# UNIT-IV

# Discrete Probability and Advanced Counting Techniques

In daily lives, many a times one needs to find out the number of all possible outcomes for a series of events. For instance, in how many ways can a panel of judges comprising of 6 men and 4 women be chosen from among 50 men and 38 women? How many different 10 lettered PAN numbers can be generated such that the first five letters are capital alphabets, the next four are digits and the last is again a capital letter. For solving these problems, mathematical theory of counting are used. **Counting** mainly encompasses fundamental counting rule, the permutation rule, and the combination rule.

**The Rules of Sum and Product**

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

**The Rule of Sum** − If a sequence of tasks T1,T2,…,TmT1,T2,…,Tm can be done in w1,w2,…wmw1,w2,…wm ways respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is w1+w2+⋯+wmw1+w2+⋯+wm. If we consider two tasks A and B which are disjoint (i.e. A∩B=∅A∩B=∅), then mathematically |A∪B|=|A|+|B||A∪B|=|A|+|B|

**The Rule of Product** − If a sequence of tasks T1,T2,…,TmT1,T2,…,Tm can be done in w1,w2,…wmw1,w2,…wm ways respectively and every task arrives after the occurrence of the previous task, then there are w1×w2×⋯×wmw1×w2×⋯×wm ways to perform the tasks. Mathematically, if a task B arrives after a task A, then |A×B|=|A|×|B||A×B|=|A|×|B|

**Example**

**Question** − A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

**Solution** − From X to Y, he can go in 3+2=53+2=5 ways (Rule of Sum). Thereafter, he can go Y to Z in 4+5=94+5=9 ways (Rule of Sum). Hence from X to Z he can go in 5×9=455×9=45 ways (Rule of Product).

**Permutations**

A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

**Examples**

From a set S ={x, y, z} by taking two at a time, all permutations are −

xy,yx,xz,zx,yz,zyxy,yx,xz,zx,yz,zy.

We have to form a permutation of three digit numbers from a set of numbers S={1,2,3}S={1,2,3}. Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

**Number of Permutations**

The number of permutations of ‘n’ different things taken ‘r’ at a time is denoted by nPrnPr

nPr=n!(n−r)!nPr=n!(n−r)!

where n!=1.2.3.…(n−1).nn!=1.2.3.…(n−1).n

**Proof** − Let there be ‘n’ different elements.

There are n number of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as [n – (r–1)] = n–r+1

So, the total no. of ways to fill up from first place up to r-th-place −

nPr=n(n−1)(n−2).....(n−r+1)nPr=n(n−1)(n−2).....(n−r+1)

=[n(n−1)(n−2)...(n−r+1)][(n−r)(n−r−1)…3.2.1]/[(n−r)(n−r−1)…3.2.1]=[n(n−1)(n−2)...(n−r+1)][(n−r)(n−r−1)…3.2.1]/[(n−r)(n−r−1)…3.2.1]

Hence,

nPr=n!/(n−r)!nPr=n!/(n−r)!

**Some important formulas of permutation**

If there are *n* elements of which a1a1 are alike of some kind, a2a2 are alike of another kind; a3a3 are alike of third kind and so on and arar are of rthrth kind, where (a1+a2+...ar)=n(a1+a2+...ar)=n.

Then, number of permutations of these n objects is = n!/[(a1!(a2!)…(ar!)]n!/[(a1!(a2!)…(ar!)].

Number of permutations of n distinct elements taking n elements at a time = nPn=n!nPn=n!

The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = n−xpr−xn−xpr−x

The number of permutations of n dissimilar elements when r specified things always come together is − r!(n−r+1)!r!(n−r+1)!

The number of permutations of n dissimilar elements when r specified things never come together is − n!–[r!(n−r+1)!]n!–[r!(n−r+1)!]

The number of circular permutations of n different elements taken x elements at time = npx/xnpx/x

The number of circular permutations of n different things = npn/nnpn/n

**Some Problems**

**Problem 1** − From a bunch of 6 different cards, how many ways we can permute it?

**Solution** − As we are taking 6 cards at a time from a deck of 6 cards, the permutation will be 6P6=6!=7206P6=6!=720

**Problem 2** − In how many ways can the letters of the word 'READER' be arranged?

**Solution** − There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be =6!/[(2!)(1!)(1!)(2!)]=180.=6!/[(2!)(1!)(1!)(2!)]=180.

**Problem 3** − In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

**Solution** − There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves =3P3=3!=6=3P3=3!=6. The remaining 3 vacant places will be filled up by 3 vowels in 3P3=3!=63P3=3!=6 ways. Hence, the total number of permutation is 6×6=366×6=36

**Combinations**

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

nCr=n!r!(n−r)!nCr=n!r!(n−r)!

**Problem 1**

Find the number of subsets of the set {1,2,3,4,5,6}{1,2,3,4,5,6} having 3 elements.

**Solution**

The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be 6C3=206C3=20.

**Problem 2**

There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

**Solution**

The number of ways to choose 3 men from 6 men is 6C36C3 and the number of ways to choose 2 women from 5 women is 5C25C2

Hence, the total number of ways is − 6C3×5C2=20×10=2006C3×5C2=20×10=200

**Problem 3**

How many ways can you choose 3 distinct groups of 3 students from total 9 students?

**Solution**

Let us number the groups as 1, 2 and 3

For choosing 3 students for 1st group, the number of ways − 9C39C3

The number of ways for choosing 3 students for 2nd group after choosing 1st group − 6C36C3

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group − 3C33C3

Hence, the total number of ways =9C3×6C3×3C3=84×20×1=1680=9C3×6C3×3C3=84×20×1=1680

**Pascal's Identity**

Pascal's identity, first derived by Blaise Pascal in 17th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: nCk=n−1Ck−1+n−1CknCk=n−1Ck−1+n−1Ck

**Proof** −

n−1Ck−1+n−1Ckn−1Ck−1+n−1Ck

=(n−1)!(k−1)!(n−k)!+(n−1)!k!(n−k−1)!=(n−1)!(k−1)!(n−k)!+(n−1)!k!(n−k−1)!

=(n−1)!(kk!(n−k)!+n−kk!(n−k)!)=(n−1)!(kk!(n−k)!+n−kk!(n−k)!)

=(n−1)!nk!(n−k)!=(n−1)!nk!(n−k)!

=n!k!(n−k)!=n!k!(n−k)!

=nCk=nCk

**Pigeonhole Principle**

In 1834, German mathematician, Peter Gustav Lejeune Dirichlet, stated a principle which he called the drawer principle. Now, it is known as the pigeonhole principle.

**Pigeonhole Principle** states that if there are fewer pigeon holes than total number of pigeons and each pigeon is put in a pigeon hole, then there must be at least one pigeon hole with more than one pigeon. If n pigeons are put into m pigeonholes where n > m, there's a hole with more than one pigeon.

**Examples**

Ten men are in a room and they are taking part in handshakes. If each person shakes hands at least once and no man shakes the same man’s hand more than once then two men took part in the same number of handshakes.

There must be at least two people in a class of 30 whose names start with the same alphabet.

**The Inclusion-Exclusion principle**

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

|A∪B|=|A|+|B|−|A∩B||A∪B|=|A|+|B|−|A∩B|

For three sets A, B and C, the principle states −

|A∪B∪C|=|A|+|B|+|C|−|A∩B|−|A∩C|−|B∩C|+|A∩B∩C||A∪B∪C|=|A|+|B|+|C|−|A∩B|−|A∩C|−|B∩C|+|A∩B∩C|

The generalized formula -

|⋃ni=1Ai|=∑1≤i<j<k≤n|Ai∩Aj|+∑1≤i<j<k≤n|Ai∩Aj∩Ak|−⋯+(−1)\n−1|A1∩⋯∩A2||⋃i=1nAi|=∑1≤i<j<k≤n|Ai∩Aj|+∑1≤i<j<k≤n|Ai∩Aj∩Ak|−⋯+(−1)\n−1|A1∩⋯∩A2|

**Problem 1**

How many integers from 1 to 50 are multiples of 2 or 3 but not both?

**Solution**

From 1 to 100, there are 50/2=2550/2=25 numbers which are multiples of 2.

There are 50/3=1650/3=16 numbers which are multiples of 3.

There are 50/6=850/6=8 numbers which are multiples of both 2 and 3.

So, |A|=25|A|=25, |B|=16|B|=16 and |A∩B|=8|A∩B|=8.

|A∪B|=|A|+|B|−|A∩B|=25+16−8=33|A∪B|=|A|+|B|−|A∩B|=25+16−8=33

**Problem 2**

In a group of 50 students 24 like cold drinks and 36 like hot drinks and each student likes at least one of the two drinks. How many like both coffee and tea?

**Solution**

Let X be the set of students who like cold drinks and Y be the set of people who like hot drinks.

So, |X∪Y|=50|X∪Y|=50, |X|=24|X|=24, |Y|=36|Y|=36

|X∩Y|=|X|+|Y|−|X∪Y|=24+36−50=60−50=10|X∩Y|=|X|+|Y|−|X∪Y|=24+36−50=60−50=10

Hence, there are 10 students who like both tea and coffee.

**Probability**

**Probability** can be conceptualized as finding the chance of occurrence of an event. Mathematically, it is the study of random processes and their outcomes. The laws of probability have a wide applicability in a variety of fields like genetics, weather forecasting, opinion polls, stock markets etc.

**Basic Concepts**

Probability theory was invented in the 17th century by two French mathematicians, Blaise Pascal and Pierre de Fermat, who were dealing with mathematical problems regarding of chance.

Before proceeding to details of probability, let us get the concept of some definitions.

**Random Experiment** − An experiment in which all possible outcomes are known and the exact output cannot be predicted in advance is called a random experiment. Tossing a fair coin is an example of random experiment.

**Sample Space** − When we perform an experiment, then the set S of all possible outcomes is called the sample space. If we toss a coin, the sample space S={H,T}S={H,T}

**Event** − Any subset of a sample space is called an event. After tossing a coin, getting Head on the top is an event.

The word "probability" means the chance of occurrence of a particular event. The best we can say is how likely they are to happen, using the idea of probability.

Probabilityofoccurenceofanevent=TotalnumberoffavourableoutcomeTotalnumberofOutcomesProbabilityofoccurenceofanevent=TotalnumberoffavourableoutcomeTotalnumberofOutcomes

As the occurrence of any event varies between 0% and 100%, the probability varies between 0 and 1.

**Steps to find the probability**

Step 1 − Calculate all possible outcomes of the experiment.

Step 2 − Calculate the number of favorable outcomes of the experiment.

Step 3 − Apply the corresponding probability formula.

**Tossing a Coin**

If a coin is tossed, there are two possible outcomes − Heads (H)(H) or Tails (T)(T)

So, Total number of outcomes = 2

Hence, the probability of getting a Head (H)(H) on top is 1/2 and the probability of getting a Tails (T)(T) on top is 1/2

**Throwing a Dice**

When a dice is thrown, six possible outcomes can be on the top − 1,2,3,4,5,61,2,3,4,5,6.

The probability of any one of the numbers is 1/6

The probability of getting even numbers is 3/6 = 1/2

The probability of getting odd numbers is 3/6 = 1/2

**Taking Cards From a Deck**

From a deck of 52 cards, if one card is picked find the probability of an ace being drawn and also find the probability of a diamond being drawn.

Total number of possible outcomes − 52

Outcomes of being an ace − 4

Probability of being an ace = 4/52 = 1/13

Probability of being a diamond = 13/52 = ¼

**Probability Axioms**

The probability of an event always varies from 0 to 1. [0≤P(x)≤1][0≤P(x)≤1]

For an impossible event the probability is 0 and for a certain event the probability is 1.

If the occurrence of one event is not influenced by another event, they are called mutually exclusive or disjoint.

If A1,A2....AnA1,A2....An are mutually exclusive/disjoint events, then P(Ai∩Aj)=∅P(Ai∩Aj)=∅ for i≠ji≠j and P(A1∪A2∪....An)=P(A1)+P(A2)+.....P(An)P(A1∪A2∪....An)=P(A1)+P(A2)+.....P(An)

**Properties of Probability**

If there are two events xx and x¯¯¯x¯which are complementary, then the probability of the complementary event is −

p(x¯¯¯)=1−p(x)p(x¯)=1−p(x)

For two non-disjoint events A and B, the probability of the union of two events −

P(A∪B)=P(A)+P(B)P(A∪B)=P(A)+P(B)

If an event A is a subset of another event B (i.e. A⊂BA⊂B), then the probability of A is less than or equal to the probability of B. Hence, A⊂BA⊂B implies P(A)≤p(B)P(A)≤p(B)

**Conditional Probability**

The conditional probability of an event B is the probability that the event will occur given an event A has already occurred. This is written as P(B|A)P(B|A).

Mathematically − P(B|A)=P(A∩B)/P(A)P(B|A)=P(A∩B)/P(A)

If event A and B are mutually exclusive, then the conditional probability of event B after the event A will be the probability of event B that is P(B)P(B).

**Problem 1**

In a country 50% of all teenagers own a cycle and 30% of all teenagers own a bike and cycle. What is the probability that a teenager owns bike given that the teenager owns a cycle?

**Solution**

Let us assume A is the event of teenagers owning only a cycle and B is the event of teenagers owning only a bike.

So, P(A)=50/100=0.5P(A)=50/100=0.5 and P(A∩B)=30/100=0.3P(A∩B)=30/100=0.3 from the given problem.

P(B|A)=P(A∩B)/P(A)=0.3/0.5=0.6P(B|A)=P(A∩B)/P(A)=0.3/0.5=0.6

Hence, the probability that a teenager owns bike given that the teenager owns a cycle is 60%.

**Problem 2**

In a class, 50% of all students play cricket and 25% of all students play cricket and volleyball. What is the probability that a student plays volleyball given that the student plays cricket?

**Solution**

Let us assume A is the event of students playing only cricket and B is the event of students playing only volleyball.

So, P(A)=50/100=0.5P(A)=50/100=0.5 and P(A∩B)=25/100=0.25P(A∩B)=25/100=0.25 from the given problem.

P⟮B|A⟯=P⟮A∩B⟯/P⟮A⟯=0.25/0.5=0.5P⟮B|A⟯=P⟮A∩B⟯/P⟮A⟯=0.25/0.5=0.5

Hence, the probability that a student plays volleyball given that the student plays cricket is 50%.

**Problem 3**

Six good laptops and three defective laptops are mixed up. To find the defective laptops all of them are tested one-by-one at random. What is the probability to find both of the defective laptops in the first two pick?

**Solution**

Let A be the event that we find a defective laptop in the first test and B be the event that we find a defective laptop in the second test.

Hence, P(A∩B)=P(A)P(B|A)=3/9×2/8=1/12P(A∩B)=P(A)P(B|A)=3/9×2/8=1/12

**Bayes' Theorem**

**Theorem** − If A and B are two mutually exclusive events, where P(A)P(A) is the probability of A and P(B)P(B) is the probability of B, P(A|B)P(A|B) is the probability of A given that B is true. P(B|A)P(B|A) is the probability of B given that A is true, then Bayes’ Theorem states −

P(A|B)=P(B|A)P(A)∑ni=1P(B|Ai)P(Ai)P(A|B)=P(B|A)P(A)∑i=1nP(B|Ai)P(Ai)

**Application of Bayes' Theorem**

In situations where all the events of sample space are mutually exclusive events.

In situations where either P(Ai∩B)P(Ai∩B) for each AiAi or P(Ai)P(Ai) and P(B|Ai)P(B|Ai) for each AiAi is known.

**Problem**

Consider three pen-stands. The first pen-stand contains 2 red pens and 3 blue pens; the second one has 3 red pens and 2 blue pens; and the third one has 4 red pens and 1 blue pen. There is equal probability of each pen-stand to be selected. If one pen is drawn at random, what is the probability that it is a red pen?

**Solution**

Let AiAi be the event that ith pen-stand is selected.

Here, i = 1,2,3.

Since probability for choosing a pen-stand is equal, P(Ai)=1/3P(Ai)=1/3

Let B be the event that a red pen is drawn.

The probability that a red pen is chosen among the five pens of the first pen-stand,

P(B|A1)=2/5P(B|A1)=2/5

The probability that a red pen is chosen among the five pens of the second pen-stand,

P(B|A2)=3/5P(B|A2)=3/5

The probability that a red pen is chosen among the five pens of the third pen-stand,

P(B|A3)=4/5P(B|A3)=4/5

According to Bayes' Theorem,

P(B)=P(A1).P(B|A1)+P(A2).P(B|A2)+P(A3).P(B|A3)P(B)=P(A1).P(B|A1)+P(A2).P(B|A2)+P(A3).P(B|A3)

=1/3.2/5+1/3.3/5+1/3.4/5=1/3.2/5+1/3.3/5+1/3.4/5

=3/5

**Recurrence Relations**

recursive techniques can derive sequences and be used for solving counting problems. The procedure for finding the terms of a sequence in a recursive manner is called **recurrence relation**. We study the theory of linear recurrence relations and their solutions. Finally, we introduce generating functions for solving recurrence relations.

**Definition**

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing FnFn as some combination of FiFi with i<ni<n).

**Example** − Fibonacci series − Fn=Fn−1+Fn−2Fn=Fn−1+Fn−2, Tower of Hanoi − Fn=2Fn−1+1Fn=2Fn−1+1

**Linear Recurrence Relations**

A linear recurrence equation of degree k or order k is a recurrence equation which is in the format xn=A1xn−1+A2xn−1+A3xn−1+…Akxn−kxn=A1xn−1+A2xn−1+A3xn−1+…Akxn−k(AnAn is a constant and Ak≠0Ak≠0) on a sequence of numbers as a first-degree polynomial.

These are some examples of linear recurrence equations −

|  |  |  |
| --- | --- | --- |
| **Recurrence relations** | **Initial values** | **Solutions** |
| Fn = Fn-1 + Fn-2 | a1 = a2 = 1 | Fibonacci number |
| Fn = Fn-1 + Fn-2 | a1 = 1, a2 = 3 | Lucas Number |
| Fn = Fn-2 + Fn-3 | a1 = a2 = a3 = 1 | Padovan sequence |
| Fn = 2Fn-1 + Fn-2 | a1 = 0, a2 = 1 | Pell number |

**How to solve linear recurrence relation**

Suppose, a two ordered linear recurrence relation is − Fn=AFn−1+BFn−2Fn=AFn−1+BFn−2 where A and B are real numbers.

The characteristic equation for the above recurrence relation is −

x2−Ax−B=0x2−Ax−B=0

Three cases may occur while finding the roots −

**Case 1** − If this equation factors as (x−x1)(x−x1)=0(x−x1)(x−x1)=0 and it produces two distinct real roots x1x1 and x2x2, then Fn=axn1+bxn2Fn=ax1n+bx2n is the solution. [Here, a and b are constants]

**Case 2** − If this equation factors as (x−x1)2=0(x−x1)2=0 and it produces single real root x1x1, then Fn=axn1+bnxn1Fn=ax1n+bnx1n is the solution.

**Case 3** − If the equation produces two distinct complex roots, x1x1 and x2x2 in polar form x1=r∠θx1=r∠θ and x2=r∠(−θ)x2=r∠(−θ), then Fn=rn(acos(nθ)+bsin(nθ))Fn=rn(acos(nθ)+bsin(nθ)) is the solution.

**Problem 1**

Solve the recurrence relation Fn=5Fn−1−6Fn−2Fn=5Fn−1−6Fn−2 where F0=1F0=1 and F1=4F1=4

**Solution**

The characteristic equation of the recurrence relation is −

x2−5x+6=0,x2−5x+6=0,

So, (x−3)(x−2)=0(x−3)(x−2)=0

Hence, the roots are −

x1=3x1=3 and x2=2x2=2

The roots are real and distinct. So, this is in the form of case 1

Hence, the solution is −

Fn=axn1+bxn2Fn=ax1n+bx2n

Here, Fn=a3n+b2n (As x1=3 and x2=2)Fn=a3n+b2n (As x1=3 and x2=2)

Therefore,

1=F0=a30+b20=a+b1=F0=a30+b20=a+b

4=F1=a31+b21=3a+2b4=F1=a31+b21=3a+2b

Solving these two equations, we get a=2a=2 and b=−1b=−1

Hence, the final solution is −

Fn=2.3n+(−1).2n=2.3n−2nFn=2.3n+(−1).2n=2.3n−2n

**Problem 2**

Solve the recurrence relation − Fn=10Fn−1−25Fn−2Fn=10Fn−1−25Fn−2 where F0=3F0=3 and F1=17F1=17

**Solution**

The characteristic equation of the recurrence relation is −

x2−10x−25=0x2−10x−25=0

So (x−5)2=0(x−5)2=0

Hence, there is single real root x1=5x1=5

As there is single real valued root, this is in the form of case 2

Hence, the solution is −

Fn=axn1+bnxn1Fn=ax1n+bnx1n

3=F0=a.50+(b)(0.5)0=a3=F0=a.50+(b)(0.5)0=a

17=F1=a.51+b.1.51=5a+5b17=F1=a.51+b.1.51=5a+5b

Solving these two equations, we get a=3a=3 and b=2/5b=2/5

Hence, the final solution is − Fn=3.5n+(2/5).n.2nFn=3.5n+(2/5).n.2n

**Problem 3**

Solve the recurrence relation Fn=2Fn−1−2Fn−2Fn=2Fn−1−2Fn−2 where F0=1F0=1 and F1=3F1=3

**Solution**

The characteristic equation of the recurrence relation is −

x2−2x−2=0x2−2x−2=0

Hence, the roots are −

x1=1+ix1=1+i and x2=1−ix2=1−i

In polar form,

x1=r∠θx1=r∠θ and x2=r∠(−θ),x2=r∠(−θ), where r=2–√r=2 and θ=π4θ=π4

The roots are imaginary. So, this is in the form of case 3.

Hence, the solution is −

Fn=(2–√)n(acos(n.⊓/4)+bsin(n.⊓/4))Fn=(2)n(acos(n.⊓/4)+bsin(n.⊓/4))

1=F0=(2–√)0(acos(0.⊓/4)+bsin(0.⊓/4))=a1=F0=(2)0(acos(0.⊓/4)+bsin(0.⊓/4))=a

3=F1=(2–√)1(acos(1.⊓/4)+bsin(1.⊓/4))=2–√(a/2–√+b/2–√)3=F1=(2)1(acos(1.⊓/4)+bsin(1.⊓/4))=2(a/2+b/2)

Solving these two equations we get a=1a=1 and b=2b=2

Hence, the final solution is −

Fn=(2–√)n(cos(n.π/4)+2sin(n.π/4))Fn=(2)n(cos(n.π/4)+2sin(n.π/4))

**Non-Homogeneous Recurrence Relation and Particular Solutions**

A recurrence relation is called non-homogeneous if it is in the form

Fn=AFn−1+BFn−2+f(n)Fn=AFn−1+BFn−2+f(n) where f(n)≠0f(n)≠0

Its associated homogeneous recurrence relation is Fn=AFn–1+BFn−2Fn=AFn–1+BFn−2

The solution (an)(an) of a non-homogeneous recurrence relation has two parts.

First part is the solution (ah)(ah) of the associated homogeneous recurrence relation and the second part is the particular solution (at)(at).

an=ah+atan=ah+at

Solution to the first part is done using the procedures discussed in the previous section.

To find the particular solution, we find an appropriate trial solution.

Let f(n)=cxnf(n)=cxn ; let x2=Ax+Bx2=Ax+B be the characteristic equation of the associated homogeneous recurrence relation and let x1x1 and x2x2 be its roots.

If x≠x1x≠x1 and x≠x2x≠x2, then at=Axnat=Axn

If x=x1x=x1, x≠x2x≠x2, then at=Anxnat=Anxn

If x=x1=x2x=x1=x2, then at=An2xnat=An2xn

**Example**

Let a non-homogeneous recurrence relation be Fn=AFn–1+BFn−2+f(n)Fn=AFn–1+BFn−2+f(n) with characteristic roots x1=2x1=2 and x2=5x2=5. Trial solutions for different possible values of f(n)f(n) are as follows −

|  |  |
| --- | --- |
| **f(n)** | **Trial solutions** |
| 4 | A |
| 5.2n | An2n |
| 8.5n | An5n |
| 4n | A4n |
| 2n2+3n+1 | An2+Bn+C |

**Problem**

Solve the recurrence relation Fn=3Fn−1+10Fn−2+7.5nFn=3Fn−1+10Fn−2+7.5n where F0=4F0=4 and F1=3F1=3

**Solution**

This is a linear non-homogeneous relation, where the associated homogeneous equation is Fn=3Fn−1+10Fn−2Fn=3Fn−1+10Fn−2 and f(n)=7.5nf(n)=7.5n

The characteristic equation of its associated homogeneous relation is −

x2−3x−10=0x2−3x−10=0

Or, (x−5)(x+2)=0(x−5)(x+2)=0

Or, x1=5x1=5 and x2=−2x2=−2

Hence ah=a.5n+b.(−2)nah=a.5n+b.(−2)n , where a and b are constants.

Since f(n)=7.5nf(n)=7.5n, i.e. of the form c.xnc.xn, a reasonable trial solution of at will be AnxnAnxn

at=Anxn=An5nat=Anxn=An5n

After putting the solution in the recurrence relation, we get −

An5n=3A(n–1)5n−1+10A(n–2)5n−2+7.5nAn5n=3A(n–1)5n−1+10A(n–2)5n−2+7.5n

Dividing both sides by 5n−25n−2, we get

An52=3A(n−1)5+10A(n−2)50+7.52An52=3A(n−1)5+10A(n−2)50+7.52

Or, 25An=15An−15A+10An−20A+17525An=15An−15A+10An−20A+175

Or, 35A=17535A=175

Or, A=5A=5

So, Fn=An5n=5n5n=n5n+1Fn=An5n=5n5n=n5n+1

The solution of the recurrence relation can be written as −

Fn=ah+atFn=ah+at

=a.5n+b.(−2)n+n5n+1=a.5n+b.(−2)n+n5n+1

Putting values of F0=4F0=4 and F1=3F1=3, in the above equation, we get a=−2a=−2 and b=6b=6

Hence, the solution is −

Fn=n5n+1+6.(−2)n−2.5nFn=n5n+1+6.(−2)n−2.5n

**Generating Functions**

**Generating Functions** represents sequences where each term of a sequence is expressed as a coefficient of a variable x in a formal power series.

Mathematically, for an infinite sequence, say a0,a1,a2,…,ak,…,a0,a1,a2,…,ak,…, the generating function will be −

Gx=a0+a1x+a2x2+⋯+akxk+⋯=∑k=0∞akxkGx=a0+a1x+a2x2+⋯+akxk+⋯=∑k=0∞akxk

**Some Areas of Application**

Generating functions can be used for the following purposes −

For solving a variety of counting problems. For example, the number of ways to make change for a Rs. 100 note with the notes of denominations Rs.1, Rs.2, Rs.5, Rs.10, Rs.20 and Rs.50

For solving recurrence relations

For proving some of the combinatorial identities

For finding asymptotic formulae for terms of sequences

**Problem 1**

What are the generating functions for the sequences {ak}{ak} with ak=2ak=2 and ak=3kak=3k?

**Solution**

When ak=2ak=2, generating function, G(x)=∑∞k=02xk=2+2x+2x2+2x3+…G(x)=∑k=0∞2xk=2+2x+2x2+2x3+…

When ak=3k,G(x)=∑∞k=03kxk=0+3x+6x2+9x3+……ak=3k,G(x)=∑k=0∞3kxk=0+3x+6x2+9x3+……

**Problem 2**

What is the generating function of the infinite series; 1,1,1,1,…1,1,1,1,…?

**Solution**

Here, ak=1ak=1, for 0≤k≤∞0≤k≤∞

Hence, G(x)=1+x+x2+x3+…⋯=1(1−x)G(x)=1+x+x2+x3+…⋯=1(1−x)

**Some Useful Generating Functions**

For ak=ak,G(x)=∑∞k=0akxk=1+ax+a2x2+……⋯=1/(1−ax)ak=ak,G(x)=∑k=0∞akxk=1+ax+a2x2+……⋯=1/(1−ax)

For ak=(k+1),G(x)=∑∞k=0(k+1)xk=1+2x+3x2……⋯=1(1−x)2ak=(k+1),G(x)=∑k=0∞(k+1)xk=1+2x+3x2……⋯=1(1−x)2

For ak=cnk,G(x)=∑∞k=0cnkxk=1+cn1x+cn2x2+……⋯+x2=(1+x)nak=ckn,G(x)=∑k=0∞cknxk=1+c1nx+c2nx2+……⋯+x2=(1+x)n

For ak=1k!,G(x)=∑∞k=0xkk!=1+x+x22!+x33!……⋯=ex