alpha}) : (\boldsymbol{s} - \boldsymbol{\alpha}) - \frac{1}{3}\sigma_y^2 = 0 \tag{19.123}$$

2. Consistency condition (plastic multiplier):

$$\dot{\lambda} = \frac{2G(\boldsymbol{s} - \boldsymbol{\alpha}) : \dot{\boldsymbol{e}}}{(\boldsymbol{s} - \boldsymbol{\alpha}) : (\boldsymbol{s} - \boldsymbol{\alpha})[2G + C]} \tag{19.124}$$

where

$$C = E_p \tag{19.125}$$

and C is related to $\boldsymbol{\alpha}$, the backstress, by

$$\dot{\boldsymbol{\alpha}} = C\dot{\boldsymbol{e}} = \dot{\lambda}C(\boldsymbol{s} - \boldsymbol{\alpha}) \tag{19.126}$$

For perfectly plastic behavior $C = 0$ and $\boldsymbol{\alpha} = 0$.

3. Tangential stress-strain relation (deviatoric):

$$\dot{\boldsymbol{s}} = \boldsymbol{G}_{ep} : \dot{\boldsymbol{e}} \tag{19.127}$$

where

$$\boldsymbol{G}_{ep} = 2G[\boldsymbol{I}_4 - \frac{2G(\boldsymbol{s}-\boldsymbol{\alpha}) \otimes (\boldsymbol{s}-\boldsymbol{\alpha})}{(\boldsymbol{s}-\boldsymbol{\alpha}) : (\boldsymbol{s}-\boldsymbol{\alpha})[2G+C]}] \tag{19.128}$$

19.10 Computer Implementation

Written by Eric Hansen

```
SUBROUTINE pd_strain(Outfid,Logfid,Pstfid,Lclfid)

! PD_STRAIN - Strain controlled parabolic Drucker-Prager model
!
! Variables required
! ------------------
! Outfid = Output file ID
! Logfid = Log file ID
! Pstfid = Post file ID
! Lclfid = Localization file ID
!
! Variables returned = none
!
! Subroutine called by
! --------------------
! p_drucker.f90 = Parabolic Drucker-Rrager model
!
! Functions/subroutines called
! ----------------------------
! alloc8.f90       = Allocate memory space in array Kmn
! el_ten1.f90      = Construct 4th order elastic stiffness tensor
!
! Variable definition
! -------------------
! Eo_ten      = Elastic stiffness tensor
! Et_ten      = Continuum tangent stiffness tensor
! alpha       = Inverse damage-effect tensor
! alpha_bar   = Damage-effect tensor
! phi_inc     = Inverse integrity tensor for each load increment
! tr_sig      = Trial stress tensor
! tr_eps      = Trial strain tensor
! phibar_inc  = Integrity tensor for current load increment
! w_ten       = Square root inverse integrity tensor
! wbar_ten    = Square root integrity tensor
! y_hat_pr    = Principal values of conjugate tensile forces
!=====================================================================

  IMPLICIT NONE

!-----------------------------------------
! Define interface with subroutine alloc8
!-----------------------------------------
  INTERFACE
    SUBROUTINE alloc8(Logfid,nrows,ncols,ptr)
      IMPLICIT NONE
```

```
      INTEGER,INTENT(IN) :: Logfid,nrows,ncols
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptr
    END SUBROUTINE
  END INTERFACE

!-------------------------------------
! Define interface with C subroutines
!-------------------------------------
  INTERFACE
    SUBROUTINE newline [C,ALIAS:'_newline'] (fid)
      INTEGER fid [REFERENCE]
    END SUBROUTINE newline
  END INTERFACE

  INTERFACE
    SUBROUTINE tab [C,ALIAS:'_tab'] (fid)
      INTEGER fid [REFERENCE]
    END SUBROUTINE tab
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtchar [C,ALIAS:'_wrtchar'] (fid, stg_len, string)
      INTEGER fid [REFERENCE]
   INTEGER stg_len [REFERENCE]
   CHARACTER*80 string [REFERENCE]
    END SUBROUTINE wrtchar
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtint [C,ALIAS:'_wrtint'] (fid, param)
      INTEGER fid [REFERENCE]
  INTEGER param [REFERENCE]
    END SUBROUTINE wrtint
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtreal [C,ALIAS:'_wrtreal'] (fid, param)
      INTEGER fid [REFERENCE]
    DOUBLEPRECISION param [REFERENCE]
    END SUBROUTINE wrtreal
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtexp [C,ALIAS:'_wrtexp'] (fid, param)
      INTEGER fid [REFERENCE]
    DOUBLEPRECISION param [REFERENCE]
    END SUBROUTINE wrtexp
  END INTERFACE

  INTERFACE
    SUBROUTINE close_file [C,ALIAS:'_close_file'] (fid)
      INTEGER fid [REFERENCE]
    END SUBROUTINE close_file
  END INTERFACE

!-------------------------------
! External function declaration
!-------------------------------
  DOUBLEPRECISION,EXTERNAL :: pd_yield
  DOUBLEPRECISION,EXTERNAL :: pd_limit_kp
  DOUBLEPRECISION,EXTERNAL :: pd_limit_cp
  DOUBLEPRECISION,EXTERNAL :: contract22
  DOUBLEPRECISION,EXTERNAL :: firstinv
  DOUBLEPRECISION,EXTERNAL :: pd_det_hp

!-------------------------------------------
! Pointer declaration/ Common pointer block
!-------------------------------------------
```

```
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptstrain
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptstress
  COMMON /pointers/ ptstrain,ptstress

  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptIDstrs
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptIDstrn
  COMMON /pointerID/ ptIDstrs,ptIDstrn

!-------------------------
! New pointer declaration
!-------------------------
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: pthist1
  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: pthist2
  COMMON /pointers1/ pthist1,pthist2

  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptmixed1
  COMMON /ptrsmix/ ptmixed1

  DOUBLEPRECISION,POINTER,DIMENSION(:,:) :: ptplasstn
  COMMON /pointerplas/ ptplasstn

!------------------
! Common variables
!------------------
  INTEGER ::       mtype,ninc1,ninc2
  INTEGER ::       nstress,nstrain,ctype
  COMMON /control/ mtype,ninc1,ninc2,nstress,nstrain,ctype

  DOUBLEPRECISION ::  young,pois
  COMMON /material/ young,pois

  DOUBLEPRECISION ::  fpc_dp,fpt_dp,ko_dp,co_dp
  COMMON /drucker/    fpc_dp,fpt_dp,ko_dp,co_dp

  INTEGER ::         lclflg,lclplane
  COMMON /qanalysis/ lclflg,lclplane

  INTEGER ::        pstpleps,psteff
  COMMON /printpst/ pstpleps,psteff

!---------------------------------
! Local Variable Type Declaration
!---------------------------------
  INTEGER,INTENT(IN) :: Outfid,Logfid,Pstfid,Lclfid

  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:,:,:) :: Eo_ten,Et_ten
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:,:) :: plastic_eps,t_plastic_eps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: eps_dot,old_eps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: tr_eps,tr_eps_e,tr_sig
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: sig_dot,old_sig,NR_sig
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: np_mat,mp_mat,barnp_mat,barmp_mat
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: iter_peps,d_plas_eps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: iden
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: dlam_deps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: temp_Es
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: eps_inc
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:) :: sig_inc
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:) :: residl,resideps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:) :: jnk1,jnk2

  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:,:) :: local_data1

  INTEGER,ALLOCATABLE,DIMENSION(:) :: local_data2

  DOUBLEPRECISION :: fail_ep,dlam_ep_in
  DOUBLEPRECISION :: limit_ep,limit_ep_k,limit_ep_c
  DOUBLEPRECISION :: I1,beta,alpha
  DOUBLEPRECISION :: determ
```

```
  DOUBLEPRECISION :: hp,barhp,nEon
  DOUBLEPRECISION :: jnk3

  INTEGER,PARAMETER :: mequil = 500
  INTEGER :: equil

  INTEGER,PARAMETER :: assoc = 1

  INTEGER :: ninc
  INTEGER :: i,j,k,l,flag
  INTEGER :: local_inc
  INTEGER :: kp_flag

  CHARACTER(LEN=80) :: value
  INTEGER :: stg_len

!========================================================================

!-----------------------------------
! Allocate space for stress history
!-----------------------------------
  CALL alloc8(Logfid,nstrain,ninc2,pthist1)

!-----------------------------------
! Allocate space for strain history
!-----------------------------------
  CALL alloc8(Logfid,nstrain,ninc2,pthist2)

!-------------------------------------------
! Allocate space for plastic strain history
!-------------------------------------------
  CALL alloc8(Logfid,nstrain,ninc2,ptplasstn)

!--------------------------------------------------
! Allocate and initialize elastic stiffness tensor
!--------------------------------------------------
  ALLOCATE(Eo_ten(3,3,3,3))
  CALL el_ten1(young,pois,Eo_ten)

!-----------------------------------------
! Allocate and initialize identity matrix
!-----------------------------------------
  ALLOCATE(iden(3,3))
  DO i = 1,3
    DO j = 1,3
      IF ( i .eq. j) THEN
        iden(i,j) = 1.0d0
      ELSE
        iden(i,j) = 0.0d0
      END IF
    END DO
  END DO

!-----------------------------------------------------------
! Allocate plastic strain and tensile plastic strain tensor
!-----------------------------------------------------------
  ALLOCATE(plastic_eps(3,3,ninc2))
  ALLOCATE(t_plastic_eps(3,3,ninc2))

!---------------------------------
! If required, allocate space for
! localization analysis results
!---------------------------------
  IF (lclflg .eq. 1) THEN
    ALLOCATE(local_data1(91,10,ninc2))
    ALLOCATE(local_data2(ninc2))
  END IF
  local_inc = 0
```

```
!-------------------------------
! Initialize flags for pst file
!-------------------------------
  pstpleps = 0
  psteff = 0

!------------------------------------------------------
! Initialize plasticity hardening/softening parameters
!------------------------------------------------------
  limit_ep_k = ko_dp
  limit_ep_c = 1.0d0
  kp_flag = 0

!=========================================================================
!                 BEGIN LOOP OVER LOAD STEPS
!=========================================================================
  DO ninc=1,ninc2

!   ---------------------------------
!   Write to log file and screen echo
!   ---------------------------------
    value = "***** Increment "
stg_len = LEN_TRIM(value)
    CALL wrtchar(Logfid, stg_len, TRIM(value))
CALL tab(Logfid)
CALL wrtint(Logfid,ninc)
value = " *****"
    stg_len = LEN_TRIM(value)
    CALL wrtchar(Logfid, stg_len, TRIM(value))
CALL newline(Logfid)
PRINT *,' '
PRINT *,'***** Increment ',ninc,' *****'
PRINT *,' '

!   -------------------------------------
!   Add current strains to strain history
!   -------------------------------------
    IF ( ninc .eq. 1 ) THEN
      pthist2(:,ninc) = ptstrain(:,ninc)
    ELSE
      pthist2(:,ninc) = pthist2(:,ninc-1) + ptstrain(:,ninc)
    END IF

!   ------------------------------
!   Fill incremental strain matrix
!   ------------------------------
    ALLOCATE(eps_dot(3,3))
    eps_dot(1,1) = ptstrain(1,ninc)
    eps_dot(2,2) = ptstrain(2,ninc)
    eps_dot(3,3) = ptstrain(3,ninc)
    eps_dot(1,2) = ptstrain(6,ninc) / 2.0d0
    eps_dot(2,1) = ptstrain(6,ninc) / 2.0d0
    eps_dot(1,3) = ptstrain(5,ninc) / 2.0d0
    eps_dot(3,1) = ptstrain(5,ninc) / 2.0d0
    eps_dot(2,3) = ptstrain(4,ninc) / 2.0d0
    eps_dot(3,2) = ptstrain(4,ninc) / 2.0d0

!   ---------------------------------------------------
!   Initialize previous increment's total strain tensor
!   ---------------------------------------------------
    ALLOCATE(old_eps(3,3))
    IF ( ninc .eq. 1 ) THEN
      DO i = 1,3
        DO j= 1,3
          old_eps(i,j) = 0.0d0
        END DO
      END DO
```

```
    ELSE
      old_eps(1,1) = pthist2(1,ninc-1)
      old_eps(2,2) = pthist2(2,ninc-1)
      old_eps(3,3) = pthist2(3,ninc-1)
      old_eps(2,3) = pthist2(4,ninc-1) / 2.0d0
      old_eps(3,2) = pthist2(4,ninc-1) / 2.0d0
      old_eps(1,3) = pthist2(5,ninc-1) / 2.0d0
      old_eps(3,1) = pthist2(5,ninc-1) / 2.0d0
      old_eps(1,2) = pthist2(6,ninc-1) / 2.0d0
      old_eps(2,1) = pthist2(6,ninc-1) / 2.0d0
    END IF

!   -------------------------------------
!   Determine trial elastic strain tensor
!   -------------------------------------
    ALLOCATE(tr_eps_e(3,3))
    IF ( ninc .eq. 1 ) THEN
      DO i = 1,3
        DO j = 1,3
          tr_eps_e(i,j) = old_eps(i,j) + eps_dot(i,j)
        END DO
      END DO
    ELSE
      DO i = 1,3
        DO j = 1,3
          tr_eps_e(i,j) = ( old_eps(i,j) - plastic_eps(i,j,ninc-1) ) + eps_dot(i,j)
        END DO
      END DO
    END IF

!   -----------------------------------
!   Determine trial total strain tensor
!   -----------------------------------
    ALLOCATE(tr_eps(3,3))
    DO i = 1,3
      DO j = 1,3
        tr_eps(i,j) = old_eps(i,j) + eps_dot(i,j)
      END DO
    END DO

!   ---------------------------
!   Store previous stress state
!   ---------------------------
    ALLOCATE(old_sig(3,3))
    IF ( ninc .ne. 1 ) THEN
      old_sig(1,1) = pthist1(1,ninc-1)
      old_sig(2,2) = pthist1(2,ninc-1)
      old_sig(3,3) = pthist1(3,ninc-1)
      old_sig(2,3) = pthist1(4,ninc-1)
      old_sig(3,2) = pthist1(4,ninc-1)
      old_sig(1,3) = pthist1(5,ninc-1)
      old_sig(3,1) = pthist1(5,ninc-1)
      old_sig(1,2) = pthist1(6,ninc-1)
      old_sig(2,1) = pthist1(6,ninc-1)
    ELSE
       DO i = 1,3
        DO j = 1,3
          old_sig(i,j) = 0.0d0
        END DO
      END DO
    END IF

!   --------------------------
!   Determine stress increment
!   --------------------------
    ALLOCATE(sig_dot(3,3))
    CALL contract42(sig_dot,Eo_ten,eps_dot)
```

```
!   ----------------------
!   Determine trial stress
!   ----------------------
    ALLOCATE(tr_sig(3,3))
    DO i = 1,3
      DO j = 1,3
        tr_sig(i,j) = old_sig(i,j) + sig_dot(i,j)
      END DO
    END DO

!   --------------------------------
!   Determine plasticity limit point
!   --------------------------------
    I1 = firstinv(3,tr_sig)
    IF ( I1 .ge. 0.0d0 ) THEN

!     Loading in tension
!     ------------------
      beta = (1.0d0/3.0d0) * (fpc_dp*fpt_dp)
  kp_flag = 1
  limit_ep_k = 1.0d0

      IF ( ninc .eq. 1 ) THEN
    limit_ep_c = 1.0d0
  ELSE
        limit_ep_c = pd_limit_cp(Logfid,ninc,tr_sig,t_plastic_eps(:,:,ninc-1))
  END IF

      IF ( limit_ep_c .lt. 0.005d0 ) THEN
        limit_ep_c = 0.005d0
      END IF

  IF ( limit_ep_c .lt. co_dp ) THEN
    limit_ep_c = co_dp
  END IF

  limit_ep = limit_ep_k * limit_ep_c * beta
  alpha = (1.0d0/3.0d0) * (fpc_dp - fpt_dp)

    ELSE

!     Loading in compression
!     ----------------------
      beta = (1.0d0/3.0d0) * (fpc_dp * fpt_dp)

  limit_ep_c = 1.0d0

      IF ( ninc .eq. 1 ) THEN
limit_ep_k = ko_dp
limit_ep = ko_dp * beta
GO TO 50
  ELSE
        limit_ep_k = pd_limit_kp(Logfid,tr_sig,plastic_eps(:,:,ninc-1))
limit_ep = limit_ep_k * beta
  END IF

  IF ( limit_ep_k .eq. 1.0d0 ) THEN
  kp_flag = 1
  END IF

!     Make sure hardening parameter does not follow descending curve after k = 1.0
!     ----------------------------------------------------------------------------
      IF ( kp_flag .eq. 1 ) THEN
        limit_ep_c = pd_limit_cp(Logfid,ninc,tr_sig,t_plastic_eps(:,:,ninc-1))

        IF ( limit_ep_c .lt. 0.005d0 ) THEN
          limit_ep_c = 0.005d0
        END IF
```

```
IF ( limit_ep_c .lt. co_dp ) THEN
  limit_ep_c = co_dp
END IF

limit_ep = limit_ep_k * limit_ep_c * beta

      END IF

  alpha = limit_ep_c * limit_ep_k * (1.0d0/3.0d0) * (fpc_dp - fpt_dp)

    END IF

50  CONTINUE

!   -----------------------
!   Evaluate yield function
!   -----------------------
    fail_ep = pd_yield(tr_sig,alpha,limit_ep)

    IF ( fail_ep .le. 0.0d0 ) THEN

!     ====================================================================
!                           Material is elastic
!     ====================================================================

!     ---------------------------------
!     Write to log file and screen echo
!     ---------------------------------
  value = "Material is not yielded"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL newline(Logfid)
  PRINT *,'Material is not yielded'

!     -----------------
!     Save stress state
!     -----------------
      pthist1(1,ninc) = tr_sig(1,1)
      pthist1(2,ninc) = tr_sig(2,2)
      pthist1(3,ninc) = tr_sig(3,3)
      pthist1(4,ninc) = tr_sig(2,3)
      pthist1(5,ninc) = tr_sig(1,3)
      pthist1(6,ninc) = tr_sig(1,2)

!     ----------------------
!     Update plastic strains
!     ----------------------
      IF ( ninc .ne. 1 ) THEN
        DO i =1,3
          DO j = 1,3
            plastic_eps(i,j,ninc) = plastic_eps(i,j,ninc-1)
            t_plastic_eps(i,j,ninc) = t_plastic_eps(i,j,ninc-1)
          END DO
        END DO
      ELSE
        DO i =1,3
          DO j = 1,3
            plastic_eps(i,j,ninc) = 0.0d0
            t_plastic_eps(i,j,ninc) = 0.0d0
          END DO
        END DO
      END IF

!     -----------------
!     Deallocate arrays
!     -----------------
      DEALLOCATE(tr_sig,tr_eps,tr_eps_e)
```

```
      DEALLOCATE(old_eps,eps_dot)
      DEALLOCATE(old_sig,sig_dot)

    ELSE IF ( fail_ep .gt. 0.0d0 ) THEN

!     ==================================================================
!                          Material has yielded
!     ==================================================================

!     ----------------------------
!     Echo to log file and screen
!     ----------------------------
  value = "Material has yielded"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL newline(Logfid)
  PRINT *,'Material has yielded'

!     ----------------------------
!     Set flag for plastic strains
!     ----------------------------
      pstpleps = 1

!     ----------------
!     Determine normal
!     ----------------
      ALLOCATE(np_mat(3,3))
      CALL pd_det_np(tr_sig,alpha,np_mat)

!     --------------------------------
!     Determine plastic flow direction
!     --------------------------------
      ALLOCATE(mp_mat(3,3))
      IF ( assoc .eq. 0 ) THEN

!       Non-associated flow
!       -------------------
    CALL pd_det_mp(Logfid,tr_sig,mp_mat)

  ELSE

!       Associated flow
!       ---------------
        CALL pd_det_np(tr_sig,alpha,mp_mat)

  END IF

!     ----------------------------
!     Determine barmp_mat = Eo : m
!     ----------------------------
      ALLOCATE(barmp_mat(3,3))
      CALL contract42(barmp_mat,Eo_ten,mp_mat)

!     --------------------------------
!     Determine barhp = hp + n : Eo : m
!     NOTE : hp = 0 for now!!!!!!
!     --------------------------------
      hp = 0.0d0
      nEon = contract22(3,np_mat,barmp_mat)
      barhp = hp - nEon

!     ----------------------------
!     Determine barnp_mat = n : Eo
!     ----------------------------
      ALLOCATE(barnp_mat(3,3))
      CALL contract24(barnp_mat,np_mat,Eo_ten)

!     ---------------------------------------------
```

```
!     Determine partial of delta lambda wrt eps_dot
!     ---------------------------------------------
      ALLOCATE(dlam_deps(3,3))
      DO i = 1,3
        DO j = 1,3
          dlam_deps(i,j) = (-1.0d0 / barhp) * barnp_mat(i,j)
        END DO
      END DO

!     ------------------------------
!     Determine initial delta lambda
!     ------------------------------
      dlam_ep_in = contract22(3,dlam_deps,eps_dot)

!     --------------------------------------------------
!     Solve for Fp = 0 (associated flow for now, replace
!     np_mat with mp_mat to get non-associated flow)
!     --------------------------------------------------
      ALLOCATE(NR_sig(3,3))
      ALLOCATE(iter_peps(3,3))
      CALL pd_solv_ep(Logfid,ninc,Eo_ten,tr_eps,plastic_eps(:,:,ninc-1), &
                  np_mat,dlam_ep_in,alpha,limit_ep,NR_sig,iter_peps)

     value = "final stress"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  call newline(Logfid)
  call wrtmatrix(Logfid,3,3,NR_sig)

!     ------------------------
!     Store final stress state
!     ------------------------
      pthist1(1,ninc) = NR_sig(1,1)
      pthist1(2,ninc) = NR_sig(2,2)
      pthist1(3,ninc) = NR_sig(3,3)
      pthist1(4,ninc) = NR_sig(2,3)
      pthist1(5,ninc) = NR_sig(1,3)
      pthist1(6,ninc) = NR_sig(1,2)

!     ------------------------------------------
!     Determine nominal plastic strain increment
!     and update total plastic strains
!     ------------------------------------------
      DO i = 1,3
        DO j = 1,3
          plastic_eps(i,j,ninc) = iter_peps(i,j)
        END DO
      END DO

!     ----------------------------------
!     Determine plastic strain increment
!     ----------------------------------
      ALLOCATE(d_plas_eps(3,3))
      DO i = 1,3
        DO j = 1,3
          d_plas_eps(i,j) = plastic_eps(i,j,ninc) - plastic_eps(i,j,ninc-1)
        END DO
      END DO

!     ------------------------------------------------------
!     If kp = 1, determine and store tensile plastic strains
!     ------------------------------------------------------
      IF ( kp_flag .eq. 1 ) THEN
        DO i = 1,3
          DO j = 1,3
            IF ( d_plas_eps(i,j) .gt. 0.0d0 ) THEN
              t_plastic_eps(i,j,ninc) = t_plastic_eps(i,j,ninc-1) + d_plas_eps(i,j)
            ELSE
```

```
              t_plastic_eps(i,j,ninc) = t_plastic_eps(i,j,ninc-1)
            END IF
          END DO
        END DO
      ELSE
        DO i = 1,3
          DO j = 1,3
            t_plastic_eps(i,j,ninc) = t_plastic_eps(i,j,ninc-1)
          END DO
        END DO
      END IF

!     --------------
!     Update normals
!     --------------
      DEALLOCATE(np_mat)
  ALLOCATE(np_mat(3,3))
      CALL pd_det_np(NR_sig,alpha,np_mat)
  DEALLOCATE(mp_mat)
  ALLOCATE(mp_mat(3,3))
  IF ( assoc .eq. 0 ) THEN

!       Non-associated flow
!       -------------------
    CALL pd_det_mp(Logfid,NR_sig,mp_mat)

  ELSE

!       Associated flow
!       ---------------
        CALL pd_det_np(NR_sig,alpha,mp_mat)

  END IF

!     -----------------------------
!     Determine hardening parameter
!     -----------------------------
      hp = pd_det_hp(Logfid,ninc,Eo_ten,NR_sig,np_mat,np_mat,plastic_eps(:,:,ninc-1), &
                     t_plastic_eps(:,:,ninc-1),limit_ep_k,limit_ep_c)

!     ------------------------------------------------
!     Determine tangent operator (Associated flow now)
!     ------------------------------------------------
  ALLOCATE(Et_ten(3,3,3,3))
      CALL pd_tangent(Logfid,Eo_ten,np_mat,np_mat,hp,Et_ten)

!     ------------------------------------------
!     If requested, perform localization analysis
!     ------------------------------------------
      IF (lclflg .eq. 1) THEN
        local_inc = local_inc + 1
        local_data2(local_inc) = ninc
        CALL acoust3d(local_inc,ninc1,local_data1,Eo_ten,Et_ten)
      END IF

!     -----------------
!     Deallocate arrays
!     -----------------
      DEALLOCATE(tr_eps,tr_eps_e)
  DEALLOCATE(old_eps,eps_dot)
      DEALLOCATE(old_sig,sig_dot)
      DEALLOCATE(tr_sig)
      DEALLOCATE(NR_sig)
      DEALLOCATE(iter_peps)
      DEALLOCATE(np_mat,mp_mat,barnp_mat,barmp_mat)
      DEALLOCATE(dlam_deps)
      DEALLOCATE(d_plas_eps)
  DEALLOCATE(Et_ten)
```

```
!   -------------------------
!   End elastic/damaged check
!   -------------------------
    END IF

!=========================================================================
!                  END LOOP OVER LOAD STEPS
!=========================================================================
  END DO

!--------------------------------------------
! Write localization file if needed
!--------------------------------------------
  IF ( lclflg .eq. 1 ) THEN

!   Screen echo
!   -----------
    PRINT *,' '
    PRINT *,'Writing Q-Analysis results to .lcl file'

!   Write localization file
!   -----------------------
    CALL write_lcl(Lclfid,local_inc,ninc1,local_data1,local_data2)

!   Log file echo
!   -------------
    value = "Q-Analysis results written to .lcl file"
    stg_len = LEN_TRIM(value)
    CALL wrtchar(Logfid, stg_len, TRIM(value))
    CALL newline(Logfid)

!   Deallocate Q-Analysis data arrays
!   ---------------------------------
    DEALLOCATE(local_data1)
    DEALLOCATE(local_data2)

  END IF

!-----------------------
! Store plastic strains
!-----------------------
  DO i = 1,ninc2
    ptplasstn(1,i) = plastic_eps(1,1,i)
    ptplasstn(2,i) = plastic_eps(2,2,i)
    ptplasstn(3,i) = plastic_eps(3,3,i)
    ptplasstn(4,i) = plastic_eps(2,3,i) * 2.0d0
    ptplasstn(5,i) = plastic_eps(1,3,i) * 2.0d0
    ptplasstn(6,i) = plastic_eps(1,2,i) * 2.0d0
  END DO
  DEALLOCATE(plastic_eps)
  DEALLOCATE(t_plastic_eps)

!------------------------------------
! Deallocate elastic stiffness tensor
!------------------------------------
  DEALLOCATE(Eo_ten)

!---------------------------
! Deallocate identity matrix
!---------------------------
  DEALLOCATE(iden)

!-----------------------------
! Write results to output file
!-----------------------------
  CALL write_out(Logfid,Outfid)
```

```
!-----------------
! Write post file
!-----------------
  CALL write_pst(Logfid,Pstfid)

!----------------
! Close log file
!----------------
  CALL close_file(Logfid)

END SUBROUTINE pd_strain
!==========================================================================
SUBROUTINE pd_solv_ep(Logfid,ninc,Eo_ten,eps,plas_eps,m_mat,dlam_in,alpha, &
                      limit_ep,sig_out,peps_out)

! PD_SOLV_EP - Solve for Fp = 0 using bisection/cutting plane algorithm
!
! Variables required
! ------------------
! Logfid     = Log file ID
! ninc       = Current load increment
! Es_ten     = Secant stiffness tensor
! eps        = Trial strain tensor
! plas_eps   = Previous increment's plastic strain tensor
! m_mat      = Current plastic flow direction
! dlam_in    = Initial delta lambda value
! limit_ep   = Material strength point
!
! Variables returned
! ------------------
! sig_out  = Final stress state
! peps_out = Final plastic strain state
!
! Subroutine called by
! --------------------
! pd_strain.f90 = Strain controlled parabolic Drucker-Prager model
! pd_mixed.f90  = Mixed controlled parabolic Drucker-Prager model
!
! Functions/subroutines called
! ----------------------------
!
!
!
! Variable definition
! -------------------
!
!
!
!==========================================================================

  IMPLICIT NONE

!------------------------------------
! Define interface with C subroutines
!------------------------------------
  INTERFACE
    SUBROUTINE newline [C,ALIAS:'_newline'] (fid)
      INTEGER fid [REFERENCE]
    END SUBROUTINE newline
  END INTERFACE

  INTERFACE
    SUBROUTINE tab [C,ALIAS:'_tab'] (fid)
      INTEGER fid [REFERENCE]
    END SUBROUTINE tab
  END INTERFACE

  INTERFACE
```

```
    SUBROUTINE wrtchar [C,ALIAS:'_wrtchar'] (fid, stg_len, string)
      INTEGER fid [REFERENCE]
   INTEGER stg_len [REFERENCE]
   CHARACTER*80 string [REFERENCE]
    END SUBROUTINE wrtchar
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtint [C,ALIAS:'_wrtint'] (fid, param)
      INTEGER fid [REFERENCE]
  INTEGER param [REFERENCE]
    END SUBROUTINE wrtint
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtreal [C,ALIAS:'_wrtreal'] (fid, param)
      INTEGER fid [REFERENCE]
    DOUBLEPRECISION param [REFERENCE]
    END SUBROUTINE wrtreal
  END INTERFACE

  INTERFACE
    SUBROUTINE wrtexp [C,ALIAS:'_wrtexp'] (fid, param)
      INTEGER fid [REFERENCE]
    DOUBLEPRECISION param [REFERENCE]
    END SUBROUTINE wrtexp
  END INTERFACE

!-------------------------------
! External function declaration
!-------------------------------
  DOUBLEPRECISION,EXTERNAL :: pd_yield
  DOUBLEPRECISION,EXTERNAL :: contract22

!------------------
! Common variables
!------------------
  INTEGER ::       mtype,ninc1,ninc2
  INTEGER ::       nstress,nstrain,ctype
  COMMON /control/ mtype,ninc1,ninc2,nstress,nstrain,ctype

!---------------------------------
! Local Variable Type Declaration
!---------------------------------
  INTEGER,INTENT(IN) :: Logfid,ninc
  DOUBLEPRECISION,DIMENSION(3,3,3,3),INTENT(IN) :: Eo_ten
  DOUBLEPRECISION,DIMENSION(3,3),INTENT(IN) :: eps,plas_eps,m_mat
  DOUBLEPRECISION,INTENT(IN) :: dlam_in,alpha,limit_ep

  DOUBLEPRECISION,DIMENSION(3,3),INTENT(OUT) :: sig_out,peps_out

  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:,:) :: iter_sig,iter_eps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:,:) :: iter_peps
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: plas_sig
  DOUBLEPRECISION,ALLOCATABLE,DIMENSION(:,:) :: peps_dot

  DOUBLEPRECISION :: f,f0,f1,f2,f_prev,f_conv
  DOUBLEPRECISION :: dlam,dlam1,dlam2

  DOUBLEPRECISION,PARAMETER :: tol = 1.0E-10

  INTEGER,PARAMETER :: miters = 500
  INTEGER :: iter
  INTEGER :: i,j,k,l
  INTEGER :: iloc

  CHARACTER(LEN=80) :: value
  INTEGER :: stg_len
```

```
!======================================================================

!-------------------------------------------
! Allocate iterative stress and strain state
!-------------------------------------------
  ALLOCATE(iter_sig(3,3,miters))
  ALLOCATE(iter_eps(3,3,miters))

!----------------------------------------
! Allocate iterative plastic strain state
!----------------------------------------
  ALLOCATE(iter_peps(3,3,miters))

!--------------------------------
! Initialize bisection parameters
!--------------------------------
  dlam = 0.0d0
  dlam1 = 0.0d0
  dlam2 = 0.0d0
  iloc = 0
  f0 = 0.0d0
  f1 = 0.0d0
  f2 = 0.0d0

!------------------------------
! Midpoint algorithm for Fp = 0
!------------------------------
  DO iter = 1,miters

!   Determine plastic strain rate
!   -----------------------------
    ALLOCATE(peps_dot(3,3))
    DO i = 1,3
      DO j = 1,3
        peps_dot(i,j) = dlam * m_mat(i,j)
      END DO
    END DO

!   Update total plastic strains
!   ----------------------------
DO i = 1,3
  DO j = 1,3
    iter_peps(i,j,iter) = plas_eps(i,j) + peps_dot(i,j)
  END DO
END DO

!   Determine new elastic strain state
!   ----------------------------------
    DO i = 1,3
  DO j = 1,3
    iter_eps(i,j,iter) = eps(i,j) - iter_peps(i,j,iter)
  END DO
END DO

!   Determine new stress state
!   --------------------------
    CALL contract42(iter_sig(:,:,iter),Eo_ten,iter_eps(:,:,iter))

!   Determine new damage function value
!   -----------------------------------
    f_prev = f
    f = pd_yield(iter_sig(:,:,iter),alpha,limit_ep)

!   Check convergence
!   -----------------
    f_conv = f - f_prev
```

```
    IF ( ABS(f) .le. tol ) THEN

!     Convergence, echo to log file and screen
!     ----------------------------------------
      value = "Bisection/cutting plane convergence for dlam_ep on iteration"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL tab(Logfid)
  CALL wrtint(Logfid,iter)
  CALL newline(Logfid)
      IF ( ctype .eq. 1 ) THEN
        PRINT *,'Bisection/cutting plane convergence for dlam_ep on iteration',iter
      END IF

!     Store final stress state
!     ------------------------
      DO i = 1,3
        DO j = 1,3
          sig_out(i,j) = iter_sig(i,j,iter)
        END DO
      END DO

!     Store final plastic strains
!     ---------------------------
      DO i = 1,3
        DO j = 1,3
          peps_out(i,j) = iter_peps(i,j,iter)
        END DO
      END DO

!     Deallocate arrays
!     -----------------
      DEALLOCATE(iter_sig,iter_eps,iter_peps)
  DEALLOCATE(peps_dot)

      GO TO 10

    END IF

!   Update delta lambda by midpoint algorithm
!   ----------------------------------------
    IF ( iter .eq. 1 ) THEN

!     1st iteration, initialize parameters
!     ------------------------------------
      f0 = f
      dlam1 = 0.0d0
      f1 = f0
      dlam = dlam_in

    ELSE

      IF ( iloc .eq. 0 ) THEN

!       Negative side of root not yet found (iloc = 0)
!       ---------------------------------------------
        IF ( f .gt. 0.0d0 ) THEN

!         Still on positive side
!         ----------------------
          dlam1 = dlam
          f1 = f

          dlam = dlam1 * ( f0/ (f0 - f1))

        ELSE

!         Negative side of root found
```

```
!         ---------------------------
          iloc = 1
          dlam2 = dlam
          f2 = f
          dlam = 0.5d0 * (dlam1 + dlam2)
        END IF

      ELSE

!       Negative side of root has been found (iloc = 1)
!       -----------------------------------------------
        IF ( f .gt. 0.0d0 ) THEN
          dlam1 = dlam
          f1 = f
        ELSE
          dlam2 = dlam
          f2 = f
        END IF
        dlam = 0.5d0 * (dlam1 + dlam2)

  END IF

    END IF

!   If no convergence, quit
!   -----------------------
    IF ( iter .eq. miters ) THEN
      PRINT *,'***** ERROR:  No convergence for increment ',ninc
      PRINT *,' *****'
      PRINT *,'***** QUITING *****'
  value = "***** ERROR:  No convergence for dlam_ep for increment "
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL wrtint(Logfid,ninc)
  value = "  *****"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL newline(Logfid)
  value = "***** QUITING *****"
  stg_len = LEN_TRIM(value)
      CALL wrtchar(Logfid, stg_len, TRIM(value))
  CALL newline(Logfid)

      STOP

    END IF

!   Deallocate arrays
!   -----------------
    DEALLOCATE(peps_dot)

  END DO

10 CONTINUE

  RETURN

END SUBROUTINE pd_solv_ep
```

Chapter 20

DAMAGE MECHANICS

The inception of damage mechanics is generally attributed to Kachanov in 1958 and his idea of a scalar damage variable related to damage on a surface in a material (?). Rabotnov furthered the studies by introducing the concept of effective stress in 1969 (?), and Lemaitre and Chaboche later expanded upon the idea of effective stress in 1985 with their strain equivalence principle (?).

Damage mechanics, like plasticity, assumes that a material experiences two distinct states: an initial linear elastic response, followed by an inelastic nonlinear damaged response. However, unlike elastoplasticity, damage mechanics assumes that the degrading stresses and strains associated with the damage response are reversible. This condition results in a decrease of the overall material stiffness once damage takes place, and unloading occurs at a reduced, secant stiffness, rather than at the initial stiffness.

This section presents background information into the theory and formulation for the Rankine-type anisotropic damage model, which provides the damage formulation for the two-surface model. It begins with a discussion of the theoretical framework for damage formulated in the spirit of plasticity. This is followed by the simplest formulation of material degradation, scalar damage, which introduces the concepts of damage variables and nominal and effective stress/strain. The formulation for the Rankine-type anisotropic damage model is then presented, followed by examples of the model behavior under a few simple loading scenarios. The section concludes with a review of previous research into anisotropic damage.

20.1 "Plasticity" format of damage mechanics

Damage mechanics may be formulated in the spirit of plasticity, adopting the concepts of a strain rate decomposition, failure condition, flow rule, and consistency (?). Following these arguments, damage mechanics may be formulated as a decomposition of the total strain rate into the sum of the elastic strain rate and the degraded strain rate,

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_e + \dot{\boldsymbol{\epsilon}}_d \tag{20.1}$$

The boundary between the regions of elastic behavior and progressive damage is governed by the failure condition, defined by

$$F_d = F_d\left(\boldsymbol{\sigma}, \mathbf{q}_d\right) = 0 \tag{20.2}$$

where $\mathbf{q}_d$ are the damage history variables which describe the evolution of the damage surface. The damage flow rule and consistency condition may be derived in a manner similar to elastoplasticity, such that

$$\dot{\boldsymbol{\epsilon}}_d = \dot{\lambda}_d \mathbf{m}_d \tag{20.3}$$

$$\dot{F}_d = \mathbf{n}_d : \dot{\boldsymbol{\sigma}} - H_d \dot{\lambda}_d = 0 \tag{20.4}$$

where in this case $\dot{\lambda}_d$ is the damage multiplier, $\mathbf{m}_d$ the damage flow direction, $\mathbf{n}_d$ the normal to the failure surface, and H_d the damage hardening parameter.

The differences between the elastoplasticity and damage mechanics formulations begin to become apparent when the stress rate equation is considered. The stress rate is defined as

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_s : (\dot{\boldsymbol{\epsilon}}_e - \dot{\boldsymbol{\epsilon}}_d) \tag{20.5}$$

In this case the stiffness $\mathbf{E}_s$ is the material secant stiffness. The use of the secant stiffness is central to the idea that the degraded strains and stresses are reversible, since the material stiffness must degrade to make this idea possible. However, degrading the material stiffness requires an additional equation for the damage formulation, an equation governing the evolution of the secant stiffness tensor $\mathbf{E}_s$. The flow equation (Eq. 20.3) is not sufficient to describe the evolution of the fourth order secant stiffness because degrading strains are a consequence of degrading stiffness, not a cause. However, the degradation of the fourth order secant stiffness is sufficient to describe the evolution of the degrading strains, and so this approach is taken to attain the relation for the evolution of $\mathbf{E}_s$.

Fig. 20.1 illustrates the elastic $d\epsilon^e$ and damage $d\epsilon^d$ strain increments for a given stress increment $d\sigma$. Notice that in damage mechanics loading occurs at the current secant stiffness for that load step,

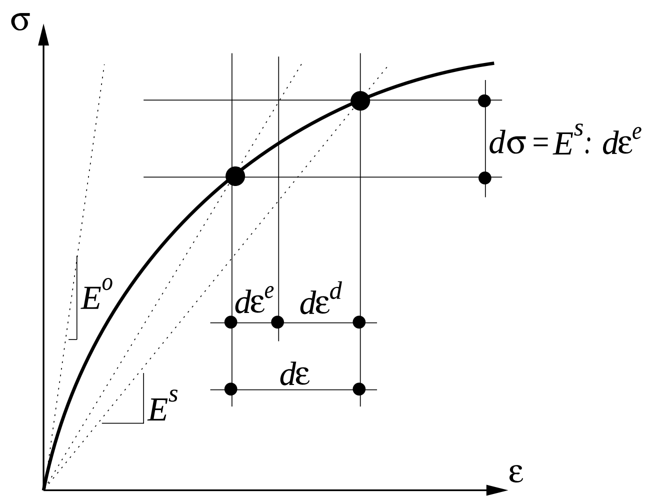


Figure 20.1: Elastic and damage strain increments

and at the end of the load step the secant is updated accordingly.

Continuum damage mechanics describes the stress-strain law in terms of the secant stiffness as

$$\boldsymbol{\sigma} = \mathbf{E}_s : \boldsymbol{\epsilon} \tag{20.6}$$

Differentiating this relation gives the rate form:

$$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{E}}_s : \boldsymbol{\epsilon} + \mathbf{E}_s : \dot{\boldsymbol{\epsilon}} \tag{20.7}$$

Introducing the stress rate equation for damage,

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_s : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_d) \tag{20.8}$$

and comparing it to Eq. 20.7 shows that

$$\dot{\mathbf{E}}_s : \boldsymbol{\epsilon} = -\mathbf{E}_s : \dot{\boldsymbol{\epsilon}}_d \tag{20.9}$$

Now, consider the fourth order identity $\mathbf{E}_s : \mathbf{C}_s = \mathbf{I}_4$, where $\mathbf{C}_s$ is the compliance tensor. The derivative of this identity is

$$\dot{\mathbf{E}}_s : \mathbf{C}_s + \mathbf{E}_s : \dot{\mathbf{C}}_s = 0 \tag{20.10}$$

Multiplying Eq. 20.10 by $\mathbf{C}_s^{-1}$ from the right side and introducing this relation into Eq. 20.9 results in the expression for the degrading strain in terms of the compliance evolution:

$$\dot{\boldsymbol{\epsilon}}_d = \dot{\mathbf{C}}_s : \mathbf{E}_s : \boldsymbol{\epsilon} = \dot{\mathbf{C}}_s : \boldsymbol{\sigma} \tag{20.11}$$

This expression may be attributed to Ortiz (Ortiz and Popov 1985), Neilsen and Schreyer (**?**), and Carol et al. (**?**). It relates the change of compliance to the change of degrading strain. Thus the damage flow rule may be determined if the evolution of the secant compliance is known.

Since the damage flow rule is now derived from the secant compliance evolution law, the compliance evolution law must be defined. Fortunately, it can be defined in a similar manner as the damage strain flow rule, such that

$$\dot{\mathbf{C}}_s = \dot{\lambda}_d \mathbf{M}_d \tag{20.12}$$

where $\dot{\lambda}_d$ is still the damage multiplier, and $\mathbf{M}_d$ is defined as the direction of the rate of change of the secant compliance, a fourth order tensor. Substituting this new relation into Eq. 20.10 results in a relation between $\mathbf{M}_d$ and the damage flow direction $\mathbf{m}_d$,

$$\mathbf{m}_d = \mathbf{M}_d : \mathbf{E}_s : \boldsymbol{\epsilon} = \mathbf{M}_d : \boldsymbol{\sigma} \tag{20.13}$$

Thus by defining the fourth order tensor $\mathbf{M}_d$, the evolution of both the secant compliance $\mathbf{C}_s$ and the degrading strains $\boldsymbol{\epsilon}_d$ may be determined.

The previous relations provide a general framework for material degradation based on a failure surface. This surface can take many forms, resulting in damage formulations ranging from simple $(1 - D)$ scalar damage to fully anisotropic damage. Before discussing anisotropic damage, however, concepts from the simpler isotropic damage will be presented.

20.1.1 Scalar damage

Before considering anisotropic damage, where material degradation is based on the direction of loading, first consider the case of simple isotropic damage, in which material degradation is expressed through a scalar parameter D. In scalar damage, the reduction of the elastic stiffness E_o to the secant stiffness E_s due to material damage is defined by

$$E_s = (1 - D)E_o \tag{20.14}$$

where the damage parameter D provides a measure of the reduction of the stiffness due to the formation of microcracks.

One of the key concepts in damage mechanics is the idea of nominal and effective stresses and strains. Material degradation may be thought of as the average effect of distributed microcracks. As microcracks form in a material subjected to load, the area of the material cross-section that remains intact and able to transmit force decreases. This decrease in "load-bearing" cross-sectional area leads to the idea of "effective" stress and strain. Effective stress and effective strain are defined as stress and strain experienced by the material skeleton between microcracks, in other words the stress and strain in that "load-bearing" cross-section. However, it is difficult, if not impossible, to measure the stresses in the material between microcracks. The stresses and strains obtained in the laboratory are measured externally and satisfy equilibrium and compatibility at the structural level. These stresses and strains are known as "nominal" stresses and strains. The realms of nominal and effective stress and strain are schematically shown in Fig. 20.2. As shown, the nominal stress and strain are measured externally on the overall specimen, while the effective stress and strain reside inside the area of microcracking. Since the undamaged material between microcracks is assumed to remain linear elastic, the relation between effective stress and strain is defined by

$$\sigma^{\text{eff}} = E_o \epsilon^{\text{eff}} \tag{20.15}$$

If the concept of "energy equivalence" (**?**) is assumed, relations between the nominal and effective stress and strain may be determined. In the energy equivalence approach, neither effective strain nor effective stress equal their nominal counterparts. Instead, the elastic energies in terms of effective

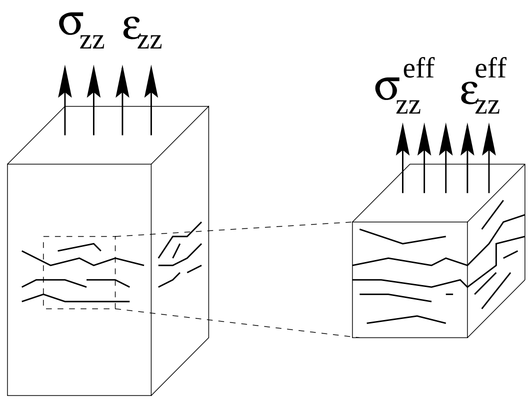


Figure 20.2: Nominal and effective stress and strain

and nominal quantities are equal. Therefore, again considering a scalar damage factor D, the nominal/effective relations for strain are

$$W = \frac{1}{2}\sigma\epsilon = W^{\rm eff} = \frac{1}{2}\sigma^{\rm eff}\epsilon^{\rm eff}$$
$$\sigma^{\rm eff}\epsilon^{\rm eff} = \sigma\epsilon \quad \text{with} \quad \sigma^{\rm eff} = E_o\epsilon^{\rm eff} \quad , \quad \sigma = (1-D)E_o\epsilon$$
$$\sigma = \sqrt{1-D}\sigma^{\rm eff} \quad ; \quad \epsilon^{\rm eff} = \sqrt{1-D}\epsilon \tag{20.16-a}$$

Defining a change of scalar variable to $\overline{\phi}$ and ϕ, such that

$$\overline{\phi} = \frac{1}{\phi} = \sqrt{1-D} \tag{20.17}$$

results in the nominal and effective stress/strain relations:

$$\sigma = \overline{\phi}\sigma^{\rm eff} \quad ; \quad \sigma^{\rm eff} = \phi\sigma \quad ; \quad \epsilon^{\rm eff} = \overline{\phi}\epsilon \quad ; \quad \epsilon = \phi\epsilon^{\rm eff} \tag{20.18}$$

While defining degradation in terms of a scalar parameter such as D provides a simple means of quantifying material damage, it is in fact too simple for many situations. The basic idea of scalar damage is that damage is isotropic; the material strength and stiffness degrades equally in all directions due to a load in any direction. However, this is generally not the case. The orientation of microcracks (and thus the direction of strength and stiffness reduction associated with the average effect of these microcracks) is related to the direction of the tensile load causing the microcracks. The material integrity in other directions should not be greatly affected by cracking in one direction. A more general damage formulation is needed, one which only considers damage in the direction(s) of loading. This direction-sensitive formulation is anisotropic damage.

Chapter 21

OTHER CONSITUTIVE MODELS

21.1 Microplane

21.1.1 Microplane Models

Models based on the microplane concept represent an alternative approach to constitutive modeling. Unlike conventional tensorial models that relate the components of the stress tensor directly to the components of the strain tensor, microplane models work with stress and strain vectors on a set of planes of various orientations (so-called *microplanes*). The basic constitutive laws are defined on the level of the microplane and must be transformed to the level of the material point using certain relations between the tensorial and vectorial components. The most natural choice would be to construct the stress and strain vector on each microplane by projecting the corresponding tensors, i.e., by contracting the tensors with the vector normal to the plane. However, it is impossible to use this procedure for both the stress and the strain and still satisfy a general law relating the vectorial components on every microplane. The original slip theory for metals worked with stress vectors as projections of the stress tensor; this is now called the *static constraint*. Most versions of the microplane model for concrete and soils have been based on the *kinematic constraint*, which defines the strain vector $\boldsymbol{e}$ on an arbitrary microplane with unit normal $\boldsymbol{n}$ as

$$\boldsymbol{e} = \boldsymbol{\varepsilon} \cdot \boldsymbol{n}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor and the dot denotes a contraction. In indicial notation, equation (21.1.1) would read[1]

$$e_i = \varepsilon_{ij} n_j$$

The microplane stress vector, $\boldsymbol{s}$, is defined as the work-conjugate variable of the microplane strain vector, $\boldsymbol{e}$. The relationship between $\boldsymbol{e}$ and $\boldsymbol{s}$ is postulated as a microplane constitutive equation. A formula linking the microplane stress vector to the macroscopic stress tensor follows from the principle of virtual work, written here as[2]

$$\boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = \frac{3}{2\pi} \int_{\Omega} \boldsymbol{s} \cdot \delta\boldsymbol{e} \, \mathrm{d}\Omega$$

where $\delta\boldsymbol{\varepsilon}$ is an arbitrary (symmetric) virtual strain tensor, and

$$\delta\boldsymbol{e} = \delta\boldsymbol{\varepsilon} \cdot \boldsymbol{n}$$

is the corresponding virtual microplane strain vector. Integration in (21.1.1) is performed over all microplanes, characterized by their unit normal vectors, $\boldsymbol{n}$. Because of symmetry, the integration domain

[1] When dealing with tensorial components, we use the so-called Einstein summation convention implying summation over twice repeated subscripts in product-like expressions. For example, subscript j on the right-hand side of (21.1.1) appears both in ε_{ij} and in n_j, and so a sum over j running from 1 to 3 is implied.

[2] The double dot between $\boldsymbol{\sigma}$ and $\delta\boldsymbol{\varepsilon}$ on the left-hand side of (21.1.1) denotes double contraction, in indicial notation written as $\sigma_{ij}\,\delta\varepsilon_{ij}$ (with summation over all i and j between 1 and 3).

Ω is taken as one half of the unit sphere, and the integral is normalized by the area of the unit hemisphere, $2\pi/3$.

Substituting (21.1.1) into (21.1.1) and taking into account the independence of variations $\delta\boldsymbol{\varepsilon}$, we obtain (after certain manipulations restoring symmetry) the following formula for the evaluation of macroscopic stress components:

$$\boldsymbol{\sigma} = \frac{3}{4\pi}\int_{\Omega}(\boldsymbol{s}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{s})\,\mathrm{d}\Omega$$

where the symbol $\otimes$ denotes the direct product of tensors ($\boldsymbol{s}\otimes\boldsymbol{n}$ is a second-order tensor with components $s_i n_j$).

In summary, a kinematically constrained microplane model is described by the kinematic constraint (21.1.1), the stress evaluation formula (21.1.1), and a suitable microplane constitutive law that relates the microplane strain vector, $\boldsymbol{e}$, to the microplane stress vector, $\boldsymbol{s}$. If this law has an explicit form (Carol, I., Bažant, Z.P. and Prat, P.C., 1992)

$$\boldsymbol{s} = \tilde{\boldsymbol{s}}(\boldsymbol{e},\boldsymbol{n})$$

then the resulting macroscopic stress-strain law can be written as

$$\boldsymbol{\sigma} = \frac{3}{4\pi}\int_{\Omega}\left[\tilde{\boldsymbol{s}}(\boldsymbol{e},\boldsymbol{n})\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\tilde{\boldsymbol{s}}(\boldsymbol{e},\boldsymbol{n})\right]\,\mathrm{d}\Omega$$

Realistic models for concrete that take into account the complex interplay between the volumetric and deviatoric components of stress and strain (Bažant and Prat 1988, **?**, Bažant 1996) usually lead to more general microplane constitutive laws of the type

$$\boldsymbol{s} = \tilde{\boldsymbol{s}}(\boldsymbol{e},\boldsymbol{n};\boldsymbol{\sigma})$$

that are affected by some components of the macroscopic stress, $\boldsymbol{\sigma}$, for example by its volumetric part. Instead of a direct evaluation of the explicit formula (21.1.1), the macroscopic stress is then computed as the solution of an implicit equation, and the stress-evaluation algorithm involves some iteration.

21.2 NonLocal

Bibliography

399, A.: n.d., *Standard Test Method for Plane Strain Fracture Toughness of Metallic Materials, E399–74*, Annual Book of ASTM Standards.

Anon.: 1985, RILEM Draft Recommendation (50–FCM): Determination of the Fracture Energy of Mortar and Concrete by Means of Three–Point Bend Tests on Notched Beams, *Materials and Structures* **18**, 287–290.

Bažant, Z.: 1996, Is no tension design of concrete or rock structures always safe?-fracture analysis, *ASCE J. of Structural Engineering* **122**(1), 2–10.

Bažant, Z. and Prat, P.: 1988, Effect of temperature and humidity on fracture energy of concrete, *ACI Materials Journal* **85**(4), 262–271.

British Standards Institution, BS 5447, London: 1977.

Broek, D.: 1986, *Elementary Engineering Fracture Mechanics, 4th Edition*, Martinus Nijhoff Publishers.

Carol, I., Bažant, Z.P. and Prat, P.C.,: 1992, Microplane type constitutive models for distributed damage and localized cracking in concrete structures, *Proc. Fracture Mechanics of Concrete Structures*, Elsevier, Breckenridge, CO, pp. 299–304.

Chen, W. and Zhang, H.: 1990, *Structural Plasticity*, Springer-Verlag.

Cherepanov, G.: 1979, *Mechanics of Brittle Fracture*, McGraw-Hill.

Erdogan, F. and Sih, G.C.: 1963, On the crack extension in plates under plane loading and transverse shear, *Journal of Basic Engineering* **85**, 519–527.

Foreman, R., Kearney, V. and Engle, R.: 1967, Numerical analysis of crack propagation in cyclic-loaded structures, *J. of Basic Engineering* **89**, 459–464.

Gdoutos, E.: 1993, *Fracture Mechanics; An Introduction*, Kluwer Academic Press.

Griffith, A.: 1921, The phenomena of rupture and flow in solids, *Phil. Trans. Roy. Soc. London* **A221**, 163–197.

Hodge, P.: 1959, *Plastic Analysis of Structures*, McGraw-Hill Book Company, New York.

Hudson, C. and Seward, S.: 1978, *A Compendium of Sources of Fracture Toughness and Fatigue Crack Growth Data for Metallic alloys*, Vol. 14, Int. J. of Fracture.

Hudson, C. and Seward, S.: 1982, A compendium of sources of fracture toughness and fatigue crack growth data for metallic alloys, *Int. J. of Fracture, Part II* **20**, R59–R117.

Hussain, M., Pu, S. and Underwood, J.: 1974, Strain energy release rate for a crack under combined mode i and mode ii, *ASTM, STP 560* pp. 2–28.

Irwin, G.: 1957, Analysis of stresses and strains near the end of a crack traversing a plate, *Transactions ASME, J. Appl. Mech.*

Irwin, G.: 1962, The crack extension force for a part through crack in a plate, *Transactions ASME, J. Appl. Mech.*

Kelly, A.: 1974, *Strong Solids*, second edn, Oxford University Press.

Miller, M. and Gallagher, J.: 1981, An analysis of several fatigue crack growth rate (fcgr) descriptions, *in* S. Hudak Jr. and R. Bucci (eds), *Fatigue Crack Growth Measurements and Data Analysis, ASTM STP 738*, ASTM, Philadelphia, pp. 205–251.

Mostovoy, e. a.: 1967, Use of crack-line loaded specimens for measuring plane strain fracture toughness, *J. of Materials* **2**(3), 661–681.

Murakami, Y.: 1987, *Stress Intensity Factors Handbook*, Pergamon Press. (2 Vols).

Newman, J.: 1971, An improved method of collocation for the stress analysis of cracked plates with various shaped boundaries, *Technical Report NASA TN D-6376*, National Aeronautics and Space Administration, Langley Research Center, Hampton, VA.

Newman, J. and Raju, I.: 1981, An emperical stress intensity factor equation for surface cracks, *Engineering Fracture Mechanics* **15**, 185–192.

Ortiz, M. and Popov, E.: 1985, Accuracy and stability of integration algorithms for elastoplastic constitutive relations, *Int. J. Num. Meth. Eng.* **21**, 1561–1576.

Ouchterlony, F.: 1982, Fracture toughness testing of rock, *Technical report*, SveDeFo, Swedish Detonic Research Foundation, Stockholm, Sweden.

Ouchterlony, F.: 1986, Suggested methods for determining fracture toughness of rock material, *Technical report*, International Society for Rock Mechanics, Commission on Testing Methods.

Paris, P. and Erdogan, F.: 1963, A critical analysis of crack propagation laws, *J. of Basic Engineering, ASME*.

Rooke, D. and Cartwright, D.: 1976, *Compendium of Stress Intensity Factors*, HMSO, London.

Sih, G., Paris, P. and Irwin, G.: 1965, On cracks in rectilinearly anisotropic bodies, *International Journal of Fracture Mechanics*.

Tada, H., Paris, P. and Irwin, G.: 1973, *The Stress Analysis of Cracks Handbook*, Del Research Corporation, Hellertown, PA.

Walker, E.: 1970, The effect of stress ratio during crack propagation and fatigue for 2024-t3 and 7075-t6 aluminum, *in* M. Rosenfeld (ed.), *Effects of Environment and Complex Load History on Fatigue Life*, ASTM STP 462.

Westergaard, H.: 1939, Bearing pressures and cracks, *J. Appl. Mech.*

Wheeler, O.: 1972, Spectrum loading and crack growth, *J. of Basic Engineering* **94**, 181–186.

Willenborg, J., Engle, R. and Wood, R.: 1971, A crack growth retardation model using an effective stress concept, *Technical report*, Air Force Flight Dynamics Laboratory Report, AFFDL-TM-71-1-FBR.